Int. J. Open Problems Complex Analysis, Vol. 4, No. 2, July 2012
ISSN 2074-2827; Copyright ©ICSRS Publication, 2012
www.i-csrs.org

Sandwich Theorems for Some Subclasses of p-Valent Functions Defined by New Differential Operator

M.K.Aouf, A.O.Mostafa, A.M.Shahin, S.M.Madian

Department of Mathematics,
Faculty of Science, Mansoura University,
Mansoura 35516, Egypt
e-mails: mkaouf127@yahoo.com
adelaeg254@yahoo.com
amshahin@mans.edu.eg
samar_math@yahoo.com

Abstract

The purpose of this paper is to derive some subordination, superordination and sandwich results, which are connected by new differential operator $D^m_{p,l,\lambda}$.

Keywords: *Analytic functions, subordination, superordination, sandwich theorems, differential operator.*
2000 Mathematical Subject Classification: 30C45.

1 Introduction

Let $H = H(U)$ denote the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and $H[a, p]$ denote the subclass of the functions $f \in H$ of the form

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \dots \quad (a \in \mathbb{C};\ p \in \mathbb{N} = \{1, 2, \dots\}).$$

Also, let $A(p)$ be the subclass of functions $f \in H$ of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}). \tag{1}$$

We write $A(1) = A$.

If f , $g \in H$ are analytic in U, we say that f is subordinate to g, or g is superordinate to f, if there exists a Schwarz function $w(z)$ in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$. In such a case we write $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$. If $g(z)$ is univalent in U, then the following equivalence relationship holds true (cf., e.g.,[8] and [14]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Let $\varphi, h \in H$ and

$$\psi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}.$$

If $\varphi(z)$ and $\psi(\varphi(z), z\varphi'(z), z^2\varphi''(z); z)$ are univalent functions in U and $\varphi(z)$ satisfies the second-order superordination

$$h(z) \prec \psi(\varphi(z), z\varphi'(z), z^2\varphi''(z); z), \tag{2}$$

then φ is called to be a solution of the differential superordination (2). A function $q \in H$ is called a subordinant of (2), if $q(z) \prec \varphi(z)$ for all the functions φ satisfying (2). A univalent subordinant $\widetilde{q}$ that satisfies $q(z) \prec \widetilde{q}(z)$ for all the subordinants q of (2), is said to be the best subordinant. Recently, Miller and Mocanu [15] obtained sufficient conditions on the functions h, q and ψ for which the following implication holds:

$$h(z) \prec \psi(\varphi(z), z\varphi'(z), z^2\varphi''(z); z) \Rightarrow q(z) \prec \varphi(z).$$

Using these results, Bulboacă [7] considered certain classes of first order differential superordinations, as well as subordination preserving integral operators [6]. Obradović and Owa [17] obtained subordination results for the quantity $\left(\frac{f(z)}{z}\right)^{\mu}$, where $\mu \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$.

For $f \in A(p)$ given by (1) and $g \in A(p)$ defined by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \tag{3}$$

the Hadamard product or (convolution) is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z).$$

Using the convolution and for $\lambda, l \geqslant 0, p \in \mathbb{N}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we define the linear operator $D^m_{p,l,\lambda}(f * g) : A(p) \to A(p)$ by:

$$
\begin{aligned}
D^0_{p,l,\lambda}(f*g)(z) &= (f*g)(z);\\
D^1_{p,l,\lambda}(f*g)(z) &= D_{p,l,\lambda}(f*g)(z) = (1-\lambda)(f*g)(z) + \frac{\lambda}{(p+l)z^{l-1}}\ \left(z^l(f*g)(z)\right)'\\
&= z^p + \sum_{k=p+1}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right) a_k b_k z^k\ ;\\
D^2_{p,l,\lambda}(f*g)(z) &= (1-\lambda)D_{p,l,\lambda}(f*g)(z) + \frac{\lambda}{(p+l)z^{l-1}}\ \left(z^l D_{p,l,\lambda}(f*g)(z)\right)'\\
&= z^p + \sum_{k=p+1}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^2 a_k b_k z^k
\end{aligned}
$$

and (in general)

$$
\begin{aligned}
D^m_{p,l,\lambda}(f*g)(z) &= (1-\lambda)D^{m-1}_{p,l,\lambda}(f*g)(z) + \frac{\lambda}{(p+l)z^{l-1}}\ \left(z^l D^{m-1}_{p,l,\lambda}(f*g)(z)\right)'\\
&= z^p + \sum_{k=p+1}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m a_k b_k z^k\ . \qquad (4)
\end{aligned}
$$

From (4), we can easily deduce that

$$\lambda z\left(D^m_{p,l,\lambda}(f*g)(z)\right)' = (p+l)D^{m+1}_{p,l,\lambda}(f*g)(z) - [p(1-\lambda)+l]\, D^m_{p,l,\lambda}(f*g)(z)\ (\lambda>0). \qquad (5)$$

We remark that:

