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Certain subclass of p-valent meromorphic functions defined by Linear operators

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Abstract

In this paper, we introduce a new subclass of $\Sigma^*_{p,j}$ by use of a linear operator,which is defined by Hadamard product and provide coefficient inequality, distortion theorem, neighborhood property, convexity and starlike radius of this subclass. Some convolution properties are also considered.

Keywords: *Hadamard product; Meromorphic functions; Convexity radius; Starlike radius; Neighborhood.*

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1 Introduction

Let $\Sigma^*_{p,j}$ denote the class of meromorphic functions

$$f(z) = \frac{1}{z^p} + \sum_{n=j}^{\infty} |a_n| z^n, \quad (j \in N^* = \{2i-1 : i \in \{1, 2, \cdots\}\}), \tag{1}$$

which are analytic and p-valent in the punctured disc $U^* = \{z : 0 < |z| < 1\} = U\backslash\{0\}$ with a simple pole at the origin with residue one there.

A function $f \in \Sigma^*_{p,j}$ is said to be meromorphically starlike of order α if it satisfies

$$Re\{-\frac{zf'(z)}{pf(z)}\} > \alpha, \tag{2}$$

for some $\alpha(0 \leq \alpha < 1)$ and for all $z \in U^*$

$f \in \Sigma^*_{p,j}$ is said to be meromorphically convex of order α if it satisfies

$$Re\{-\left(\frac{1}{p} + \frac{zf''(z)}{pf'(z)}\right)\} > \alpha, \tag{3}$$

for some $\alpha(0 \leq \alpha < 1)$ and for all $z \in U^*$

Some subclasses of $\Sigma^*_{p,j}$ when $p = 1, j = 1$ were introduced by Miller[1],Cho et al.[2]and Aouf[3,4].

For $f(z) = \frac{1}{z^p} + \sum_{n=j}^{\infty} |a_n|z^n$ and $g(z) = \frac{1}{z^p} + \sum_{n=j}^{\infty} |b_n|z^n$ the Hadamard product (or convolution)is defined by

$$(f * g)(z) = \frac{1}{z^p} + \sum_{n=j}^{\infty} |a_n||b_n|z^n = (g * f)(z).$$

The linear operator $L_p(a, c)$ is defined as follows (see Saitoh[5], Liu and Srivastava[6])

$$L_p(a, c)f(z) = \varphi_p(a, c; z) * f(z) \qquad \left(f(z) \in \Sigma^*_{p,j}\right), \tag{4}$$

and $\varphi_p(a, c; z)$ is defined by

$$\varphi_p(a, c; z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n-1} \quad (z \in U^* a \in R; c \in R\backslash z_0^-; z_0^- := \{0, -1, -2, \cdots\}) \tag{5}$$

, where $(\nu)_k$ is the pochhammer symbol defined (in terms of the Gamma function) by

$$(\nu)_n = \frac{\Gamma(\nu + n)}{\Gamma(\nu)} = \begin{cases} 1, & n = 0 \\ \nu(\nu + 1) \cdots (\nu + n - 1), & n \in N. \end{cases}$$

Aouf, Silverman and Srivastava [7] by means of the linear operator $L_p(a, c)$ defined the classes $P_{a,c}(A, B; p, \lambda)$ and $P^+_{a,c}(A, B; p, \lambda)$, further investigated the properties of these tow classes . In [8] Aouf and El-Ashwah investigated several properties for the class $\Sigma^*_m(a, c, A, B, \alpha)$ of univalent meromorphic functions with positive coefficients. Aouf [9] defined the subclasses of $\sum_p^{(A,B)}$ by use of derivative operator and discussed the correspondig properties. In this paper we will use operator $L_p(a,c)$ to define a subclass of p-valent meromorphic function class $\Sigma^*_{p,j}$ as below:

For $p \in N,\ a > 0,\ c > 0, 0 \le \alpha \le \frac{(p+m-1)!}{(p-1)!},\ m \in N^*,\ m \le j$ and for the parameters $\lambda,\ A$ and B such that

$$-1 \le A < B \le 1,\quad 0 < B \le 1,\quad \lambda \ge 1, \tag{6}$$

we say that a function $f(z) \in \Sigma^*_{p,j}$ is in the class $\Sigma^*_{p,j}(a,c,A,B,\alpha,m;\lambda)$ if it satisfies the following subordination:

$$\frac{1}{-\frac{(p+m-1)!}{(p-1)!}+\alpha}[(\lambda-1)(p+m-1)z^{p+m-1}(L_p(a,c)f(z))^{(m-1)}+\lambda z^{p+m}(L_p(a,c)f(z))^{(m)}+\alpha] \prec \frac{1+Az}{1+Bz}. \tag{7}$$

Or equivalently, if the following inequality holds true:

$$\left|\frac{(\lambda-1)(p+m-1)z^{p+m-1}(L_p(a,c)f(z))^{(m-1)}+\lambda z^{p+m}(L_p(a,c)f(z))^{(m)}+\frac{(p+m-1)!}{(p-1)!}}{B\left[(\lambda-1)(p+m-1)z^{p+m-1}(L_p(a,c)f(z))^{(m-1)}+\lambda z^{p+m}(L_p(a,c)f(z))^{(m)}\right]+\left[B\frac{(p+m-1)!}{(p-1)!}+(A-B)\left(\frac{(p+m-1)!}{(p-1)!}-\alpha\right)\right]}\right| < 1. \tag{8}$$

From the above definition we can imply that the function class $\Sigma^*_m(a,c,A,B,\alpha)$ in [8] is the special case of $\Sigma^*_{p,j}(a,c,A,B,\alpha,m;\lambda)$ in our present paper because $\Sigma^*_{1,j}(a,c,A,B,\alpha,m;1) = \Sigma^*_m(a,c,A,B,\alpha)$.

