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Certain classes of analytic functions defined by convolution with varying argument of coefficients

Huo Tang, Guan-tie Deng and Shu-hai Li

School of Mathematical Sciences, Beijing Normal University
Beijing 100875, China
School of Mathematics and Statistics, Chifeng University
Chifeng 024000, Inner Mongolia, China
e-mail: thth2009@163.com, thth2009@tom.com
School of Mathematical Sciences, Beijing Normal University
Beijing 100875, China
e-mail: denggt@bnu.edu.cn
School of Mathematics and Statistics, Chifeng University
Chifeng 024000, Inner Mongolia, China
e-mail: lishms66@sina.com

Abstract

In this paper, using Hadmard product, we introduce certain new classes of analytic functions in the open unit disk. Such results as inclusion and subordination relationships, characterization and coefficient estimates, growth and distortion theorems, extreme points, closure theorems and radius of starlikeness and convexity belonging to the class $TS(f, g; \lambda, \alpha, \beta, A, B)$ ***are obtained. Further subordination results for the class*** $S(f, g; \lambda, \alpha, \beta, A, B)$ ***are derived.***

Keywords: *analytic functions, Hadmard product, varying argument, coefficient estimates, subordination*

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1 Introduction

Let H denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z \in C : |z| < 1\}$.

Also, we denote by T, the class of functions $f(z) \in H$ of the form (1.1) for which there exists a real number η such that

$$\arg(a_k) = \pi + (1-k)\eta \quad (k = 2, 3, \cdots), \tag{1.2}$$

which was introduced by Silverman [1](see also [2]) and called the class of functions with varying argument of coefficients.

For functions $f \in H$ given by (1.1) and $g \in H$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad (b_k \geq 0;\ z \in U), \tag{1.3}$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z) \quad (z \in U). \tag{1.4}$$

For two functions f and g, analytic in U, we say that the function f is subordinate to g in U, and write $f(z) \prec g(z)$, if there exists a Schwarz function ω, which is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in U$, such that $f(z) = g(\omega(z))$ $(z \in U)$. In particular, if the function g is univalent in U, then we have the following equivalence

$$f(z) \prec g(z) \quad (z \in U) \Leftrightarrow f(0) = g(0) \quad \textit{and} \quad f(U) \subset g(U).$$

Following Goodman [3,4], Rønning [5] and Kanas and Wisniowska [6] (see also [7],[8]), Hams et al. [9] define two subclasses of H as follows.

For $-1 < \gamma \leq 1$ and $\beta \geq 0$, a function $f \in H$ is said to be in the class

(i) β-uniformly starlike functions of order γ is denoted by $US(\beta, \gamma)$, if it satisfies the condition

$$Re\left(\frac{zf'(z)}{f(z)} - \gamma\right) > \beta\left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in U), \tag{1.5}$$

and (ii) β-uniformly convex functions of order γ is denoted by $UK(\beta, \gamma)$, if it satisfies the condition

$$Re\left(1 + \frac{zf''(z)}{f'(z)} - \gamma\right) > \beta\left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in U). \tag{1.6}$$

Indeed, it follows from (1.5) and (1.6) that

$$f \in UK(\beta,\gamma) \Leftrightarrow zf' \in US(\beta,\gamma).$$

Motivated by above definitions, we define a new class of analytic functions related to Hadmard products.

Definition 1.1. For $\alpha \geq 1, \beta \geq 0, 0 \leq \lambda \leq 1, -1 \leq B < A \leq 1$ and for all $z \in U$, let $S(f,g;\lambda,\alpha,\beta,A,B)$ denote the subclass of H consisting of functions $f(z)$ of the form (1.1) and $g(z)$ of the form (1.3) and satisfying the following subordination:

$$\left[\alpha\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}-(\alpha-1)\right]$$
$$-\beta\left|\alpha\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}-\alpha\right| \prec \frac{1+Az}{1+Bz}. \tag{1.7}$$

We also let $TS(f,g;\lambda,\alpha,\beta,A,B) = T \cap S(f,g;\lambda,\alpha,\beta,A,B)$.

For suitable choices of the function g and by specializing the parameters λ,α,β,A,B involved in the class $S(f,g;\lambda,\alpha,\beta,A,B)$, we obtain the following subclasses.