(i) For $g(z) = z^p(1-z)^{-1}$ or $b_k = 1$ $(k \geq p+1)$, we have $D^m_{p,l,\lambda}(f*g)(z) = I^m_p(\lambda,l)f(z)$, where the operator $I^m_p(\lambda,l)$ was introduced and studied by Catas [9] which contains the operators D^m_p (see [4] and [12]) and D^m_λ (see [1]);

(ii) For $b_k = \dfrac{(\alpha_1)_{k-p}...(\alpha_q)_{k-p}}{(\beta_1)_{k-p}...(\beta_s)_{k-p}(1)_{k-p}}$, we have $D^m_{p,l,\lambda}(f*g)(z) = I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)$, where the operator $I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)$ was introduced and studied by El-Ashwah and Aouf [11] ($\alpha_1,\alpha_2,...,\alpha_q$ and $\beta_1,\beta_2,...,\beta_s$ are real or complex numbers, $\beta_j \notin \mathbb{Z}_0^- = \{0,-1,-2,...\}$, $j = 1,2,...,s$, $q \leq s+1$, $s,q \in \mathbb{N}_0$) and

$$(d)_k = \begin{cases} 1 & (k=0; d\in\mathbb{C}^*)\\ d(d+1)...(d+k-1) & (k\in\mathbb{N}; d\in\mathbb{C}); \end{cases}$$

(iii) For $m = 0$ and $b_k = \dfrac{\Gamma(p+\alpha+\beta)\Gamma(k+\beta)}{\Gamma(p+\beta)\Gamma(k+\alpha+\beta)}$ $(\alpha \geq 0, \beta > -p, p \in \mathbb{N})$, we have $D^m_{p,l,\lambda}(f*g)(z) = Q^\alpha_{p,\beta}f(z)$, where the operator $Q^\alpha_{p,\beta}$ was introduced by Liu and Owa [13] and reduces to the generalized Bernardi-Libera-Livingston operator $F_{c,p}$ for $\alpha = 1$ and $\beta = c$ $(c > -p, p \in \mathbb{N})$ (see [10]).

In this paper, we obtain sufficient conditions for analytic functions $f, g \in A(p)$ defined by using the operator $D^m_{p,l,\lambda}$ to satisfy:

$$q_1(z) \prec \left(\frac{D^m_{p,l,\lambda}(f * g)(z)}{z^p}\right)^\mu \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U.

2 Definitions and Preliminaries

To prove our results we shall need the following definition and lemmas.

Definition 1 [**15**]**.** *Let $\mathcal{Q}$ be the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where*

$$E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Lemma 1 [**14**]. *Let q be univalent in the unit disc U and let θ and ψ be analytic in a domain D containing $q(U)$, with $\psi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\psi(q(z))$, $S(z) = \theta(q(z)) + Q(z)$ and suppose that*

(i) Q is a starlike function in U,

(ii) $\mathrm{Re}\left\{\dfrac{zS'(z)}{Q(z)}\right\} > 0,\ z \in U.$

If φ is analytic in U with $\varphi(0) = q(0)$, $\varphi(U) \subseteq D$ and

$$\theta(\varphi(z)) + z\varphi'(z)\psi(\varphi(z)) \prec \theta(q(z)) + zq'(z)\psi(q(z)), \tag{6}$$

then $\varphi(z) \prec q(z)$ and q is the best dominant of (6).

Lemma 2 [**8**]**.** *Let q be a univalent function in the unit disc U and let θ and ψ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\mathrm{Re}\left\{\dfrac{\theta'(q(z))}{\psi(q(z))}\right\} > 0$ *for $z \in U$,*

(ii) $zq'(z)\psi(q(z))$ is starlike in U.

If $\varphi(z) \in H[q(0), 1] \cap \mathcal{Q}$, with $\varphi(U) \subseteq D$, $\theta(\varphi(z)) + z\varphi'(z)\psi(\varphi(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\psi(q(z)) \prec \theta(\varphi(z)) + z\varphi'(z)\psi(\varphi(z)), \tag{7}$$

then $q(z) \prec \varphi(z)$ and q is the best subordinant of (7).

Lemma 3 [**18**]**.** The function $q(z) = (1 - z)^{-2ab}$ $(a, b \in \mathbb{C}^*)$ is univalent in U if and only if $|2ab - 1| \leq 1$ or $|2ab + 1| \leq 1$.

3 Subordination results

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\mu, \delta \in \mathbb{C}^*$, $\sigma, \upsilon \in \mathbb{C}$, $p \in \mathbb{N}, m \in \mathbb{N}_0$, $\lambda > 0, l \geqslant 0, z \in U$, $f, g \in A(p)$ are given by (1) and (3), respectively, and the powers are considered the principal ones.

