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On inextensible flows of $\mathsf{B}-\mathbf{M}_2$ developable surfaces of biharmonic $\mathsf{B}-$ slant helices according to Bishop frame in the $\widetilde{\mathrm{SL}_2(\mathbb{R})}$

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Abstract

In this paper, we study inextensible flows of $\mathsf{B}-\mathbf{M}_2$ developable surfaces of biharmonic $\mathsf{B}-$slant helices in the $\widetilde{\mathrm{SL}_2(\mathbb{R})}$. We obtain partial differential equations about $\mathsf{B}-\mathbf{M}_2$ developable surfaces of biharmonic $\mathsf{B}-$slant helices in terms of their curvature and torsion. Finally, we find explicit equations of one-parameter family of the $\mathsf{B}-\mathbf{M}_2$ developable surface associated with unit speed non-geodesic biharmonic B-slant helix in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.

***Keywords**: Biharmonic curve, $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, Curvatures, Developable surface.*

1 Introduction

A smooth map $\phi : N \to M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} \left| \mathrm{T}(\phi) \right|^2 dv_h,$$

where $\mathrm{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ.

The Euler--Lagrange equation of the bienergy is given by $\mathrm{T}_2(\phi) = 0$. Here the section $\mathrm{T}_2(\phi)$ is defined by

$$\mathrm{T}_2(\phi) = -\Delta_{\phi}\mathrm{T}(\phi) + \mathrm{tr}R(\mathrm{T}(\phi), d\phi)d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

This study is organised as follows: Firstly, we study inextensible flows of $\mathrm{B}-\mathbf{M}_2$ developable of biharmonic $\mathrm{B}-$slant helices in the $\widetilde{SL_2(\mathbb{R})}$. Secondly, we obtain partial differential equations about $\mathrm{B}-\mathbf{M}_2$ developable of biharmonic $\mathrm{B}-$slant helices in terms of their curvature and torsion. Finally, we find explicit equations of one-parameter family of the $\mathrm{B}-\mathbf{M}_2$ developable surface associated with unit speed non-geodesic biharmonic B-slant helix in $\widetilde{SL_2(\mathbb{R})}$.

2 $\widetilde{SL_2(\mathbb{R})}$

We identify $\widetilde{SL_2(\mathbb{R})}$ with

$$\mathbb{R}^3_+ = \{(x,y,z) \in \mathbb{R}^3 : z > 0\}$$

endowed with the metric

$$g = ds^2 = (dx + \frac{dy}{z})^2 + \frac{dy^2 + dz^2}{z^2}.$$

The following set of left-invariant vector fields forms an orthonormal basis for $\widetilde{SL_2(\mathbb{R})}$

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \mathbf{e}_2 = z\frac{\partial}{\partial y} - \frac{\partial}{\partial x}, \mathbf{e}_3 = z\frac{\partial}{\partial z}. \tag{2.1}$$

The characterising properties of g defined by

$$g(\mathbf{e}_1,\mathbf{e}_1) = g(\mathbf{e}_2,\mathbf{e}_2) = g(\mathbf{e}_3,\mathbf{e}_3) = 1,$$
$$g(\mathbf{e}_1,\mathbf{e}_2) = g(\mathbf{e}_2,\mathbf{e}_3) = g(\mathbf{e}_1,\mathbf{e}_3) = 0.$$

The Riemannian connection ∇ of the metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y)$$
$$- g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]),$$

which is known as Koszul's formula.

Using the Koszul's formula, we obtain

$$\nabla_{\mathbf{e}_1}\mathbf{e}_1 = 0, \qquad \nabla_{\mathbf{e}_1}\mathbf{e}_2 = \frac{1}{2}\mathbf{e}_3, \quad \nabla_{\mathbf{e}_1}\mathbf{e}_3 = -\frac{1}{2}\mathbf{e}_2,$$

$$\nabla_{\mathbf{e}_2}\mathbf{e}_1 = \frac{1}{2}\mathbf{e}_3, \quad \nabla_{\mathbf{e}_2}\mathbf{e}_2 = \mathbf{e}_3, \nabla_{\mathbf{e}_2}\mathbf{e}_3 = -\frac{1}{2}\mathbf{e}_1 - \mathbf{e}_2, \tag{2.2}$$
$$\nabla_{\mathbf{e}_3}\mathbf{e}_1 = -\frac{1}{2}\mathbf{e}_2, \quad \nabla_{\mathbf{e}_3}\mathbf{e}_2 = \frac{1}{2}\mathbf{e}_1, \quad \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0.$$