Since $\Sigma^*_{1,j}(a,c,A,B,\alpha,m;1) = \Sigma^*_m(a,c,A,B,\alpha)$,then like [8] we have the following subclasses which were studied in many earlier works:

(1) $\Sigma^*_{1,j}(a,a,-\beta,\beta,0,m;1) = T_m(\beta)(m \in N^*, 0 < \beta \le 1)$ (Kim et al.[10]).

(2) $\Sigma^*_{1,j}(a,a,-\beta,(2\gamma-1)\beta,\alpha,1;1) = \Sigma_1(\alpha,\beta,\gamma)$ $(0 \le \alpha < 1; 0 < \beta \le 1; \frac{1}{2} \le \gamma \le 1)$ (Cho et al. [2]).

(3) $\Sigma^*_{1,j}(a,a,-A,-B,0,1;1) = \Sigma_1(A,B)$ $(-1 \le B < A \le 1; -1 \le B < 0)$ (Cho[11]).

(4) $\Sigma^*_{1,j}(a,a,-A,-B,\alpha,1;1) = \Sigma(\alpha,1,A,B,1)$ $(0 \le \alpha < 1; -1 \le A < B \le 1; 0 < B \le 1)$ (Aouf[4]).

The purpose of this paper is to give various properties of $\Sigma^*_{p,j}(a,c,A,B,\alpha,m;\lambda)$. We will extend the results of basic paper [8].

2 Coefficient inequality for class $\Sigma^*_{p,j}(a,c,A,B,\alpha,m;\lambda)$

Theorem 2.1 *Suppose* $f(z) = \frac{1}{z^p} + \sum_{n=j}^{\infty}|a_n|z^n$, *then* $f(z) \in \Sigma^*_{p,j}(a,c,A,B,\alpha,m;\lambda)$ *if and only if*

$$\sum_{n=j}^{\infty}\phi_n(p,B,a,c,\alpha,m;\lambda)|a_n| \le (B-A), \tag{9}$$

where

$$\phi_n(p,B,a,c,\alpha,m;\lambda) = \frac{(1+B)\left[p(\lambda-1)+\lambda n-m+1\right](a)_{n+1}n!}{(c)_{n+1}(n-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right]}. \tag{10}$$

Proof For $f(z)=\frac{1}{z^p}+\sum_{n=j}^{\infty}|a_n|z^n$, we denote

$$\begin{aligned}
& \left|F_{p,m}(a,c,\lambda,z)+\frac{(p+m-1)!}{(p-1)!}\right| \\
- & \left|B\left[F_{p,m}(a,c,\lambda,z)\right]+\left[B\frac{(p+m-1)!}{(p-1)!}+(A-B)\left(\frac{(p+m-1)!}{(p-1)!}-\alpha\right)\right]\right| \\
= & \left|\sum_{n=j}^{\infty}|a_n|\frac{(a)_{n+1}n!\left[p(\lambda-1)+\lambda n-m+1\right]}{(c)_{n+1}(n-m+1)!}z^{n+p}\right| \\
- & \left|(B-A)\left(\frac{(p-m+1)!}{(p-1)!}-\alpha\right)-B\sum_{n=j}^{\infty}|a_n|\frac{(a)_{n+1}n!\left[p(\lambda-1)+\lambda n-m+1\right]}{(c)_{n+1}(n-m+1)!}z^{n+p}\right| \\
\le & \sum_{n=j}^{\infty}|a_n|\frac{(a)_{n+1}n!\left[p(\lambda-1)+\lambda n-m+1\right]}{(c)_{n+1}(n-m+1)!}|z|^{n+p} \\
- & \left[(B-A)\left(\frac{(p-m+1)!}{(p-1)!}-\alpha\right)-B\sum_{n=j}^{\infty}|a_n|\frac{(a)_{n+1}n!\left[p(\lambda-1)+\lambda n-m+1\right]}{(c)_{n+1}(n-m+1)!}|z|^{n+p}\right] \\
\le & \sum_{n=j}^{\infty}(1+B)|a_n|\frac{(a)_{n+1}n!\left[p(\lambda-1)+\lambda n-m+1\right]}{(c)_{n+1}(n-m+1)!}-(B-A)\left(\frac{(p-m+1)!}{(p-1)!}-\alpha\right) \quad (z\in\partial U),
\end{aligned}$$

where

$$\begin{aligned}
F_{p,m}(a,c,\lambda,z) &= (\lambda-1)(p+m-1)z^{p+m-1}\left(L_p(a,c)f(z)\right)^{(m-1)} \\
&+ \lambda z^{p+m}\left(L_p(a,c)f(z)\right)^{(m)}.
\end{aligned}$$