Theorem 1. *Let $q(z)$ be convex univalent in U with $q(0) = 1$ and satisfies*

$$\operatorname{Re}\left\{\frac{\sigma + 2\upsilon q(z)}{\delta}\right\} > 0. \tag{8}$$

Let

$$\chi(f, g, \sigma, \upsilon, \delta, \mu, p, \lambda, l, m)(z) = \sigma\left(\frac{D^m_{p,l,\lambda}(f * g)(z)}{z^p}\right)^{\mu} + \upsilon\left(\frac{D^m_{p,l,\lambda}(f * g)(z)}{z^p}\right)^{2\mu}$$

$$+\delta\mu\frac{(p+l)}{\lambda}\left(\frac{D^m_{p,l,\lambda}(f * g)(z)}{z^p}\right)^{\mu}\left(\frac{D^{m+1}_{p,l,\lambda}(f * g)(z)}{D^m_{p,l,\lambda}(f * g)(z)} - 1\right). \tag{9}$$

If $q(z)$ satisfies the following subordination:

$$\chi(f, g, \sigma, \upsilon, \delta, \mu, p, \lambda, l, m)(z) \prec \sigma q(z) + \upsilon(q(z))^2 + \delta z q'(z), \tag{10}$$

then

$$\left(\frac{D^m_{p,l,\lambda}(f * g)(z)}{z^p}\right)^{\mu} \prec q(z) \tag{11}$$

and $q(z)$ is the best dominant.

Proof. Define $\varphi(z)$ by

$$\varphi(z) = \left(\frac{D^m_{p,l,\lambda}(f * g)(z)}{z^p}\right)^{\mu} \quad (z \in U). \tag{12}$$

Then the function $\varphi(z)$ is analytic in U and $\varphi(0) = 1$. Therefore, differentiating (12) logarithmically with respect to z, we deduce that

$$\frac{z\varphi'(z)}{\varphi(z)} = \mu\left(\frac{z\left(D^m_{p,l,\lambda}(f * g)(z)\right)'}{D^m_{p,l,\lambda}(f * g)(z)} - p\right). \tag{13}$$

From (13) and by using (5), a simple computation shows that

$$\sigma\varphi(z) + \upsilon(\varphi(z))^2 + \delta z\varphi'(z) = \sigma\left(\frac{D^m_{p,l,\lambda}(f * g)(z)}{z^p}\right)^{\mu} + \upsilon\left(\frac{D^m_{p,l,\lambda}(f * g)(z)}{z^p}\right)^{2\mu}$$

$$+\delta\mu\frac{(p+l)}{\lambda}\left(\frac{D^m_{p,l,\lambda}(f * g)(z)}{z^p}\right)^{\mu}\left(\frac{D^{m+1}_{p,l,\lambda}(f * g)(z)}{D^m_{p,l,\lambda}(f * g)(z)} - 1\right), \tag{14}$$

hence the subordination (10) is equivalent to

$$\sigma\varphi(z) + \upsilon(\varphi(z))^2 + \delta z\varphi'(z) \prec \sigma q(z) + \upsilon(q(z))^2 + \delta zq'(z).$$

The above subordination can be written as (6), when $\theta(w) = \sigma w + \upsilon w^2$ and $\psi(w) = \delta$. Note that $\psi(w) \neq 0$ and $\theta(w)$, $\psi(w)$ are analytic in $\mathbb{C}$. Setting

$$Q(z) = zq'(z)\psi(q(z)) = \delta zq'(z) \tag{15}$$

and

$$S(z) = \theta(q(z)) + Q(z) = \sigma q(z) + \upsilon(q(z))^2 + \delta zq'(z), \tag{16}$$

we can verify that $Q(z)$ is starlike univalent in U and

$$\operatorname{Re}\left\{\frac{zS'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\frac{\sigma + 2\upsilon q(z)}{\delta} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0. \tag{17}$$

The theorem follows by applying Lemma 1.

Theorem 2. *Let $q(z)$ be convex univalent in U and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. Further assume that*

$$\operatorname{Re}\left\{\frac{\upsilon q(z)}{\delta} - \frac{zq'(z)}{q(z)}\right\} > 0. \tag{18}$$

If

$$\sigma+\upsilon\left(\frac{D^m_{p,l,\lambda}(f*g)(z)}{z^p}\right)^{\mu}+\delta\mu\frac{(p+l)}{\lambda}\left(\frac{D^{m+1}_{p,l,\lambda}(f*g)(z)}{D^m_{p,l,\lambda}(f*g)(z)}-1\right) \prec \sigma+\upsilon q(z)+\delta\frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{D^m_{p,l,\lambda}(f*g)(z)}{z^p}\right)^{\mu} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Let $\theta(w) = \sigma + \upsilon w$ and $\psi(w) = \frac{\delta}{w}$, we have $\psi(w) \neq 0$ and $\theta(w)$ is analytic in $\mathbb{C}$ and $\psi(w)$ is analytic in $\mathbb{C}^*$. Hence the result follows as an application of Lemma 1 for $\varphi(z) = \left(\frac{D^m_{p,l,\lambda}(f*g)(z)}{z^p}\right)^{\mu}$.