(1) If $g(z) = z+\sum_{k=2}^{\infty} z^k$ (or $b_k = 1$), then $S(f, z+\sum_{k=2}^{\infty} z^k;\lambda,\alpha,\beta,A,B) = S(\lambda,\alpha,\beta,A,B)$

$$= \Big\{ f \in H : \left[\alpha\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-(\alpha-1)\right]$$
$$-\beta\left|\alpha\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-\alpha\right| \prec \frac{1+Az}{1+Bz},$$
$$\alpha \geq 1, \beta \geq 0, 0 \leq \lambda \leq 1, -1 \leq B < A \leq 1, z \in U \Big\}.$$

In particular, $S(0,1,\beta,1-2\gamma,-1) = US(\beta,\gamma)$ and $S(1,1,\beta,1-2\gamma,-1) = UK(\beta,\gamma)$ (see [9]);

(2) If $g(z) = z+\sum_{k=2}^{\infty}[1+(\mu\delta k+\mu-\delta)(k-1)]^m z^k$ (or $b_k = [1+(\mu\delta k+\mu-\delta)(k-1)]^m$, $m \in N_0 = N \cup \{0\}, 0 \leq \delta \leq \mu \leq 1$), then $S(f, z+\sum_{k=2}^{\infty}[1+(\mu\delta k+\mu-\delta)(k-1)]^m z^k;\lambda,\alpha,\beta,A,B) = S(\mu,\delta,m;\lambda,\alpha,\beta,A,B)$

$$= \Big\{ f \in H : \left[\alpha\frac{z\left(D^m_{\mu,\delta}f(z)\right)'+\lambda z^2\left(D^m_{\mu,\delta}f(z)\right)''}{(1-\lambda)\left(D^m_{\mu,\delta}f(z)\right)+\lambda z\left(D^m_{\mu,\delta}f(z)\right)'}-(\alpha-1)\right]$$
$$-\beta\left|\alpha\frac{z\left(D^m_{\mu,\delta}f(z)\right)'+\lambda z^2\left(D^m_{\mu,\delta}f(z)\right)''}{(1-\lambda)\left(D^m_{\mu,\delta}f(z)\right)+\lambda z\left(D^m_{\mu,\delta}f(z)\right)'}-\alpha\right| \prec \frac{1+Az}{1+Bz},$$
$$\alpha \geq 1, \beta \geq 0, 0 \leq \lambda \leq 1, m \in N_0, 0 \leq \delta \leq \mu \leq 1, -1 \leq B < A \leq 1, z \in U \Big\}.$$

In particular, $S(\mu,\delta,m;\lambda,\alpha,\beta,1-2\gamma,-1)=G^m(\mu,\delta;\lambda,\alpha,\beta,\gamma)$ (see [10]), where the operator $D^m_{\mu,\delta}$ was introduced and studied by Raducanu and Orhan [10], for $\delta=0$, the operator $D^m_{\mu,0}=D^m_\mu$ was introduced and studied by Al-Oboudi [11] and for $\mu=1$ and $\delta=0$, the operator $D^m_{1,0}=D^m$ was defined by Salagean [12].

(3) If $g(z)=z+\sum_{k=2}^{\infty}\Psi_k z^k$ (or $b_k=\Psi_k$), where

$$\Psi_k=\frac{(\alpha_1)_{k-1},\cdots,(\alpha_q)_{k-1}}{(\beta_1)_{k-1},\cdots,(\beta_s)_{k-1}(k-1)!} \tag{1.8}$$

$$(\alpha_i>0, i=1,2,\cdots,q;\ \beta_j>0, j=1,2,\cdots,s;\ q\le s+1;\ q,s\in N_0),$$

then $S(f,z+\sum_{k=2}^{\infty}\Psi_k z^k;\lambda,\alpha,\beta,A,B)=S_{q,s}([\alpha_1];\lambda,\alpha,\beta,A,B)$

$$=\Big\{f\in H:\left[\alpha\frac{z\left(H_{q,s}(\alpha_1)f(z)\right)'+\lambda z^2\left(H_{q,s}(\alpha_1)f(z)\right)''}{(1-\lambda)\left(H_{q,s}(\alpha_1)f(z)\right)+\lambda z\left(H_{q,s}(\alpha_1)f(z)\right)'}-(\alpha-1)\right]$$

$$-\beta\left|\alpha\frac{z\left(H_{q,s}(\alpha_1)f(z)\right)'+\lambda z^2\left(H_{q,s}(\alpha_1)f(z)\right)''}{(1-\lambda)\left(H_{q,s}(\alpha_1)f(z)\right)+\lambda z\left(H_{q,s}(\alpha_1)f(z)\right)'}-\alpha\right|\prec\frac{1+Az}{1+Bz},$$

$$\alpha\ge 1,\beta\ge 0,0\le\lambda\le 1,-1\le B<A\le 1,z\in U\Big\},$$

where $H_{q,s}(\alpha_1)$ is the Dziok-Srivastava operator [13] (see also [14]), which contains well known operators such as Carlson-Shaffer linear operator [15], the Bernardi-Libera-Livingston operator [16], the Srivastava-Owa fractional derivative operator [17], the Choi-Saigo-Srivastava operator [18], the Cho-Kwon-Srivastava operator [19], the Ruscheweyh derivative operator [20], the Noor integral operator [21], and other operators.