By the Maximum Modulus Theorem, for any $z\in U$ we have

$$\begin{aligned}
& \left|F_{p,m}(a,c,\lambda,z)+\frac{(p+m-1)!}{(p-1)!}\right| \\
- & \left|B\left[F_{p,m}(a,c,\lambda,z)\right]+\left[B\frac{(p+m-1)!}{(p-1)!}+(A-B)\left(\frac{(p+m-1)!}{(p-1)!}-\alpha\right)\right]\right| \\
\le & \sum_{n=j}^{\infty}(1+B)|a_n|\frac{(a)_{n+1}n!\left[p(\lambda-1)+\lambda n-m+1\right]}{(c)_{n+1}(n-m+1)!}-(B-A)\left(\frac{(p-m+1)!}{(p-1)!}-\alpha\right) \\
\le & \ 0,
\end{aligned}$$

this implies $f(z)\in\Sigma^*_{p,j}(a,c,A,B,\alpha,m;\lambda)$.

Conversely, suppose $f(z)\in\Sigma^*_{p,j}(a,c,A,B,\alpha,m;\lambda)$, then we have

$$\left|\frac{\sum_{n=j}^{\infty}|a_n|\frac{(a)_{n+1}n![p(\lambda-1)+\lambda n-m+1]}{(c)_{n+1}(n-m+1)!}z^{n+p}}{B\sum_{n=j}^{\infty}|a_n|\frac{(a)_{n+1}n![p(\lambda-1)+\lambda n-m+1]}{(c)_{n+1}(n-m+1)!}z^{n+p}+(A-B)\left(\frac{(p-m+1)!}{(p-1)!}-\alpha\right)}\right|<1.$$

Since $|Rez|\le|z|$ for all z, choosing z to be real and letting $z\to 1^-$ through real value, then we have

$$\frac{\sum_{n=j}^{\infty}|a_n|\frac{(a)_{n+1}n![p(\lambda-1)+\lambda n-m+1]}{(c)_{n+1}(n-m)!(n-m+1)}|z|^{n+p}}{(B-A)\left(\frac{(p-m+1)!}{(p-1)!}-\alpha\right)-B\sum_{n=j}^{\infty}|a_n|\frac{(a)_{n+1}n![p(\lambda-1)+\lambda n-m+1]}{(c)_{n+1}(n-m)!(n-m+1)}|z|^{n+p}}<1.$$

So we have

$$\sum_{n=j}^{\infty}(1+B)|a_n|\frac{(a)_{n+1}n!\left[p(\lambda-1)+\lambda n-m+1\right]}{(c)_{n+1}(n-m)!(n-m+1)}\le(B-A)\left(\frac{(p-m+1)!}{(p-1)!}-\alpha\right),$$

this completes the proof of theorem 2.1.

Corollary 2.2 *Let* $f(z)=\frac{1}{z^p}+\sum_{n=j}^{\infty}|a_n|z^n$, *if* $f(z)\in\Sigma_{p,j}^{*}(a,c,A,B,\alpha,m;\lambda)$, *then*

$$|a_n|\leq\frac{(B-A)\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right]}{(1+B)\frac{(a)_{n+1}n![p(\lambda-1)+\lambda n-m+1]}{(c)_{n+1}(n-m+1)!}},$$

the result is sharp when

$$f(z)=\frac{1}{z^p}+\frac{(B-A)\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right]}{(1+B)\frac{(a)_{n+1}n![p(\lambda-1)+\lambda n-m+1]}{(c)_{n+1}(n-m+1)!}}z^n.$$

Remark 2.3 *Theorem 2.1 in this paper is the extension of theorem 1 in [8].*

3 Distortion Theorem

Theorem 3.1 *If* $f(z)=\frac{1}{z^p}+\sum_{n=j}^{\infty}|a_n|z^n$ *be in the class* $\Sigma_{p,j}^{*}(a,c,A,B,\alpha,m;\lambda)$, *then*

$$\begin{aligned}&\left\{\frac{(p+k-1)!}{(p-1)!}F_{p,j}(A,B,a,c,m,k,\lambda,\alpha)|z|^{j+p}\right\}|z|^{-(k+p)}\\ \leq\;& |f^{(k)}(z)|\\ \leq\;& \left\{\frac{(p+k-1)!}{(p-1)!}+F_{p,j}(A,B,a,c,m,k,\lambda,\alpha)|z|^{j+p}\right\}|z|^{-(k+p)},\end{aligned}\tag{11}$$

where $F_{p,j}(A,B,a,c,m,k,\lambda,\alpha)=\frac{(B-A)(c)_{j+1}(j-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right]}{(1+B)(a)_{j+1}(j-k)![p(\lambda-1)+\lambda j-m+1]}$. *Furthermore* $a>c>0$, $\frac{(p+m-1)!}{(p-1)!}-\frac{(1+B)(a)_{j+i}(j-k)![p(\lambda-1)+\lambda j-m+1](p+k-1)!}{(B-A)(c)_{j+i}(j-m+1)(p-1)!}\leq\alpha<\frac{(p+m-1)!}{(p-1)!}$, $j,m\in N^*$, $k\in N_0$; $k\leq m-1\leq j-1$.