Taking $q(z) = \frac{1}{(1-z)^{2\mu b}}$ $(\mu, b \in C^*)$, $\delta = \frac{1}{\mu b}$, $\lambda = \sigma = p = 1$, $g(z) = z(1-z)^{-1}$ *or* $b_k = 1$ $(k \geq 2)$ and $m = \nu = \ell = 0$ in Theorem 2, we obtain the result obtained by Obradovic et al. [16, Theorem 1].

Corollary 1. *Let $\mu, b \in C^*$ such that $|2\mu b - 1|$ or $|2\mu b + 1| \leq 1$. Let $f(z) \in A$ and suppose that $\frac{f(z)}{z} \neq 0$ for all $z \in U$. If*

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1+z}{1-z}\ ,$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec (1-z)^{-2\mu b}$$

and $(1-z)^{-2\mu b}$ is the best dominant.

Remark 1. For $\mu = 1$, Corollary 1, reduces to the recent result of Srivastava and Lashin [19, Corollary1].

Taking $q(z) = (1+Bz)^{\frac{\mu(A-B)}{B}}$, $-1 \leq B < A \leq 1, B \neq 0,\ \mu \in C^*,\ \lambda = \sigma = \delta = p = 1,\ m = \nu = \ell = 0$ and $g(z) = z(1-z)^{-1}$ *or* $b_k = 1 (k \geq 2)$ in Theorem 2, we obtain the following corollary.

Corollary 2. *Let* $-1 \leq B < A \leq 1$, *with* $B \neq 0$, *and suppose that*

$$\left|\frac{\mu(A-B)}{B} - 1\right| \leq 1$$

or

$$\left|\frac{\mu(A-B)}{B} + 1\right| \leq 1 \ .$$

If $f(z) \in A$ such that $\frac{f(z)}{z} \neq 0$ for all $z \in U$, and let $\mu \in C^*$. If

$$1 + \mu\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1 + [B + \mu(A-B)]\,z}{1+Bz} \ ,$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec (1+Bz)^{\frac{\mu(A-B)}{B}}$$

and $(1+Bz)^{\frac{\mu(A-B)}{B}}$ is the best dominant.

Remark 2. For $\mu = 1$, Corollary 2, reduces to the recent result of Obradovic and Owa [17].

Putting $q(z) = (1-z)^{-2\mu b \cos\rho e^{-i\rho}}$ $(\mu, b \in C^*; |\rho| < \frac{\pi}{2})$, $\delta = \frac{e^{i\rho}}{\mu b \cos\rho}, \sigma = p = \lambda = 1$, $g(z) = z(1-z)^{-1}$ *or* $b_k = 1$ $(k \geq 2)$ and $m = \nu = \ell = 0$ in Theorem 2, we obtain the next result due to Aouf et al. [2, Theorem 1].

Corollary 3 [2]. *Let* $\mu, b \in C^*$ *and* $|\rho| < \frac{\pi}{2}$, *and suppose that* $|2\mu b \cos\rho e^{-i\rho} - 1| \leq 1$ *or* $|2\mu b \cos\rho e^{-i\rho} + 1| \leq 1$. *Let* $f(z) \in A$ *such that* $\frac{f(z)}{z} \neq 0$ *for all* $z \in U$. *If*

$$1 + \frac{e^{-i\rho}}{b\cos\rho}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1+z}{1-z} \ ,$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec (1-z)^{-2\mu b \cos \rho e^{-i\rho}}$$

and $(1-z)^{-2\mu b \cos \rho e^{-i\rho}}$ is the best dominant.

Using arguments similar to those of the proof of Theorem 1, we obtain the following result.

Theorem 3. *Let $q(z)$ be convex univalent in U with $q(0)=1$, satisfies (8) and*

$$\phi(f,g,\sigma,\upsilon,\delta,\mu,p,\lambda,l,m)(z) = \sigma\left(\frac{z^p}{D^m_{p,l,\lambda}(f*g)(z)}\right)^{\mu} + \upsilon\left(\frac{z^p}{D^m_{p,l,\lambda}(f*g)(z)}\right)^{2\mu}$$

$$+\delta\mu\frac{(p+l)}{\lambda}\left(\frac{z^p}{D^m_{p,l,\lambda}(f*g)(z)}\right)^{\mu}\left(1-\frac{D^{m+1}_{p,l,\lambda}(f*g)(z)}{D^m_{p,l,\lambda}(f*g)(z)}\right). \tag{19}$$

If

$$\phi(f,g,\sigma,\upsilon,\delta,\mu,p,\lambda,l,m)(z) \prec \sigma q(z) + \upsilon(q(z))^2 + \delta z q'(z),$$

then

$$\left(\frac{z^p}{D^m_{p,l,\lambda}(f*g)(z)}\right)^{\mu} \prec q(z)$$

and $q(z)$ is the best dominant.

Putting $\sigma = 1$, $\upsilon = 0$ *and* $\delta = \frac{\beta}{\mu}$ $(\beta \in \mathbb{C}^*)$ in Theorem 3, we obtain the following result.