(4) If $g(z)=z+\sum_{k=2}^{\infty}I^m(\rho,l)z^k$ (or $b_k=I^m(\rho,l)$), where

$$I^m(\rho,l)=\left[\frac{1+l+\rho(k-1)}{1+l}\right]^m \quad (\rho\ge 0,l\ge 0,m\in N_0), \tag{1.9}$$

then $S(f,z+\sum_{k=2}^{\infty}I^m(\rho,l)z^k;\lambda,\alpha,\beta,A,B)=S(\rho,l,m;\lambda,\alpha,\beta,A,B)$

$$=\Big\{f\in H:\left[\alpha\frac{z\left(I^m(\rho,l)f(z)\right)'+\lambda z^2\left(I^m(\rho,l)f(z)\right)''}{(1-\lambda)\left(I^m(\rho,l)f(z)\right)+\lambda z\left(I^m(\rho,l)f(z)\right)'}-(\alpha-1)\right]$$

$$-\beta\left|\alpha\frac{z\left(I^m(\rho,l)\right)'+\lambda z^2\left(I^m(\rho,l)f(z)\right)''}{(1-\lambda)\left(I^m(\rho,l)f(z)\right)+\lambda z\left(I^m(\rho,l)f(z)\right)'}-\alpha\right|\prec\frac{1+Az}{1+Bz},$$

$$\alpha\ge 1,\beta\ge 0,0\le\lambda\le 1,\rho\ge 0,l\ge 0,m\in N_0,-1\le B<A\le 1,z\in U\Big\},$$

where the operator $I^m(\rho,l)$ was introduced and studied by Catas [22], which contains (as its special cases) the Cho-Srivastava operator [23], the Al-Oboudi operator [11] and the Salagean operator [12].

In this paper, we obtain a sufficient coefficient condition for functions f given by (1.1) to be in the class $S(f,g;\lambda,\alpha,\beta,A,B)$ and a necessary and sufficient coefficient condition for functions in the class $TS(f,g;\lambda,\alpha,\beta,A,B)$. Growth and distortion theorems, extreme points, closure theorems and radius of starlikeness and convexity for functions in $TS(f,g;\lambda,\alpha,\beta,A,B)$ are given. Finally, we investigate subordination results for the class $S(f,g;\lambda,\alpha,\beta,A,B)$.

2 Inclusion and subordination relationships

To prove our main result, we need the following lemma.

Lemma 2.1 ([24]). Let $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$. Then

$$\frac{1+A_1z}{1+B_1z} \prec \frac{1+A_2z}{1+B_2z}. \tag{2.1}$$

Theorem 2.1. Let $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$. Then

$$S(f,g;\lambda,\alpha,\beta,A_1,B_1) \subset S(f,g;\lambda,\alpha,\beta,A_2,B_2). \tag{2.2}$$

Proof. Suppose that $f \in S(f,g;\lambda,\alpha,\beta,A_1,B_1)$, in view of Definition 1.1, we have

$$\left[\alpha\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}-(\alpha-1)\right]$$
$$-\beta\left|\alpha\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}-\alpha\right| \prec \frac{1+A_1z}{1+B_1z}.$$

Thus, by Lemma 2.1, we obtain that

$$\left[\alpha\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}-(\alpha-1)\right]$$
$$-\beta\left|\alpha\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}-\alpha\right| \prec \frac{1+A_1z}{1+B_1z} \prec \frac{1+A_2z}{1+B_2z},$$

which implies that $f \in S(f,g;\lambda,\alpha,\beta,A_2,B_2)$. Hence we complete the proof.

3 Characterization and coefficient estimates

First we obtain a sufficient condition for functions in the class $S(f,g;\lambda,\alpha,\beta,A,B)$.

Theorem 3.1. Let $f \in H$ given by (1.1). If

$$\sum_{k=2}^{\infty}[1+\lambda(k-1)][\alpha(k-1)(1+(1+|B|)\beta)+|A-\alpha Bk+B(\alpha-1)|]b_k|a_k| \le A-B, \tag{3.1}$$

then $f \in S(f, g; \lambda, \alpha, \beta, A, B)$.

Proof. Assume that the inequality (3.1) holds true for $\alpha \geq 1, \beta \geq 0, 0 \leq \lambda \leq 1, -1 \leq B < A \leq 1, z \in U$. For $f \in H$, let

$$p(z) = \left[\alpha \frac{zG'(z)}{G(z)} - (\alpha - 1)\right] - \beta \left|\alpha \frac{zG'(z)}{G(z)} - \alpha\right|, \tag{3.2}$$

where

$$G(z) = (1 - \lambda)(f * g)(z) + \lambda z (f * g)'(z). \tag{3.3}$$

It is sufficient to show that

$$\left|\frac{p(z) - 1}{A - Bp(z)}\right| < 1 \quad (z \in U). \tag{3.4}$$

We note that

$$\left|\frac{p(z) - 1}{A - Bp(z)}\right| = \left|\frac{\alpha(zG'(z) - G(z)) - \alpha\beta e^{i\theta} \left|zG'(z) - G(z)\right|}{AG(z) - B[\alpha(zG'(z) - G(z)) + G - \alpha\beta e^{i\theta} \left|zG'(z) - G(z)\right|]}\right|$$