The result is sharp for the function f *given by*

$$f(z)=\frac{1}{z^p}+\frac{(B-A)\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right]}{(1+B)\frac{(a)_{j+1}j![p(\lambda-1)+\lambda j-m+1]}{(c)_{j+1}(j-m+1)!}}z^j.$$

Proof Let $f(z)=\frac{1}{z^p}+\sum_{n=j}^{\infty}|a_n|z^n$, $0\leq k\leq(m-1)$, then we have

$$f^{(k)}(z)=\frac{(-1)^k(p+k-1)!}{z^{p+k}(p-1)!}+\sum_{n=j}^{\infty}|a_n|\frac{n!}{(n-k)!}z^{n-k},\tag{12}$$

and we can imply

$$\begin{aligned}&(1+B)\frac{(a)_{j+1}(j-k)!\,[p(\lambda-1)+\lambda j-m+1]}{(c)_{j+1}(j-m+1)!}\sum_{n=j}^{\infty}\frac{n!}{(n-k)!}|a_n|\\ \leq\;&\sum_{n=j}^{\infty}(1+B)\frac{(a)_{n+1}n!\,[p(\lambda-1)+\lambda n-m+1]}{(c)_{n+1}(n-m+1)!}|a_n|.\end{aligned}$$

By theorem 2.1 we obtain

$$\sum_{n=j}^{\infty} \frac{n!}{(n-k)!}|a_n| \leq (B-A)\left[\frac{(p+m-1)!}{(p-1)!} - \alpha\right] \frac{(c)_{j+1}(j-m+1)!}{(1+B)(a)_{j+1}(j-k)!\,[p(\lambda-1)+\lambda j - m + 1]}, \quad (13)$$

by (12)and (13) we can imply (11).

4 Radius of starlikeness and convexity

Theorem 4.1 *Let the function $f(z)$ defined by (1) be in the class $\Sigma^*_{p,j}(a,c,A,B,\alpha,m;\lambda)$. Then $f(z)$ is starlike of $\xi(0 \leq \xi < 1)$ in $|z| < r_\xi(A,B,a,c,p,\alpha,m;\lambda)$ where*

$$r_\xi(a,c,A,B,p,\alpha,m;\lambda) = inf_{n\geq j}\left\{\frac{p(1-\xi)\phi_n(p,B,a,c,\alpha,m;\lambda)}{(B-A)(n+2p-\xi p)}\right\}^{\frac{1}{n+p}}, \quad (14)$$

and $\phi_n(p,B,a,c,\alpha,m;\lambda)$ is defined by (10).

Proof It sufficient to show that

$$\left|1+\frac{zf'(z)}{pf(z)}\right| < 1-\xi \quad \text{for} \quad |z| < r_\xi(a,c,A,B,p,\alpha,m;\lambda).$$

Since

$$\left|1+\frac{zf'(z)}{pf(z)}\right| = \left|\frac{\sum_{n=j}^{\infty}(n+p)|a_n|z^{n+p}}{p+\sum_{n=j}^{\infty}p|a_n|z^{n+p}}\right| \leq \frac{\sum_{n=j}^{\infty}(n+p)|a_n||z|^{n+p}}{p-\sum_{n=j}^{\infty}p|a_n||z|^{n+p}},$$

to prove (14) it sufficient to prove

$$\frac{\sum_{n=j}^{\infty}(n+p)|a_n||z|^{n+p}}{p-\sum_{n=j}^{\infty}p|a_n||z|^{n+p}} < 1-\xi.$$

It is equivalent to

$$\sum_{n=j}^{\infty}\frac{(n+2p-p\xi)|a_n|}{p(1-\xi)}|z|^{n+p} \leq 1. \quad (15)$$

By theorem 2.1 we have

$$\sum_{n=j}^{\infty}\frac{\phi_n(p,B,a,c,\alpha,m;\lambda)}{(B-A)}|a_n| \leq 1.$$

Hence (15) will be true if

$$\frac{(n+2p-p\xi)}{p(1-\xi)}|z|^{n+p} \leq \frac{\phi_n(p,B,a,c,\alpha,m;\lambda)}{B-A},$$

it is equivalent to

$$|z| \leq \left\{\frac{p(1-\xi)\phi_n(p,B,a,c,\alpha,m;\lambda)}{(B-A)(n+2p-p\xi)}\right\}^{\frac{1}{n+p}}.$$

This completes the proof.

Theorem 4.2 *Let the function $f(z)$ defined by (1) be in the class $\Sigma^*_{p,j}(a,c,A,B,\alpha,m;\lambda)$. Then $f(z)$ is convex of $\eta(0 \leq \eta < 1)$ in $|z| < r_\eta(A,B,a,c,p,\alpha,m;\lambda)$ where*

$$r_\eta(A,B,a,c,p,\alpha,m;\lambda) = inf_{n\geq j}\left\{\frac{p(1-\eta)\phi_n(p,B,a,c,\alpha,m;\lambda)}{n(B-A)(n+2p-p\eta)}\right\}^{\frac{1}{n+p}}, \quad (16)$$

and $\phi_n(p,B,a,c,\alpha,m;\lambda)$ is defined by (10).