Moreover we put

$$R_{ijk} = R(\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_k, \quad R_{ijkl} = R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l),$$

where the indices i, j, k and l take the values $1,2$ and 3

$$R_{1212} = R_{1313} = \frac{1}{4}, \; R_{2323} = -\frac{7}{4}. \tag{2.3}$$

3 Biharmonic $\mathrm{B}-$Slant Helices in $\widetilde{\mathrm{SL}_2(\mathrm{R})}$

Assume that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet--Serret equations:

$$\begin{aligned} &\nabla_{\mathbf{T}}\mathbf{T} = \kappa\mathbf{N}, \\ &\nabla_{\mathbf{T}}\mathbf{N} = -\kappa\mathbf{T} + \tau\mathbf{B}, \\ &\nabla_{\mathbf{T}}\mathbf{B} = -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ its torsion and

$$g_{\widetilde{\mathrm{SL}_2(\mathrm{R})}}(\mathbf{T},\mathbf{T}) = 1, g_{\widetilde{\mathrm{SL}_2(\mathrm{R})}}(\mathbf{N},\mathbf{N}) = 1, g_{\widetilde{\mathrm{SL}_2(\mathrm{R})}}(\mathbf{B},\mathbf{B}) = 1, \tag{3.2}$$
$$g_{\widetilde{\mathrm{SL}_2(\mathrm{R})}}(\mathbf{T},\mathbf{N}) = g_{\widetilde{\mathrm{SL}_2(\mathrm{R})}}(\mathbf{T},\mathbf{B}) = g_{\widetilde{\mathrm{SL}_2(\mathrm{R})}}(\mathbf{N},\mathbf{B}) = 0.$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} &\nabla_{\mathbf{T}}\mathbf{T} = k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\ &\nabla_{\mathbf{T}}\mathbf{M}_1 = -k_1\mathbf{T}, \\ &\nabla_{\mathbf{T}}\mathbf{M}_2 = -k_2\mathbf{T}, \end{aligned} \tag{3.3}$$

where

$$g_{\widetilde{\mathrm{SL}_2(\mathrm{R})}}(\mathbf{T},\mathbf{T}) = 1, g_{\widetilde{\mathrm{SL}_2(\mathrm{R})}}(\mathbf{M}_1,\mathbf{M}_1) = 1, g_{\widetilde{\mathrm{SL}_2(\mathrm{R})}}(\mathbf{M}_2,\mathbf{M}_2) = 1, \tag{3.4}$$
$$g_{\widetilde{\mathrm{SL}_2(\mathrm{R})}}(\mathbf{T},\mathbf{M}_1) = g_{\widetilde{\mathrm{SL}_2(\mathrm{R})}}(\mathbf{T},\mathbf{M}_2) = g_{\widetilde{\mathrm{SL}_2(\mathrm{R})}}(\mathbf{M}_1,\mathbf{M}_2) = 0.$$

Here, we shall call the set $\{\mathbf{T},\mathbf{M}_1,\mathbf{M}_1\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures and $\Upsilon(s)=\arctan\frac{k_2}{k_1}$, $\tau(s)=\Upsilon'(s)$ and $\kappa(s)=\sqrt{k_1^2+k_2^2}$.

Bishop curvatures are defined by

$$k_1=\kappa(s)\cos\Upsilon(s),$$
$$k_2=\kappa(s)\sin\Upsilon(s).$$

The relation matrix may be expressed as

$$\mathbf{T}=\mathbf{T},$$
$$\mathbf{N}=\cos\Upsilon(s)\mathbf{M}_1+\sin\Upsilon(s)\mathbf{M}_2,$$
$$\mathbf{B}=-\sin\Upsilon(s)\mathbf{M}_1+\cos\Upsilon(s)\mathbf{M}_2.$$