$$= \left|\frac{\sum_{k=2}^{\infty} \alpha M a_k b_k z^{k-1} - \alpha\beta e^{i\theta} \left|M a_k b_k z^{k-1}\right|}{(A - B) + \sum_{k=2}^{\infty}[1 + \lambda(k-1)][A - \alpha Bk + B(\alpha - 1)] a_k b_k z^{k-1} + \alpha\beta B e^{i\theta} \left|M a_k b_k z^{k-1}\right|}\right|$$

$$\leq \frac{\sum_{k=2}^{\infty} \alpha M b_k |a_k| |z|^{k-1} + \alpha\beta \sum_{k=2}^{\infty} M b_k |a_k| |z|^{k-1}}{(A - B) - \sum_{k=2}^{\infty}[1 + \lambda(k-1)][A - \alpha Bk + B(\alpha - 1)] b_k |a_k| |z|^{k-1} - \alpha\beta |B| \sum_{k=2}^{\infty} M b_k |a_k| |z|^{k-1}}$$

$$\leq \frac{\sum_{k=2}^{\infty} \alpha M b_k |a_k| + \alpha\beta \sum_{k=2}^{\infty} M b_k |a_k|}{(A - B) - \sum_{k=2}^{\infty}[1 + \lambda(k-1)][A - \alpha Bk + B(\alpha - 1)] b_k |a_k| - \alpha\beta |B| \sum_{k=2}^{\infty} M b_k |a_k|},$$

where $M = (k-1)[1 + \lambda(k-1)]$.

The last expression is bounded above by 1, if

$$\sum_{k=2}^{\infty} [1 + \lambda(k-1)][\alpha(k-1)(1 + (1 + |B|)\beta) + |A - \alpha Bk + B(\alpha - 1)|] b_k |a_k| \leq A - B,$$

and hence the proof is completed.

Next, we obtain a necessary and sufficient condition for the functions in the class $TS(f, g; \lambda, \alpha, \beta, A, B)$.

Theorem 3.2. Let $f \in H$ given by (1.1) and satisfy (1.2). Then $f \in TS(f, g; \lambda, \alpha, \beta, A, B)$ if and only if the inequality (3.1) holds true.

Proof. In view of Theorem 3.1, we need only to prove the necessary part. If $f \in TS(f, g; \lambda, \alpha, \beta, A, B)$, then from (1.1), (1.7) and (3.4), we find that

$$= \left| \frac{\sum_{k=2}^{\infty} \alpha M a_k b_k z^{k-1} - \alpha\beta e^{i\theta} \left|\sum_{k=2}^{\infty} M a_k b_k z^{k-1}\right|}{(A-B) + \sum_{k=2}^{\infty} [1+\lambda(k-1)][A - \alpha Bk + B(\alpha-1)] a_k b_k z^{k-1} + \alpha\beta B e^{i\theta} \left|\sum_{k=2}^{\infty} M a_k b_k z^{k-1}\right|} \right|$$

$$< 1 \quad (M = (k-1)[1+\lambda(k-1)]; \ z \in U).$$

Setting $z = re^{i\eta}$ $(0 \le r < 1)$ in the above inequality and applying (1.2), we have

$$\frac{\sum_{k=2}^{\infty} \alpha M b_k |a_k| r^{k-1} + \alpha\beta \sum_{k=2}^{\infty} M b_k |a_k| r^{k-1}}{(A-B) - \sum_{k=2}^{\infty} [1+\lambda(k-1)][A - \alpha Bk + B(\alpha-1)] b_k |a_k| r^{k-1} - \alpha\beta |B| \sum_{k=2}^{\infty} M b_k |a_k| r^{k-1}}$$

$$< 1 \quad (M = (k-1)[1+\lambda(k-1)]).$$

Thus, by a simple computation, we obtain

$$\sum_{k=2}^{\infty} [1+\lambda(k-1)][\alpha(k-1)(1+(1+|B|)\beta) + |A - \alpha Bk + B(\alpha-1)|] b_k |a_k| r^{k-1} < A - B,$$

which, upon letting $r \to 1^-$, readily yields the desired inequality (3.1).