Proof It sufficient to show that

$$\left|1+\frac{1}{p}+\frac{zf''(z)}{pf'(z)}\right| < 1-\eta \quad \text{for} \quad |z| < r_\eta(A,B,a,c,p,\alpha,m;\lambda).$$

Since

$$\begin{aligned}\left|1+\frac{1}{p}+\frac{zf''(z)}{pf'(z)}\right| &= \left|\frac{\sum_{n=j}^{\infty} n(n+p)|a_n|z^{n+p}}{-p+\sum_{n=j}^{\infty} np|a_n|z^{n+p}}\right| \\ &\leq \frac{\sum_{n=j}^{\infty} n(n+p)|a_n||z|^{n+p}}{p-\sum_{n=j}^{\infty} np|a_n||z|^{n+p}},\end{aligned}$$

to prove (16) it sufficient to prove

$$\frac{\sum_{n=j}^{\infty} n(n+p)|a_n||z|^{n+p}}{p-\sum_{n=j}^{\infty} np|a_n||z|^{n+p}} < 1-\eta.$$

It is equivalent to

$$\sum_{n=j}^{\infty}\frac{n(n+2p-p\eta)}{p(1-\eta)}|z|^{n+p} \leq 1. \quad (17)$$

By theorem 2.1 we have

$$\sum_{n=j}^{\infty}\frac{\phi_n(p,B,a,c,\alpha,m;\lambda)}{(B-A)}|a_n| \leq 1.$$

Hence (17) will be true if

$$\frac{n(n+2p-p\eta)|a_n|}{p(1-\eta)}|z|^{n+p} \leq \frac{\phi_n(p,B,a,c,\alpha,m;\lambda)}{B-A},$$

it is equivalent to

$$|z| \leq \left\{ \frac{p(1-\eta)\phi_n(p,B,a,c,\alpha,m;\lambda)}{n(B-A)(n+2p-p\eta)} \right\}^{\frac{1}{n+p}}.$$

This completes the proof.

5 δ-Neighborhoods of $\Sigma^*_{p,j}(a,c,A,B,\alpha,m;\lambda)$

Based on the earlier works of Aouf, Silverman and Srivastava[7] Goodman[12] and by Altintas et al.[13] ,we introduce the δ-neighborhood of a function $f(z) \in \Sigma^*_{p,j}$ of the form (1) and present the relationship between δ-neighborhood and corresponding function class.

Definition 5.1 *The δ-neighborhood of a function $f(z) \in \Sigma^*_{p,j}$ of the form (1) is defined as follows:*

$$N^+_\delta(f) = \left\{ g : g(z) = \frac{1}{z^p} - \sum_{n=j}^{\infty} |b_n| z^n \in \Sigma^*_{p,j} \text{ and } \sum_{n=j}^{\infty} t_k \left| |b_n| - |a_n| \right| < \delta \right\}, \tag{18}$$

where

$$t_k = \frac{n!(1+B)\left[p(\lambda-1)+\lambda n - m + 1\right])(a)_{n+1}}{(B-A)(n-m+1)!\left[\frac{(p+m-1)!}{(p-1)!} - \alpha\right](c)_{n+1}},$$

and $a > 0, c > 0, -1 \leq A < B \leq 1, 0 < B \leq 1, \delta > 0, 0 \leq \alpha < \frac{(p+m-1)!}{(p-1)!}, j, m \in N^*, j \geq m$.

Theorem 5.2 *Let the function $f(z)$ defined by (1) be in the class $\Sigma^*_{p,j}(a+1,c,A,B,\alpha,m;\lambda)$, then*

$$N^+_\delta(f) \subset \Sigma^*_{p,j}(a,c,A,B,\alpha,m;\lambda), \qquad \left(\delta := \frac{j+1}{a+j+1}\right).$$

This result is the best possible in the sense that δ can not be increased.

Proof It is easily seen from (8) that $f(z) \in \Sigma^*_{p,j}(a,c,A,B,\alpha,m;\lambda)$ if and only if for any complex number σ with $|\sigma| = 1$

$$\frac{(\lambda-1)(p+m-1)z^{p+m-1}(L_p(a,c)f(z))^{(m-1)} + \lambda z^{p+m}(L_p(a,c)f(z))^{(m)} + \frac{(p+m-1)!}{(p-1)!}}{B\left[(\lambda-1)(p+m-1)z^{p+m-1}(L_p(a,c)f(z))^{(m-1)} + \lambda z^{p+m}(L_p(a,c)f(z))^{(m)}\right] + \left[B\frac{(p+m-1)!}{(p-1)!} + (A-B)\left(\frac{(p+m-1)!}{(p-1)!} - \alpha\right)\right]}$$
$$\neq \sigma. \tag{19}$$

Which is equivalent to

$$\frac{(f*h)(z)}{z^{-p}} \neq 0, \quad (z \in U), \tag{20}$$

where

$$\begin{aligned} h(z) &= \frac{1}{z^p} + \sum_{n=j}^{\infty} c_n z^n \\ &= \frac{1}{z^p} + \sum_{n=j}^{\infty} \frac{n!(1-\sigma B)\left[p(\lambda-1)+\lambda n-m+1\right](a)_{n+1}}{\sigma(B-A)(n-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right](c)_{n+1}} z^n. \end{aligned}$$