On the other hand, using above equation we have

$$\mathbf{T}=\mathbf{T},$$
$$\mathbf{M}_1=\cos\Upsilon(s)\mathbf{N}-\sin\Upsilon(s)\mathbf{B}$$
$$\mathbf{M}_2=\sin\Upsilon(s)\mathbf{N}+\cos\Upsilon(s)\mathbf{B}.$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned}\mathbf{T}&=T^1e_1+T^2e_2+T^3e_3,\\ \mathbf{M}_1&=M_1^1\mathbf{e}_1+M_1^2\mathbf{e}_2+M_1^3\mathbf{e}_3,\\ \mathbf{M}_2&=M_2^1\mathbf{e}_1+M_2^2\mathbf{e}_2+M_2^3\mathbf{e}_3.\end{aligned}\tag{3.5}$$

Theorem 3.1. ([9]) $\gamma:I\to\widetilde{\mathrm{SL}_2(\mathbb{R})}$ *is a biharmonic curve according to Bishop frame if and only if*

$$\begin{aligned}&k_1^2+k_2^2=\text{constant}\neq0,\\ &k_1''-\left[k_1^2+k_2^2\right]k_1=-k_1\left[\frac{15}{4}M_2^1-\frac{1}{4}\right]-2k_2M_1^1M_2^1,\\ &k_2''-\left[k_1^2+k_2^2\right]k_2=2k_1M_1^1M_2^1-k_2\left[\frac{15}{4}M_1^1-\frac{1}{4}\right].\end{aligned}\tag{3.6}$$

To separate a slant helix according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as B-slant helix.

Theorem 3.2. ([9]) *Let* $\gamma : I \to \widetilde{\mathrm{SL}_2(\mathbb{R})}$ *be a unit speed non-geodesic biharmonic curve. Then the position vector of* γ *is*

$$
\begin{aligned}
\gamma(s) = {} & [\frac{1}{\mathsf{Q}_1}\cos \mathsf{W} \sin[\mathsf{Q}_1 s + \mathsf{Q}_2] + \frac{1}{\mathsf{Q}_1}\cos \mathsf{W} \cos[\mathsf{Q}_1 s + \mathsf{Q}_2] + \mathsf{Q}_4 \\
& - \frac{1}{\mathsf{Q}_1^2 + \sin^2 \mathsf{W}} \cos \mathsf{W} (\mathsf{Q}_1 \cos[\mathsf{Q}_1 s + \mathsf{Q}_2] \\
& + \sin \mathsf{W} \sin[\mathsf{Q}_1 s + \mathsf{Q}_2]) + \frac{\mathsf{Q}_5}{\mathsf{Q}_3} e^{\sin \mathsf{W} s}]\mathbf{e}_1 \\
& [-\frac{1}{\mathsf{Q}_1^2 + \sin^2 \mathsf{W}} \cos \mathsf{W} (\mathsf{Q}_1 \cos[\mathsf{Q}_1 s + \mathsf{Q}_2] \\
& + \sin \mathsf{W} \sin[\mathsf{Q}_1 s + \mathsf{Q}_2]) + \frac{\mathsf{Q}_5}{\mathsf{Q}_3} e^{\sin \mathsf{W} s}]\mathbf{e}_2 + \mathbf{e}_3,
\end{aligned}
\tag{3.7}
$$

where $\mathsf{Q}_1, \mathsf{Q}_2, \mathsf{Q}_3, \mathsf{Q}_4, \mathsf{Q}_5$ *are constants of integration*.

4 Inextensible Flows of $\mathsf{B}-\mathbf{M}_2$ Developable Surfaces of Biharmonic $\mathsf{B}-$Slant Helices in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$

To separate a $\mathbf{M}_2$ developable according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for this surface as $\mathsf{B}-\mathbf{M}_2$ developable.

The purpose of this section is to study $\mathsf{B}-\mathbf{M}_2$ developable surfaces of $\mathsf{B}-$slant helices in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.

The $\mathsf{B}-\mathbf{M}_2$ developable of γ is a ruled surface

$$\mathsf{A}(s,u) = \gamma(s) + u\mathbf{M}_2. \tag{4.1}$$

Definition 4.1. *A surface evolution* $\mathsf{A}(s,u,t)$ *and its flow* $\frac{\partial \mathsf{A}}{\partial t}$ *are said to be inextensible if its first fundamental form* $\{\mathbf{E}, \mathbf{F}, \mathbf{G}\}$ *satisfies*

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial \mathbf{F}}{\partial t} = \frac{\partial \mathbf{G}}{\partial t} = 0. \tag{4.2}$$

Definition 4.2. *We can define the following one-parameter family of* $\mathrm{B}-\mathbf{M}_2$ *developable ruled surface*

$$\mathrm{A}(s,u,t)=\gamma(s,t)+u\mathbf{M}_2(s,t). \tag{4.3}$$

Hence, we have the following theorem.