Corollary 3.1. If $f \in TS(f, g; \lambda, \alpha, \beta, A, B)$, then

$$|a_k| \le \frac{A-B}{[1+\lambda(k-1)][\alpha(k-1)(1+(1+|B|)\beta) + |A - \alpha Bk + B(\alpha-1)|] b_k} \quad (k \ge 2). \tag{3.5}$$

The equality in (3.5) holds true for the function given by

$$f_{k,\eta}(z) = z - \frac{(A-B)e^{i(1-k)\eta}}{[1+\lambda(k-1)][\alpha(k-1)(1+(1+|B|)\beta) + |A - \alpha Bk + B(\alpha-1)|] b_k} z^k \quad (z \in U). \tag{3.6}$$

4 Growth and distortion theorems

Theorem 4.1. Let $f \in TS(f, g; \lambda, \alpha, \beta, A, B)$ and $|z| = r < 1$. If the sequence $\{M_k(\lambda, \alpha, \beta, A, B)\}_{k=2}^{\infty}$ is nondecreasing, then

$$r - \frac{A-B}{M_2(\lambda, \alpha, \beta, A, B)} r^2 \le |f(z)| \le r + \frac{A-B}{M_2(\lambda, \alpha, \beta, A, B)} r^2. \tag{4.1}$$

Moreover, if the sequence $\left\{\frac{M_k(\lambda,\alpha,\beta,A,B)}{k}\right\}_{k=2}^{\infty}$ is nondecreasing, then

$$1 - \frac{2(A-B)}{M_2(\lambda, \alpha, \beta, A, B)} r \le |f'(z)| \le 1 + \frac{2(A-B)}{M_2(\lambda, \alpha, \beta, A, B)} r, \tag{4.2}$$

where

$$M_k(\lambda,\alpha,\beta,A,B) = [1+\lambda(k-1)][\alpha(k-1)(1+(1+|B|)\beta)+|A-\alpha Bk+B(\alpha-1)|]b_k \ (k\geq 2). \tag{4.3}$$

The result is sharp. The extremal functions are the functions $f_{2,\eta}(z)$ of the form (3.6).

Proof. Since $f\in TS(f,g;\lambda,\alpha,\beta,A,B)$, from Theorem 3.2 it follows that

$$M_2(\lambda,\alpha,\beta,A,B)\sum_{k=2}^{\infty}|a_k| \leq \sum_{k=2}^{\infty} M_k(\lambda,\alpha,\beta,A,B)|a_k| \leq A-B,$$

which is equivalent to

$$\sum_{k=2}^{\infty}|a_k| \leq \frac{A-B}{M_2(\lambda,\alpha,\beta,A,B)}. \tag{4.4}$$

Using (1.1) and (4.4), we have

$$|f(z)| \leq |z|+|z|^2\sum_{k=2}^{\infty}|a_k| \leq r+\frac{A-B}{M_2(\lambda,\alpha,\beta,A,B)}r^2$$

and

$$|f(z)| \geq |z|-|z|^2\sum_{k=2}^{\infty}|a_k| \geq r-\frac{A-B}{M_2(\lambda,\alpha,\beta,A,B)}r^2.$$

Similarly, we also have

$$\frac{M_2(\lambda,\alpha,\beta,A,B)}{2}\sum_{k=2}^{\infty}k|a_k| \leq \sum_{k=2}^{\infty} M_k(\lambda,\alpha,\beta,A,B)|a_k| \leq A-B,$$

which yields

$$\sum_{k=2}^{\infty}k|a_k| \leq \frac{2(A-B)}{M_2(\lambda,\alpha,\beta,A,B)}.$$

Thus,

$$|f'(z)| \leq 1+\sum_{k=2}^{\infty}k|a_k||z|^{k-1} \leq 1+|z|\sum_{k=2}^{\infty}k|a_k| \leq 1+\frac{2(A-B)}{M_2(\lambda,\alpha,\beta,A,B)}r$$

and

$$|f'(z)| \geq 1-\sum_{k=2}^{\infty}k|a_k||z|^{k-1} \geq 1-|z|\sum_{k=2}^{\infty}k|a_k| \geq 1-\frac{2(A-B)}{M_2(\lambda,\alpha,\beta,A,B)}r.$$

This completes the proof of Theorem 4.1.

5 Extreme points

Now, we determine extreme points for the class $TS(f, g; \lambda, \alpha, \beta, A, B)$.

Theorem 5.1. Let the functions

$$f_1(z) = z \quad \textit{and} \quad f_{k,\eta}(z) = z - \frac{(A-B)e^{i(1-k)\eta}}{M_k(\lambda, \alpha, \beta, A, B)} z^k \quad (k \geq 2) \tag{5.1}$$

with $M_k(\lambda, \alpha, \beta, A, B)$ defined as in (4.3). Then $f \in TS(f, g; \lambda, \alpha, \beta, A, B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_{k,\eta}(z), \quad \mu_k \geq 0 \quad \textit{and} \quad \sum_{k=1}^{\infty} \mu_k = 1. \tag{5.2}$$