Now if $f(z) \in \Sigma^*_{p,j}(a+1, c, A, B, \alpha, m; \lambda)$,then we have

$$\begin{aligned} \left|\frac{f(z)*h(z)}{z^{-p}}\right| &= \left|1+\sum_{n=j}^{\infty} c_n |a_n| z^{n+p}\right| \\ &\geq 1-\sum_{n=j}^{\infty} \frac{n!(1+B)\left[p(\lambda-1)+\lambda n-m+1\right](a)_{n+1}}{(B-A)(n-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right](c)_{n+1}} \\ &\geq 1-\frac{a}{a+j+1}\sum_{n=j}^{\infty} \frac{n!(1+B)\left[p(\lambda-1)+\lambda n-m+1\right](a+1)_{n+1}}{(B-A)(n-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right](c)_{n+1}} \\ &\geq 1-\frac{a}{a+j+1} = \frac{j+i}{a+j+1} = \delta. \end{aligned} \tag{21}$$

For any $g \in N^+_\delta(f)$,we have

$$\begin{aligned} \left|\frac{[f(z)-g(z)]*h(z)}{z^{-p}}\right| &= \left|\sum_{n=j}^{\infty} (|b_n|-|a_n|)\,|c_n| z^{n+p}\right| \\ &\leq \sum_{h=j}^{\infty} ||b_n|-|a_n||\,|c_n| \\ &\leq \delta. \end{aligned} \tag{22}$$

From (21),(22) we can imply $\frac{g(z)*h(z)}{z^{-p}} \neq 0$,it means $g(z) \in \Sigma^*_{p,j}(a, c, A, B, \alpha, m; \lambda)$.

To show the sharpness, we consider the functions f and g given by

$$f(z) = \frac{1}{z^p} + \frac{(B-A)(j-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right](c)_{j+1}}{j!(1+B)\left[p(\lambda-1)+\lambda j-m+1\right](a)_{j+1}} z^j, \tag{23}$$

$$g(z) = \frac{1}{z^p} + \left[\frac{B-A}{\phi_j(p, B, a, c, \alpha, m; \lambda)} + \frac{(B-A)(j-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right]\delta'(c)_{j+1}}{j!(1+B)\left[p(\lambda-1)+\lambda j-m+1\right](a)_{j+1}}\right] z^j. \tag{24}$$

where $\delta' > \delta = \frac{j+1}{a+j+1}$.

Clearly $g \in N^{+}_{\delta'}(f)$,but by theorem 1 $g(z)\overline{\in}\Sigma^{*}_{p,j}(a,c,A,B,\alpha,m;\lambda)$, since

$$\begin{aligned}
& \frac{\phi_j(p,B,a,c,\alpha,m;\lambda)}{B-A}\cdot\left\{\frac{B-A}{\phi_j(p,B,a,c,\alpha,m;\lambda)}+\frac{(B-A)(j-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right]\delta'(c)_{j+1}}{j!(1+B)\left[p(\lambda-1)+\lambda j-m+1\right](a)_{j+1}}\right\}\\
= & \frac{(a)_{j+1}}{(a+1)_{j+1}}+\delta'\\
= & \frac{a}{a+j+1}+\delta'\\
> & \frac{a}{a+j+1}+\frac{j+1}{a+j+1}=1.
\end{aligned}$$

6 Properties associated with modified Hadamard product

Following early works by Aouf,Silverman and Srivastava in [8], we provide the properties of modified Hadamard product of $\Sigma^{*}_{p,j}(a,c,A,B,\alpha,m;\lambda)$

For the function

$$f_k(z)=\frac{1}{z^p}+\sum_{n=j}^{\infty}|a_{n,k}|z^n \qquad (k=1,2;p\in N^{*}), \tag{25}$$

the modified Hadamard product of the functions $f_1(z)$ and $f_2(z)$ was denoted by $(f_1 \bullet f_2)(z)$ and defined as follows

$$(f_1\bullet f_2)(z)=\frac{1}{z^p}+\sum_{n=j}^{\infty}|a_{n,1}||a_{n,2}|z^n=(f_2\bullet f_1)(z).$$

Theorem 6.1 *Let $f_k(z)$ $(k=1,2)$ given by (25) be in the class $\Sigma^{*}_{p,j}(a,c,A,B,\alpha,m;\lambda)$,then*

$$(f_1\bullet f_2)(z)\in\Sigma^{*}_{p,j}(a,c,A,B,\gamma,m;\lambda),$$

where

$$\gamma:=\frac{(p+m-1)!}{(p-1)!}-\frac{(B-A)(j-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right]^2(c)_{j+1}}{j!(1+B)\left[p(\lambda-1)+\lambda j-m+1\right](a)_{j+1}}.$$

The result is sharp for the functions $f_k(z)$ $(k=1,2)$ given by

$$f_k(z)=\frac{1}{z^p}+\frac{(B-A)(j-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right](c)_{j+1}}{j!(1+B)\left[p(\lambda-1)+\lambda j-m+1\right](a)_{j+1}}. \tag{26}$$