Theorem 4.3. *Let* A *is one-parameter family of the* $\mathrm{B}-\mathbf{M}_2$ *developable surface associated with unit speed non-geodesic biharmonic* B *-slant helix in* $\widetilde{\mathrm{SL}_2(\mathrm{R})}$. *Then* $\frac{\partial \mathrm{A}}{\partial t}$ *is inextensible if and only if*

$$\begin{aligned}&\frac{\partial}{\partial t}[(1-uk_2(t))\cos \mathrm{W}(t)\cos[\mathrm{Q}_1(t)s+\mathrm{Q}_2(t)]]^2\\&+\frac{\partial}{\partial t}[(1-uk_2(t))\cos \mathrm{W}(t)\sin[\mathrm{Q}_1(t)s+\mathrm{Q}_2(t)]]^2\\&=-\frac{\partial}{\partial t}[(1-uk_2(t))\sin \mathrm{W}(t)]^2.\end{aligned} \tag{4.4}$$

Proof. Assume that $\mathrm{A}(s,u,t)$ be a one-parameter family of ruled surface.
From our assumption, we get the following equation

$$\begin{aligned}\mathbf{M}_1 &= \sin \mathrm{W}(t)\cos[\mathrm{Q}_1(t)s+\mathrm{Q}_2(t)]\mathbf{e}_1+\sin \mathrm{W}(t)\sin[\mathrm{Q}_1(t)s+\mathrm{Q}_2(t)]\mathbf{e}_2\\&+\cos \mathrm{W}(t)\mathbf{e}_3,\end{aligned} \tag{4.5}$$

where $\mathrm{Q}_1(t),\mathrm{Q}_2(t)$ are smooth functions of time.
On the other hand, using Bishop formulas (3.3) and (2.1), we have

$$\mathbf{M}_2=\sin[\mathrm{Q}_1(t)s+\mathrm{Q}_2(t)]\mathbf{e}_1-\cos[\mathrm{Q}_1(t)s+\mathrm{Q}_2(t)]\mathbf{e}_2. \tag{4.6}$$

Using above equation and (4.5), we get

$$\begin{aligned}\mathbf{T} &= \cos \mathrm{W}(t)\cos[\mathrm{Q}_1(t)s+\mathrm{Q}_2(t)]\mathbf{e}_1+\cos \mathrm{W}(t)\sin[\mathrm{Q}_1(t)s+\mathrm{Q}_2(t)]\mathbf{e}_2\\&-\sin \mathrm{W}(t)\mathbf{e}_3.\end{aligned} \tag{4.7}$$

Furthermore, we have the natural frame $\{\mathrm{R}_s,\mathrm{R}_u\}$ given by

$$\begin{aligned}\mathrm{H}_s &= (1-uk_1(t))\cos \mathrm{W}(t)\cos[\mathrm{Q}_1(t)s+\mathrm{Q}_2(t)]\mathbf{e}_1\\&+(1-uk_1(t))\cos \mathrm{W}(t)\sin[\mathrm{Q}_1(t)s+\mathrm{Q}_2(t)]\mathbf{e}_2-(1-uk_1(t))\sin \mathrm{W}(t)\mathbf{e}_3,\end{aligned}$$

and

$$\mathrm{A}_u=\sin[\mathrm{Q}_1(t)s+\mathrm{Q}_2(t)]\mathbf{e}_1-\cos[\mathrm{Q}_1(t)s+\mathrm{Q}_2(t)]\mathbf{e}_2.$$

The components of the first fundamental form are

$$\mathbf{E}=g(\mathrm{A}_s,\mathrm{A}_s)=[(1-uk_2(t))\cos \mathrm{W}(t)\cos[\mathrm{Q}_1(t)s+\mathrm{Q}_2(t)]]^2 \tag{4.8}$$

$$+[(1-uk_2(t))\cos \mathsf{W}(t)\sin[\mathsf{Q}_1(t)s+\mathsf{Q}_2(t)]]^2+[(1-uk_2(t))\sin \mathsf{W}(t)]^2,$$

$$\mathbf{F} = g(\mathsf{H}_s,\mathsf{H}_u) = 0, \tag{4.9}$$
$$\mathbf{G} = g(\mathsf{H}_u,\mathsf{H}_u) = 1.$$