Proof. Assume that $f(z)$ can be written as in (5.2). Then

$$f(z) = \mu_1 z + \sum_{k=2}^{\infty} \mu_k \left[z - \frac{(A-B)e^{i(1-k)\eta}}{M_k(\lambda, \alpha, \beta, A, B)} z^k \right] = z - \sum_{k=2}^{\infty} \frac{(A-B)e^{i(1-k)\eta}}{M_k(\lambda, \alpha, \beta, A, B)} \mu_k z^k \ (z \in U).$$

Since

$$\sum_{k=2}^{\infty} M_k(\lambda, \alpha, \beta, A, B) \left| \frac{(A-B)e^{i(1-k)\eta}}{M_k(\lambda, \alpha, \beta, A, B)} \mu_k \right| = \sum_{k=2}^{\infty} (A-B)\mu_k = (A-B)(1-\mu_1) \leq A-B,$$

it follows, from Theorem 3.2, that $f(z) \in TS(f, g; \lambda, \alpha, \beta, A, B)$.

Conversely, if $f(z) \in TS(f, g; \lambda, \alpha, \beta, A, B)$, then, by using (3.5), we may set

$$\mu_k = \frac{M_k(\lambda, \alpha, \beta, A, B)}{A-B} a_k \quad \textit{and} \quad \mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k \ \ (k \geq 2).$$

Then $f(z) = \sum_{k=1}^{\infty} \mu_k f_{k,\eta}(z)$ and this completes the proof.

Corollary 5.1. The extreme points of the class $TS(f, g; \lambda, \alpha, \beta, A, B)$ are the functions given by (5.1).

6 Closure throrems

Let the functions $f_j \in H$ $(j = 1, 2, \cdots, p)$ with (1.2) defined by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \quad (z \in U). \tag{6.1}$$

Then we obtain the closure theorems of the class $TS(f, g; \lambda, \alpha, \beta, A, B)$.

Theorem 6.1. Let the functions f_j $(j = 1, 2, \cdots, p)$ given by (6.1) be in the class $TS(f, g; \lambda, \alpha, \beta, A, B)$ and $c_j \geq 0$ $(j = 1, 2, \cdots, p)$ such that $\sum_{j=1}^{p} c_j = 1$. Then the function $h(z)$ defined by

$$h(z) = \sum_{j=1}^{p} c_j f_j \tag{6.2}$$

is also in the class $TS(f, g; \lambda, \alpha, \beta, A, B)$.

Proof. In view of (6.1) and (6.2), we can write

$$h(z) = \sum_{j=1}^{p} c_j \left[z + \sum_{k=2}^{\infty} a_{k,j} z^k \right] = z + \sum_{k=2}^{\infty} \left(\sum_{j=1}^{p} c_j a_{k,j} \right) z^k.$$

Since the functions $f_j \in TS(f, g; \lambda, \alpha, \beta, A, B)$, for every $j = 1, 2, \cdots, p$, we have

$$\sum_{k=2}^{\infty} M_k(\lambda, \alpha, \beta, A, B)|a_{k,j}| \leq A - B.$$

Hence, we get

$$\sum_{k=2}^{\infty} M_k(\lambda, \alpha, \beta, A, B) \left| \sum_{j=1}^{p} c_j a_{k,j} \right| \leq \sum_{j=1}^{p} c_j (A - B) \leq A - B,$$

which implies that $h(z) \in TS(f, g; \lambda, \alpha, \beta, A, B)$.

Corollary 6.1. The class $TS(f, g; \lambda, \alpha, \beta, A, B)$ is closed under convex linear combination.

Proof. Suppose that the functions f_j $(j = 1, 2)$ given by (6.1) be in the class $TS(f, g; \lambda, \alpha, \beta, A, B)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = cf_1(z) + (1 - c)f_2(z) \quad (0 \leq c \leq 1)$$

is also in the class $TS(f, g; \lambda, \alpha, \beta, A, B)$. In fact, by taking $p = 2, c_1 = c$ and $c_2 = 1 - c$ in Theorem 6.1, we immediately get the required result.

7 Radius of starlikeness and convexity

We begin this section with the following theorem.

Theorem 7.1. Let the function $f(z)$ given by (1.1) with (1.2) be in the class $TS(f, g; \lambda, \alpha, \beta, A, B)$. Then $f(z)$ is starlike of order σ $(0 \leq \sigma < 1)$ in $|z| < r_1(\lambda, \alpha, \beta, A, B)$, where

$$r_1(\lambda, \alpha, \beta, A, B) = \inf_{k \geq 2} \left\{ \frac{(1 - \sigma)M_k(\lambda, \alpha, \beta, A, B)}{(k - \sigma)(A - B)} \right\}^{\frac{1}{k-1}} \tag{7.1}$$

with $M_k(\lambda, \alpha, \beta, A, B)$ defined as in (4.3). The result is sharp for the function $f_{k,\eta}(z)$ given by (3.6).