Corollary 4. *Let $q(z)$ be convex univalent in U with $q(0)=1$, satisfies*

$$\operatorname{Re}\left\{\frac{\mu}{\beta}\right\} > 0 \tag{20}$$

and

$$\psi(f,g,\beta,\mu,p,\lambda,l,m)(z) = \left(1+\frac{\beta(p+l)}{\lambda}\right)\left(\frac{z^p}{D^m_{p,l,\lambda}(f*g)(z)}\right)^{\mu} +$$

$$-\frac{\beta(p+l)}{\lambda}\left(\frac{z^p}{D^m_{p,l,\lambda}(f*g)(z)}\right)^{\mu}\frac{D^{m+1}_{p,l,\lambda}(f*g)(z)}{D^m_{p,l,\lambda}(f*g)(z)}. \tag{21}$$

If

$$\psi(f,g,\beta,\mu,p,\lambda,l,m)(z) \prec q(z) + \frac{\beta}{\mu} z q'(z),$$

then

$$\left(\frac{z^p}{D^m_{p,l,\lambda}(f*g)(z)}\right)^{\mu} \prec q(z)$$

and $q(z)$ is the best dominant.

4 Superordination results

Theorem 4. *Let $q(z)$ be convex univalent in U with $q(0) = 1$ and satisfies (8). If $0 \neq \left(\frac{D^m_{p,l,\lambda}(f*g)(z)}{z^p}\right)^\mu \in H[q(0), 1] \cap \mathcal{Q}$ and $\chi(f, g, \sigma, \upsilon, \delta, \mu, p, \lambda, l, m)(z)$ is univalent in U, then*

$$\sigma q(z) + \upsilon(q(z))^2 + \delta zq'(z) \prec \chi(f, g, \sigma, \upsilon, \delta, \mu, p, \lambda, l, m)(z), \tag{22}$$

implies

$$q(z) \prec \left(\frac{D^m_{p,l,\lambda}(f * g)(z)}{z^p}\right)^\mu, \tag{23}$$

where $\chi(f, g, \sigma, \upsilon, \delta, \mu, p, \lambda, l, m)(z)$ is defined by (9) and $q(z)$ is the best subordinant.

Proof. Let $\varphi(z)$ defined by (12), we see that (13) holds and the subordination (22) is equivalent to

$$\sigma q(z) + \upsilon(q(z))^2 + \delta zq'(z) \prec \sigma\varphi(z) + \upsilon(\varphi(z))^2 + \delta z\varphi'(z),$$

this can be written as (7), when $\theta(w) = \sigma w + \upsilon w^2$ and $\psi(w) = \delta$. Note that $\theta(w)$, $\psi(w)$ are analytic in $\mathbb{C}$. Hence the assertion (23) follows by an application of Lemma 2. This completes the proof of Theorem 4.

Theorem 5. *Let $q(z)$ be convex univalent in U and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. Further assume that*

$$\operatorname{Re}\left\{\frac{\upsilon}{\delta}q(z)\right\} > 0. \tag{24}$$

Let

$$\sigma + \upsilon\left(\frac{D^m_{p,l,\lambda}(f * g)(z)}{z^p}\right)^\mu + \delta\mu\frac{(p + l)}{\lambda}\left(\frac{D^{m+1}_{p,l,\lambda}(f * g)(z)}{D^m_{p,l,\lambda}(f * g)(z)} - 1\right),$$

is univalent in U. If $0 \neq \left(\frac{D^m_{p,l,\lambda}(f*g)(z)}{z^p}\right)^\mu \in H[q(0), 1] \cap \mathcal{Q}$, then

$$\sigma+\upsilon q(z)+\delta\frac{zq'(z)}{q(z)} \prec \sigma+\upsilon\left(\frac{D^m_{p,l,\lambda}(f * g)(z)}{z^p}\right)^\mu+\delta\mu\frac{(p + l)}{\lambda}\left(\frac{D^{m+1}_{p,l,\lambda}(f * g)(z)}{D^m_{p,l,\lambda}(f * g)(z)} - 1\right),$$

implies

$$q(z) \prec \left(\frac{D^m_{p,l,\lambda}(f * g)(z)}{z^p}\right)^\mu$$

where $q(z)$ is the best subordinant.

Proof. Let $\theta(w) = \sigma + \upsilon w$ and $\psi(w) = \frac{\delta}{w}$. Note that $\psi(w) \neq 0$ $(w \in \mathbb{C}^*)$ and $\theta(w)$ is analytic in $\mathbb{C}$ and $\psi(w)$ is analytic in $\mathbb{C}^*$. Hence the result follows by an application of Lemma 2 for $\varphi(z) = \left(\frac{D^m_{p,l,\lambda}(f*g)(z)}{z^p}\right)^\mu$.

Using arguments similar to those of the proof of Theorem 1 and then by applying Lemma 2 we obtain the following result.