Proof By theorem 2.1 we need to find the largest γ such that

$$\sum_{n=j}^{\infty}\frac{n!(1+B)\left[p(\lambda-1)+\lambda n-m+1\right](a)_{n+1}}{(B-A)(n-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\gamma\right](c)_{n+1}}|a_{n,1}|\cdot|a_{n,2}|\leq 1.$$

Since $f_k(z)\in\Sigma^*_{p,j}(a,c,A,B,\alpha,m;\lambda)$, then we see that

$$\sum_{n=j}^{\infty}\frac{n!(1+B)\left[p(\lambda-1)+\lambda n-m+1\right](a)_{n+1}}{(B-A)(n-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right](c)_{n+1}}|a_{n,k}|\leq 1,\quad (k=1,2),$$

by Cauchy-Schwartz inequality, we obtain

$$\sum_{n=j}^{\infty}\frac{n!(1+B)\left[p(\lambda-1)+\lambda n-m+1\right](a)_{n+1}}{(B-A)(n-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right](c)_{n+1}}\sqrt{|a_{n,1}|\cdot|a_{n,2}|}\leq 1. \tag{27}$$

This implies that we only need to show that

$$\frac{1}{\left[\frac{(p+m-1)!}{(p-1)!}-\gamma\right]}|a_{n,1}|\cdot|a_{n,2}|\leq\frac{1}{\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right]}\sqrt{|a_{n,1}|\cdot|a_{n,2}|},$$

or equivalently that

$$\sqrt{|a_{n,1}|\cdot|a_{n,2}|}\leq\frac{\left[\frac{(p+m-1)!}{(p-1)!}-\gamma\right]}{\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right]}.$$

By making use of the inequality (27), it is sufficient to prove that

$$\frac{(B-A)(n-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right](c)_{n+1}}{n!(1+B)\left[p(\lambda-1)+\lambda n-m+1\right](a)_{n+1}}\leq\frac{\left[\frac{(p+m-1)!}{(p-1)!}-\gamma\right]}{\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right]}. \tag{28}$$

From (28), we have

$$\gamma\leq\frac{(p+m-1)!}{(p-1)!}-\frac{(B-A)(n-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right]^2(c)_{n+1}}{n!(1+B)\left[p(\lambda-1)+\lambda n-m+1\right](a)_{n+1}}.$$

Define the function $\Phi(n)$ by

$$\Phi(n):=\frac{(p+m-1)!}{(p-1)!}-\frac{(B-A)(n-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right]^2(c)_{n+1}}{n!(1+B)\left[p(\lambda-1)+\lambda n-m+1\right](a_{n+1})},$$

we see that $\Phi(n)$ is an increasing function of $a > c > 0,\ 0 \le \alpha \le \frac{(p+m-1)!}{(p-1)!}, j \ge m$ .Therefore, we conclude that

$$\gamma \le \phi(j) = \frac{(p+m-1)!}{(p-1)!} - \frac{(B-A)(j-m+1)! \left[\frac{(p+m-1)!}{(p-1)!} - \alpha\right]^2 (c)_{j+1}}{j!(1+B)\left[p(\lambda-1)+\lambda j - m + 1\right](a_{j+1})}.$$

By using arguments similar to those in the proof of theorem 6.1 we can derive the following result.

Theorem 6.2 *Let $f_1(z)$ defined by (25) be in the class $\Sigma^*_{p,j}(a, c, A, B, \alpha_1, m; \lambda)$, $f_2(z)$ defined by (25) be in the class $\Sigma^*_{p,j}(a, c, A, B, \alpha_2, m; \lambda)$,then*

$$(f_1 \bullet f_2)(z) \in \Sigma^*_{p,j}(a, c, A, B, \tau, m; \lambda),$$

where

$$\tau := \frac{(p+m-1)!}{(p-1)!} - \frac{(B-A)(j-m+1)! \left[\frac{(p+m-1)!}{(p-1)!} - \alpha_1\right] \left[\frac{(p+m-1)!}{(p-1)!} - \alpha_2\right] (c)_{j+1}}{j!(1+B)\left[p(\lambda-1)+\lambda j - m + 1\right](a_{j+1})}.$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_1(z) = \frac{1}{z^p} + \frac{(B-A)(j-m+1)! \left[\frac{(p+m-1)!}{(p-1)!} - \alpha_1\right] (c)_{j+1}}{j!(1+B)\left[p(\lambda-1)+\lambda j - m + 1\right](a)_{j+1}},$$

$$f_2(z) = \frac{1}{z^p} + \frac{(B-A)(j-m+1)! \left[\frac{(p+m-1)!}{(p-1)!} - \alpha_2\right] (c)_{j+1}}{j!(1+B)\left[p(\lambda-1)+\lambda j - m + 1\right](a)_{j+1}}.$$