Using second and third equation of above system, we have

$$\frac{\partial \mathbf{F}}{\partial t} = 0,$$
$$\frac{\partial \mathbf{G}}{\partial t} = 0.$$

Hence, $\frac{\partial \mathsf{A}}{\partial t}$ is inextensible if and only if (4.4) is satisfied. This concludes the proof of theorem.

Theorem 4.4. *Let* A *is one-parameter family of the* $\mathsf{B}-\mathbf{M}_2$ *developable surface associated with unit speed non-geodesic biharmonic* B*-slant helix in* $\widetilde{\mathsf{SL}_2(\mathsf{R})}$. *Then, the parametric equations of* $\mathsf{B}-$ *tangent developable of* γ *are*

$$\mathsf{A}(s,u,t)=[-\frac{k_2(t)}{k_1(t)}\sin \mathsf{W}(t)\frac{1}{\mathsf{Q}_1(t)}\sin[\mathsf{Q}_1(t)s+\mathsf{Q}_2(t)]$$
$$-\frac{k_2(t)}{k_1(t)}\sin \mathsf{W}(t)\frac{1}{\mathsf{Q}_1}\cos[\mathsf{Q}_1(t)s+\mathsf{Q}_2(t)]+\mathsf{Q}_4(t)$$
$$+\frac{k_2(t)}{k_1(t)}\sin \mathsf{W}(t)\frac{1}{\mathsf{Q}_1^2(t)+\sin^2 \mathsf{W}(t)}(\mathsf{Q}_1(t)\cos[\mathsf{Q}_1(t)s+\mathsf{Q}_2(t)]$$
$$+\sin \mathsf{W}(t)\sin[\mathsf{Q}_1(t)s+\mathsf{Q}_2(t)])+\frac{\mathsf{Q}_5(t)}{\mathsf{Q}_3(t)}e^{\sin \mathsf{W}(t)s}+u\sin[\mathsf{Q}_1(t)s+\mathsf{Q}_2(t)]]\mathbf{e}_1$$
$$[\frac{k_2(t)}{k_1(t)}\sin \mathsf{W}(t)\frac{1}{\mathsf{Q}_1^2(t)+\sin^2 \mathsf{W}(t)}(\mathsf{Q}_1(t)\cos[\mathsf{Q}_1(t)s+\mathsf{Q}_2(t)]$$
$$+\sin \mathsf{W}(t)\sin[\mathsf{Q}_1(t)s+\mathsf{Q}_2(t)])+\frac{\mathsf{Q}_5(t)}{\mathsf{Q}_3(t)}e^{\sin \mathsf{W}(t)s}-u\cos[\mathsf{Q}_1(t)s+\mathsf{Q}_2(t)]]\mathbf{e}_2$$
$$+\mathbf{e}_3,$$

where $\mathsf{Q}_1,\mathsf{Q}_2,\mathsf{Q}_3,\mathsf{Q}_4,\mathsf{Q}_5$ *are smooth functions of time.*

Proof. By the Bishop formula, we have above system. This concludes the proof of Theorem.

We can use Mathematica in above theorem, yields

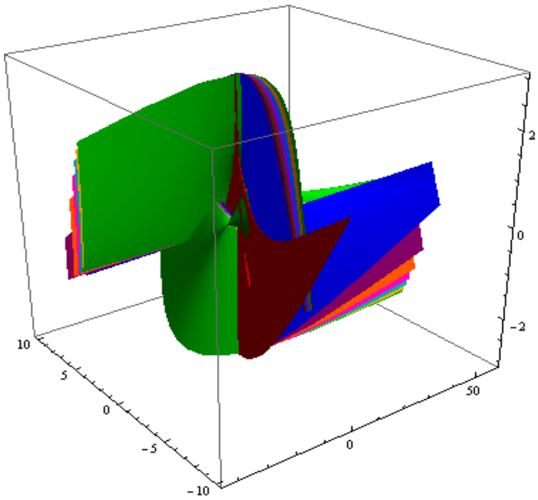

Fig.1.