Proof. It suffices to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \sigma \quad (0 \leq \sigma < 1;\ |z| < r_1(\lambda, \alpha, \beta, A, B)), \tag{7.2}$$

or, equivalently

$$\sum_{k=2}^{\infty} \left(\frac{k - \sigma}{1 - \sigma} \right) |a_k||z|^{k-1} \leq 1. \tag{7.3}$$

By Theorem 3.2, the ineauality (7.3) would hold true if

$$\left(\frac{k - \sigma}{1 - \sigma} \right) |z|^{k-1} \leq \frac{M_k(\lambda, \alpha, \beta, A, B)}{A - B},$$

or if

$$|z| \leq \left\{ \frac{(1 - \sigma) M_k(\lambda, \alpha, \beta, A, B)}{(k - \sigma)(A - B)} \right\}^{\frac{1}{k-1}} \quad (k \geq 2).$$

Thus, we complete the proof.

Similary, we can prove the following theorem.

Theorem 7.2. Let the function $f(z)$ given by (1.1) with (1.2) be in the class $TS(f, g; \lambda, \alpha, \beta, A, B)$. Then $f(z)$ is convex of order ξ $(0 \leq \xi < 1)$ in $|z| < r_2(\lambda, \alpha, \beta, A, B)$, where

$$r_2(\lambda, \alpha, \beta, A, B) = \inf_{k \geq 2} \left\{ \frac{(1 - \xi) M_k(\lambda, \alpha, \beta, A, B)}{k(k - \xi)(A - B)} \right\}^{\frac{1}{k-1}} \tag{7.4}$$

with $M_k(\lambda, \alpha, \beta, A, B)$ defined as in (4.3). The result is sharp for the function $f_{k,\eta}(z)$ given by (3.6).

8 Subordination results

In order to prove our main result, we recall here the following definition and lemma.

Definition 8.1 (Subordinating Factor Sequence [25]). A sequence $\{d_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever f of the form (1.1) is analytic, univalent and convex in U, we have the subordination given by

$$\sum_{k=1}^{\infty} d_k a_k z^k \prec f(z) \quad (z \in U;\ a_1 = 1). \tag{8.1}$$

Lemma 8.1 (Wilf [25]). The sequence $\{d_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if

$$Re\left\{1+2\sum_{k=1}^{\infty} d_k z^k\right\} > 0 \quad (z \in U). \tag{8.2}$$

Let $S^*(f,g;\lambda,\alpha,\beta,A,B)$ denote the class of functions $f(z) \in H$ whose coefficients satisfy the condition (3.1). We note that $S^*(f,g;\lambda,\alpha,\beta,A,B) \subseteq S(f,g;\lambda,\alpha,\beta,A,B)$.

Employing the technique used earlier by Attiya [26], Srivastava and Attiya [27] and Aouf [28], we prove

Theorem 8.1. Let $f(z) \in S^*(f,g;\lambda,\alpha,\beta,A,B)$, $b_k \geq b_2 > 0$ $(k \geq 2)$ and $-1 \leq B < A \leq 1$. Suppose that K denote the class of functions $f(z) \in H$ which are convex in U. Then for every function $\phi(z) \in K$, we have

$$\frac{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}{2[(A-B)+(1+\lambda)(\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|)b_2]}(f*\phi)(z) \prec \phi(z), \tag{8.3}$$

and

$$Re\{f(z)\} > -\frac{(A-B)+(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2} \quad (z \in U). \tag{8.4}$$

The constant $\frac{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}{2[(A-B)+(1+\lambda)(\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|)b_2]}$ is the best estimate.

Proof. Let $f(z) \in S^*(f,g;\lambda,\alpha,\beta,A,B)$ and suppose that

$$\phi(z) = z + \sum_{k=2}^{\infty} d_k z^k \in K.$$

Then, for $f \in H$ given by (1.1), we have

$$\frac{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}{2[(A-B)+(1+\lambda)(\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|)b_2]}(f*\phi)(z)$$

$$= \frac{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}{2[(A-B)+(1+\lambda)(\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|)b_2]}\left(z+\sum_{k=2}^{\infty} a_k d_k z^k\right). \tag{8.5}$$

Thus, by Definition 8.1, the subordination result (8.3) will be true if the sequence

$$\left\{\frac{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}{2[(A-B)+(1+\lambda)(\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|)b_2]}a_k\right\}_{k=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 8.1, this is equivalent to the following inequality

$$Re\left\{1+\sum_{k=1}^{\infty}\frac{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}{(A-B)+(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}a_k z^k\right\}>0. \tag{8.6}$$