Theorem 6.3 *Let $f_k(z)$ $(k = 1, 2)$ defined by (25) be in the class $\Sigma^*_{p,j}(a, c, A, B, \alpha, m; \lambda)$,then the function $h(z)$ defined by*

$$h(z) = \frac{1}{z^p} + \sum_{n=j}^{\infty} \left(|a_{n,1}|^2 + |a_{n,2}|^2\right) z^n,$$

belongs to the class $\Sigma^*_{p,j}(a, c, A, B, \chi, m; \lambda)$ where

$$\chi := \frac{(p+m-1)!}{(p-1)!} - \frac{2(B-A)(j-m+1)! \left[\frac{(p+m-1)!}{(p-1)!} - \alpha\right]^2 (c)_{j+1}}{j!(1+B)\left[p(\lambda-1)+\lambda j - m + 1\right](a)_{j+1}}.$$

This result is sharp for the functions given by (26)

Proof By theorem 2.1 we want to find the largest χ such that

$$\sum_{n=j}^{\infty}\frac{n!(1+B)\left[p(\lambda-1)+\lambda n-m+1\right](a)_{n+1}}{(B-A)(n-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\chi\right](c)_{n+1}}\left(|a_{n,1}|^2+|a_{n,2}|^2\right)\leq 1. \tag{29}$$

Since $f_k(z)\in\Sigma^*_{p,j}(a,c,A,B,\alpha,m;\lambda)$, $(k=1,2)$ we readily see that

$$\sum_{n=j}^{\infty}\frac{n!(1+B)\left[p(\lambda-1)+\lambda n-m+1\right](a)_{n+1}}{(B-A)(n-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right](c)_{n+1}}|a_{n,k}|\leq 1,\qquad (k=1,2). \tag{30}$$

From (30) we have

$$\sum_{n=j}^{\infty}\left[\frac{n!(1+B)\left[p(\lambda-1)+\lambda n-m+1\right](a)_{n+1}}{(B-A)(n-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right](c)_{n+1}}\right]^2|a_{n,k}|^2\leq 1,\qquad (k=1,2),$$

then we have

$$\sum_{n=j}^{\infty}\left[\frac{n!(1+B)\left[p(\lambda-1)+\lambda n-m+1\right](a)_{n+1}}{(B-A)(n-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right](c)_{n+1}}\right]^2\left(|a_{n,1}|^2+|a_{n,2}|^2\right)\leq 2. \tag{31}$$

From (31), if we want to prove (29), it sufficient to prove there exists the largest χ such that

$$\frac{1}{\left[\frac{(p+m-1)!}{(p+1)!}-\chi\right]}\leq\frac{n!(1+B)\left[p(\lambda-1)+\lambda n-m+1\right](a)_{n+1}}{2(B-A)(n-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right]^2(c)_{n+1}},$$

that is

$$\chi\leq\frac{(p+m-1)!}{(p-1)!}-\frac{2(B-A)(n-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right]^2(c)_{n+1}}{n!(1+B)\left[p(\lambda-1)+\lambda n-m+1\right](a)_{n+1}}.$$

Now we define $\Psi(n)$ by

$$\Psi(n)=\frac{(p+m-1)!}{(p-1)!}-\frac{2(B-A)(n-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right]^2(c)_{n+1}}{n!(1+B)\left[p(\lambda-1)+\lambda n-m+1\right](a)_{n+1}},$$

we see that $\Psi(n)$ is an increasing function of n.Therefore, we conclude that

$$\chi\leq\Psi(j)=\frac{(p+m-1)!}{(p-1)!}-\frac{2(B-A)(j-m+1)!\left[\frac{(p+m-1)!}{(p-1)!}-\alpha\right]^2(c)_{j+1}}{j!(1+B)\left[p(\lambda-1)+\lambda j-m+1\right](a)_{j+1}}.$$

which completes the proof of theorem 6.3.

7 Open Problem

Let $\sum_{p,j}^{\xi}$ denote the class of meromorphic functions

$$f(z) = \frac{a_0}{(z-\xi)^p} + \sum_{n=j}^{\infty} |a_n| z^n, \quad (a_0 > 0, 0 \leq \xi < 1, j \in N^* = \{2i-1 : i \in N\}),$$

which are analytic and p-valent in the punctured disc $U_{\xi}^* = \{z : 0 < |z-\xi| < 1 + |\xi|\}$ with a simple pole at the origin with residue one there.

Let $LS_{p,a,c}(\alpha, \mu; A, B)$ denote the subclass of $\sum_{p,j}^{\xi}$ consisting of function $f(z)$ which satisfy the following inequality:

$$\frac{1}{-\frac{(p+m-1)!a_0}{(p-1)!} + \alpha} F_{a,\xi}(p,\lambda,\alpha) - \mu |frac1 - \frac{(p+m-1)!a_0}{(p-1)!} + \alpha F_{a,\xi}(p,\lambda,\alpha) - 1| \prec \frac{1+Az}{1+Bz},$$

where

$$\begin{aligned} F_{a,\xi}(p,\lambda,\alpha) &= (\lambda-1)(p+m-1)(z-\xi)^{p+m-1}(L_p(a,c)f(z))^{(m-1)} \\ &+ \lambda(z-\xi)^{p+m}(L_p(a,c)f(z))^{(m)} + \alpha. \end{aligned}$$

Discuss coefficient inequality, distortion theorem, neighborhood properties, convexity and starlike radius of subclass $LS_{p,a,c}(\alpha, \mu; A, B)$.
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