Theorem 6. *Let $q(z)$ be convex univalent in U with $q(0) = 1$ and satisfies (8). If $0 \neq \left(\frac{z^p}{D^m_{p,l,\lambda}(f*g)(z)}\right)^\mu \in H[q(0),1] \cap \mathcal{Q}$ and $\phi(f,g,\sigma,\upsilon,\delta,\mu,p,\lambda,l,m)(z)$ is univalent in U, then*

$$\sigma q(z) + \upsilon(q(z))^2 + \delta z q'(z) \prec \phi(f,g,\sigma,\upsilon,\delta,\mu,p,\lambda,l,m)(z),$$

implies

$$q(z) \prec \left(\frac{z^p}{D^m_{p,l,\lambda}(f*g)(z)}\right)^\mu,$$

where $\phi(f,g,\sigma,\upsilon,\delta,\mu,p,\lambda,l,m)(z)$ is defined by (19) and $q(z)$ is the best subordinant.

Putting $\sigma = 1,\ \upsilon = 0$ *and* $\delta = \frac{\beta}{\mu}\,(\beta \in \mathbb{C}^*)$ in Theorem 6, we obtain the following result.

Corollary 5. *Let $q(z)$ be convex univalent in U with $q(0) = 1$ and satisfies (20). If $0 \neq \left(\frac{z^p}{D^m_{p,l,\lambda}(f*g)(z)}\right)^\mu \in H[q(0),1] \cap \mathcal{Q}$ and $\psi(f,g,\beta,\mu,p,\lambda,l,m)(z)$ is univalent in U, then*

$$q(z) + \frac{\beta}{\mu} z q'(z) \prec \psi(f,g,\beta,\mu,p,\lambda,l,m)(z),$$

implies

$$q(z) \prec \left(\frac{z^p}{D^m_{p,l,\lambda}(f*g)(z)}\right)^\mu,$$

where $\psi(f,g,\beta,\mu,p,\lambda,l,m)(z)$ is defined by (21) and $q(z)$ is the best subordinant.

5 Sandwich results

By combining Theorem 1 with Theorem 4, we obtain the following sandwich theorem:

Theorem 7. *Let* $q_1(z)$ *and* $q_2(z)$ *be convex univalent in* U, *satisfying* $\mathrm{Re}\left\{\frac{\sigma+2\upsilon q_i(z)}{\delta}\right\}>0$ for $i=1,2$ *such that* $q_1(0)=q_2(0)=q(0)=1$. *If* $\left(\frac{D^m_{p,l,\lambda}(f*g)(z)}{z^p}\right)^\mu \in H[q(0),1]\cap\mathcal{Q}$ *and* $\chi(f,g,\sigma,\upsilon,\delta,\mu,p,\lambda,l,m)(z)$ *is univalent in* U *where* $\chi(f,g,\sigma,\upsilon,\delta,\mu,p,\lambda,l,m)(z)$ *is defined by (9), then*

$$\begin{aligned}\sigma q_1(z)+\upsilon(q_1(z))^2+\delta z q_1'(z) &\prec \chi(f,g,\sigma,\upsilon,\delta,\mu,p,\lambda,l,m)(z)\\ &\prec \sigma q_2(z)+\upsilon(q_2(z))^2+\delta z q_2'(z),\end{aligned}$$

implies

$$q_1(z)\prec\left(\frac{D^m_{p,l,\lambda}(f*g)(z)}{z^p}\right)^\mu\prec q_2(z),$$

where $q_1(z)$ and $q_2(z)$ *are respectively the best subordinant and best dominant.*

By combining Theorem 2 with Theorem 5, we obtain the following sandwich theorem:

Theorem 8. *Let* $q_1(z)$ *and* $q_2(z)$ *be convex univalent in* U, *satisfying (24) and (18) respectively such that* $q_1(0)=q_2(0)=q(0)=1$. *Suppose* $\frac{zq_i'(z)}{q_i(z)}$ *be starlike univalent in* U for $i=1,2$. *Let*

$$\eta(f,g,\sigma,\upsilon,\delta,\mu,p,\lambda,l,m)(z)=\sigma+\upsilon\left(\frac{D^m_{p,l,\lambda}(f*g)(z)}{z^p}\right)^\mu+\delta\mu\frac{(p+l)}{\lambda}\left(\frac{D^{m+1}_{p,l,\lambda}(f*g)(z)}{D^m_{p,l,\lambda}(f*g)(z)}-1\right),$$

be univalent in U. *If* $0\neq\left(\frac{D^m_{p,l,\lambda}(f*g)(z)}{z^p}\right)^\mu\in H[q(0),1]\cap\mathcal{Q}$, *then*

$$\sigma+\upsilon q_1(z)+\delta\frac{zq_1'(z)}{q_1(z)}\prec\eta(f,g,\sigma,\upsilon,\delta,\mu,p,\lambda,l,m)(z)\prec\sigma+\upsilon q_2(z)+\delta\frac{zq_2'(z)}{q_2(z)},$$

implies

$$q_1(z)\prec\left(\frac{D^m_{p,l,\lambda}(f*g)(z)}{z^p}\right)^\mu\prec q_2(z),$$

where $q_1(z)$ and $q_2(z)$ *are respectively the best subordinant and best dominant.*