Now, since the equality (4.3) is an increasing function of k ($k \geq 2$), we have

$$Re\left\{1+\sum_{k=1}^{\infty}\frac{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}{(A-B)+(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}a_k z^k\right\}$$

$$= Re\Big\{1+\frac{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}{(A-B)+(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}z$$

$$+\frac{1}{(A-B)+(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}\cdot$$

$$\sum_{k=2}^{\infty}(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2 a_k z^k\Big\}$$

$$\geq 1-\frac{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}{(A-B)+(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}r$$

$$-\frac{1}{(A-B)+(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}\sum_{k=2}^{\infty}M_k(\lambda,\alpha,\beta,A,B)|a_k|r^k$$

$$\geq 1-\frac{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}{(A-B)+(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}r-$$

$$\frac{A-B}{(A-B)+(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}r = 1-r>0 \ \ (|z|=r<1),$$

where we have used the assertion (3.1) of Theorem 3.1. Thus (8.6) holds true in U. This prove the inequality (8.3). The inequality (8.4) follows by taking the convex function $\phi(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k \in K$. To prove the sharpness of the constant $\frac{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}{2[(A-B)+(1+\lambda)(\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|)b_2]}$, we consider the function $f_0(z) \in S^*(f,g;\lambda,\alpha,\beta,A,B)$ given by

$$f_0(z)=z-\frac{A-B}{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}z^2$$

$$(-1\leq B<A\leq 1;\ z\in U). \tag{8.7}$$

Thus, from (8.3), we have

$$\frac{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}{2[(A-B)+(1+\lambda)(\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|)b_2]}f_0(z)\prec\frac{z}{1-z}. \tag{8.8}$$

Moreover, it can be verified for the function $f_0(z)$ given by (8.7) that

$$\min_{|z|\le r}\left\{Re\frac{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}{2[(A-B)+(1+\lambda)(\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|)b_2]}f_0(z)\right\}=-\frac{1}{2}. \tag{8.9}$$

This shows that the constant $\frac{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]b_2}{2[(A-B)+(1+\lambda)(\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|)b_2]}$ is the best possible. This completes the proof of Theorem 8.1.

Putting $g(z)=z+\sum_{k=2}^{\infty}\Psi_k z^k$ (or $b_k=\Psi_k$), where Ψ_k is defined by (1.8), in Theorems 3.1 and 8.1, we obtain the following corollary:

Corollary 8.1. Let f defined by (1.1) be in the class $S^*_{q,s}([\alpha_1];\lambda,\alpha,\beta,A,B)$ and satisfy the condition

$$\sum_{k=2}^{\infty}[1+\lambda(k-1)][\alpha(k-1)(1+(1+|B|)\beta)+|A-\alpha Bk+B(\alpha-1)|]\Psi_k|a_k|\le A-B.$$

Then for every function $\phi(z)\in K$, we have

$$\frac{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]\Psi_2}{2[(A-B)+(1+\lambda)(\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|)\Psi_2]}(f*\phi)(z)\prec\phi(z),$$

and

$$Re\{f(z)\}>-\frac{(A-B)+(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]\Psi_2}{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]\Psi_2}\quad(z\in U).$$

The constant $\frac{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|]\Psi_2}{2[(A-B)+(1+\lambda)(\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|)\Psi_2]}$ is the best estimate.

Putting $g(z)=z+\sum_{k=2}^{\infty}I^m(\rho,l)z^k$ (or $b_k=I^m(\rho,l)$), where $I^m(\rho,l)$ is defined by (1.9), in Theorems 3.1 and 8.1, we obtain the following corollary:

Corollary 8.2. Let f defined by (1.1) be in the class $S^*(\rho,l,m;\lambda,\alpha,\beta,A,B)$ and satisfy the condition

$$\sum_{k=2}^{\infty}[1+\lambda(k-1)][\alpha(k-1)(1+(1+|B|)\beta)+|A-\alpha Bk+B(\alpha-1)|]\left[\frac{1+l+\rho(k-1)}{1+l}\right]^m|a_k|$$
$$\le A-B.$$

Then for every function $\phi(z)\in K$, we have

$$\frac{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|](1+l+\rho)^m}{2[(A-B)(1+l)^m+(1+\lambda)(\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|)(1+l+\rho)^m]}$$
$$\cdot(f*\phi)(z)\prec\phi(z),$$

and

$$Re\{f(z)\}>-\frac{(A-B)(1+l)^m+(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|](1+l+\rho)^m}{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|](1+l+\rho)^m}.$$

The constant $\frac{(1+\lambda)[\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|](1+l+\rho)^m}{2[(A-B)(1+l)^m+(1+\lambda)(\alpha(1+(1+|B|)\beta)+|A-B-\alpha B|)(1+l+\rho)^m]}$ is the best estimate.

Remarks. Specializing the function g and the parameters $\lambda, \alpha, \beta, A, B$ involved in the results presented in this paper, we can obtain the corresponding results for the corresponding operators and classes (1)-(4) defined in the introduction.

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9 Open Problem

The authors suggest to study the properties of partial sum and Hadmard product for the function classes $TS(f, g; \lambda, \alpha, \beta, A, B)$ and $S(f, g; \lambda, \alpha, \beta, A, B)$.

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