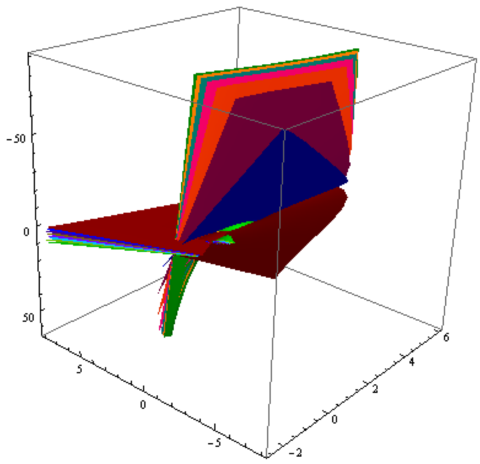

Fig.2.

Fig. 1,2: *The equation (4.10) is illustrated colour Red, Blue, Purple, Orange, Magenta, Cyan, Yellow, Green at the time* $t=1$, $t=1.2$, $t=1.4$, $t=1.6$, $t=1.8$, $t=2$, $t=2.2$, $t=2.4$, *respectively*.

5 Open Problem

The authors can be resarch inextensible flows of $\mathsf{B}-\mathbf{M}_1$ developable surfaces of biharmonic $\mathsf{B}-$slant helices in the $\widetilde{\mathsf{SL}_2(\mathsf{R})}$.

References

[1] L. R. Bishop: *There is More Than One Way to Frame a Curve,* Amer. Math. Monthly 82 (3) (1975) 246-251.

[2] R. Caddeo and S. Montaldo: *Biharmonic submanifolds of* $\mathbb{S}^3$, Internat. J. Math. 12(8) (2001), 867--876.

[3] B. Y. Chen: *Some open problems and conjectures on submanifolds of finite type,* Soochow J. Math. 17 (1991), 169--188.

[4] I. Dimitric: *Submanifolds of* $\mathbb{E}^m$ *with harmonic mean curvature vector,* Bull. Inst. Math. Acad. Sinica 20 (1992), 53--65.

[5] J. Eells and L. Lemaire: *A report on harmonic maps,* Bull. London Math. Soc. 10 (1978), 1--68.

[6] J. Eells and J. H. Sampson: *Harmonic mappings of Riemannian manifolds,* Amer. J. Math. 86 (1964), 109--160.

[7] G. Y.Jiang: *2-harmonic isometric immersions between Riemannian manifolds,* Chinese Ann. Math. Ser. A 7(2) (1986), 130--144.

[8] G. Y. Jiang: *2-harmonic maps and their first and second variational formulas,* Chinese Ann. Math. Ser. A 7(4) (1986), 389--402.

[9] T. Körpınar, E. Turhan: *Biharmonic B-Slant Helices According To Bishop Frame In The* $\widetilde{SL_2(\mathbb{R})}$, Bol. Soc. Paran. Mat. 31 (2) (2013), 39--45.

[10] DY. Kwon , FC. Park, DP Chi: *Inextensible flows of curves and developable surfaces,* Appl. Math. Lett. 18 (2005), 1156-1162.

[11] E. Loubeau and S. Montaldo: *Biminimal immersions in space forms,* preprint, 2004, math.DG/0405320 v1.

[12] I. Sato: On a structure similar to the almost contact structure, Tensor, (N.S.), 30 (1976), 219-224.

[13] T. Takahashi: Sasakian ϕ-symmetric spaces, Tohoku Math. J., 29 (1977), 91-113.

[14] E. Turhan, T. Körpınar: *On Characterization Of Timelike Horizontal Biharmonic Curves In The Lorentzian Heisenberg Group* $Heis^3$, Zeitschrift für Naturforschung A- A Journal of Physical Sciences 65a (2010), 641-648.

[15] E. Turhan and T. Körpınar: *On Characterization Canal Surfaces around Timelike Horizontal Biharmonic Curves in Lorentzian Heisenberg Group* $Heis^3$, Zeitschrift für Naturforschung A- A Journal of Physical Sciences 66a (2011), 441-449.

[16] D.J. Unger: *Developable surfaces in elastoplastic fracture mechanics,* Int. J. Fract. 50 (1991), 33--38.