Remark 3. *Putting* $\sigma=p=1,\ \upsilon=0$ *and* $g(z)=z(1-z)^{-1}$ *or* $b_k=1(k\geq 2)$ *in Theorem 8 we obtain result obtained by Aouf et al. [5; Theoeom 6].*

By combining Theorem 3 with Theorem 6, we obtain the following sandwich result:

Corollary 6. *Let* $q_1(z)$ *and* $q_2(z)$ *be convex univalent in* U, *satisfying (8) such that* $q_1(0)=q_2(0)=q(0)=1$. *Suppose* $\frac{zq_i'(z)}{q_i(z)}$ *be starlike univalent in*

U for $i = 1, 2$. *Let* $\phi(f, g, \sigma, \upsilon, \delta, \mu, p, \lambda, l, m)(z)$ *be univalent in* U. *If* $0 \neq \left(\frac{z^p}{D^m_{p,l,\lambda}(f * g)(z)}\right)^\mu \in H[q(0), 1] \cap \mathcal{Q}$, *then*

$$\sigma q_1(z) + \upsilon(q_1(z))^2 + \delta z q_1'(z) \prec \phi(f, g, \sigma, \upsilon, \delta, \mu, p, \lambda, l, m)(z)$$
$$\prec \sigma q_2(z) + \upsilon(q_2(z))^2 + \delta z q_2'(z),$$

implies

$$q_1(z) \prec \left(\frac{z^p}{D^m_{p,l,\lambda}(f * g)(z)}\right)^\mu \prec q_2(z),$$

where $\phi(f, g, \sigma, \upsilon, \delta, \mu, p, \lambda, l, m)(z)$ *is defined by (19) and* $q_1(z)$ and $q_2(z)$ *are respectively the best subordinant and best dominant.*

Remark 4. *Putting* $\sigma = p = 1,\ \upsilon = 0,\ \delta = \frac{\beta}{\mu}$ $(\beta \in \mathbb{C}^*)$ *and* $g(z) = z(1-z)^{-1}$ *or* $b_k = 1$ $(k \geq 2)$ *in Corollary 6 we obtain the result obtained by Aouf and El-Ashwah [3; Theoeom 7].*

By combining Corollary 4 and Corollary 5, we obtain the following sandwich result:

Corollary 7. *Let* $q_1(z)$ *and* $q_2(z)$ *be convex univalent in* U, *satisfying (20) such that* $q_1(0) = q_2(0) = q(0) = 1$. *Suppose* $\frac{zq_i'(z)}{q_i(z)}$ *be starlike univalent in* U for $i = 1, 2$. *Let* $\psi(f, g, \beta, \mu, p, \lambda, l, m)(z)$ *be univalent in* U. *If* $0 \neq \left(\frac{z^p}{D^m_{p,l,\lambda}(f * g)(z)}\right)^\mu \in H[q(0), 1] \cap \mathcal{Q}$, *then*

$$q_1(z) + \frac{\beta}{\mu} z q_1'(z) \prec \psi(f, g, \beta, \mu, p, \lambda, l, m)(z) \prec q_2(z) + \frac{\beta}{\mu} z q_2'(z),$$

implies

$$q_1(z) \prec \left(\frac{z^p}{D^m_{p,l,\lambda}(f * g)(z)}\right)^\mu \prec q_2(z),$$

where $\psi(f, g, \beta, \mu, p, \lambda, l, m)(z)$ *is defined by (21) and* $q_1(z)$ and $q_2(z)$ *are respectively the best subordinant and best dominant.*

Remark 5. (i) *Taking* $g(z) = z^p(1-z)^{-1}$ *or* $b_k = 1$ $(k \geq p+1)$ *in the above results, we obtain the results corresponding to the operator* $I^m_p(\lambda, l)$;

(ii) *Taking* $b_k = \frac{(\alpha_1)_{k-p} \ldots (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \ldots (\beta_s)_{k-p}(1)_{k-p}}$, $(\alpha_1, \alpha_2, \ldots, \alpha_q$ *and* $\beta_1, \beta_2, \ldots, \beta_s$ *are real or complex numbers* , $\beta_j \notin \mathbb{Z}_0^-, j = 1, 2, \ldots, s, q \leq s+1, s, q \in \mathbb{N}_0)$ *in the above results, we obtain the results corresponding to the operator* $I^{m,l}_{p,q,s,\lambda}(\alpha_1, \beta_1)$;

(iii) *Taking* $m = 0$ *and* $b_k = \frac{\Gamma(p+\alpha+\beta)\Gamma(k+\beta)}{\Gamma(p+\beta)\Gamma(k+\alpha+\beta)}(\alpha \geq 0, \beta > -1, p \in \mathbb{N})$, *in the above results, we obtain the results corresponding to the operator* $Q^\alpha_{p,\beta}$.

Remark 6. *(i) Putting* $\sigma = p = 1$, $\upsilon = 0$, $\delta = \frac{\beta}{\mu}$ $(\beta \in \mathbb{C}^*)$ *and* $g(z) = z(1-z)^{-1}$ *or* $b_k = 1 (k \geq 2)$ *in Theorems 1, 4 and 7, respectively, we obtain the results obtained by Aouf et. al [5; Theorems 1, 3 and 5, respectively];*

(ii) Putting $\sigma = p = 1$, $\upsilon = 0$, $\delta = \beta$ $(\beta \in \mathbb{C}^*)$ *and* $g(z) = z(1-z)^{-1}$ *or* $b_k = 1 (k \geq 2)$ *in Theorems 2 and 5, respectively, we obtain the results obtained by Aouf et. al [5; Theorems 2 and 4 ,respectively];*

(iii) Putting $\sigma = 1$, $\upsilon = 0$, $\delta = \frac{\beta}{\mu}$ $(\beta \in \mathbb{C}^*)$ *and* $g(z) = z(1-z)^{-1}$ *or* $b_k = 1 (k \geq 2)$ *in Theorems 3 and 6, respectively, we obtain the results obtained by Aouf and El-Ashwah [3; Theorems 1 and 4 ,respectively].*

6 Open Problem

Find sufficient conditions for analytic functions $f, g \in A(p)$ defined by using the operator $D^m_{p,l,\lambda}(f * g)(z)$ to satisfy:

$$q_1(z) \prec \frac{z^{p+1}(D^m_{p,l,\lambda}(f * g)(z))'}{p\{D^m_{p,l,\lambda}(f * g)(z)\}^2} \prec q_2(z)),$$

where q_1 and q_2 are given univalent functions in U.

References

[1] F. M. Al-Oboudi, On univalent functions defined by a generalized Sălăgean operator, Internat. J. Math. Math. Sci., 27 (2004), 1429-1436.

[2] M. K. Aouf , F. M. Al.-Oboudi and M. M. Haidan, On some results for λ-spirallike and λ-Robertson functions of complex order, Publ. Institute Math. Belgrade, 77 (2005), no. 91, 93–98.

[3] M. K. Aouf and R. M. El-Ashwah, Differential sandwich theorem for certain subclasses of analytic functions involving an extended multiplier transformation, Tokyo J. Math., 33 (2010), no. 1, 73-88.

[4] M. K. Aouf and A. O. Mostafa, On a subclass of n-p-valent prestarlike functions, Comput. Math. Appl., 55 (2008), no. 4, 851-861.

[5] M. K. Aouf, A. Shamandy, R. M. El-Ashwah and E. E. Ali, On sandwich theorems for some subclasses of analytic functions involving an extended multiplier transformations, Mat. Vesnik (To appear).

[6] T. Bulboaca, A class of superordination-preserving integral operators, Indag. Math. (N. S.), 13 (2002), no. 3, 301-311.

[7] T. Bulboacă, Classes of first order differential superordinations, Demonstratio Math., 35 (2002), no. 2, 287-292.

[8] T. Bulboacă, Differential Subordinations and Superordinations, Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.

[9] A. Catas, On certain classes of p-valent functions defined by multiplier transformations, in: Proc. Book of the International Symposium on Geometric Function Theory and Applications, Istanbul, Turkey, August 2007, 241-250.

[10] J. H. Choi, M. Saigo and H. M. Srivastava, Some inclusion properties of a certain family of integral operators, J. Math. Anal. Appl., 276 (2002), 432-445.

[11] R. M. El-Ashwah and M. K. Aouf, Differential subordination and superordination for certain subclasses of p-valent functions, Math. Comput. Modelling, 51 (2010), 349-360.

[12] M. Kamali and H. Orhan, On a subclass of certain starlike functions with negative coefficients, Bull. Korean Math. Soc., 41 (2004), no. 1, 53-71.

[13] J.-L. Liu and S. Owa, Properties of certain integral operators, Internat. J. Math. Math. Sci., 3 (2004), no.1, 69-75.

[14] S. S. Miller and P. T. Mocanu, Differential Subordination : Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York and Basel, 2000.

[15] S. S. Miller and P. T. Mocanu, Subordinates of differential superordinations, Complex Variables, 48 (2003), no. 10, 815-826.

[16] M. Obradović, M. K. Aouf and S. Owa, On some results for starlike functions of complex order, Publ. Inst. Math. Belgrade, 46 (1989), no. 60, 79–85.

[17] M. Obradović and S. Owa, On certain properties for some classes of starlike functions, J. Math. Anal. Appl., 145 (1990), no. 2, 357–364.

[18] W. C. Royster, On the univalence of a certain integral, Michigan Math. J., 12 (1965), 385–387.

[19] H. M. Srivastava and A. Y. Lashin, Some applications of the Briot-Bouquet differential subordination, J. Ineq. Pure Appl. Math., 6 (2005), no. 2, Art. 41, 1–7.