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The Univalence Conditions For a New Integral Operator

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Abstract

The main object of the present paper is to discuss some univalence conditions for a new integral operator. Several other closely-related results are also considered.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$$

and satisfy the following usual normalization condition

$$f(0) = f'(0) - 1 = 0.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are univalent in $\mathcal{U}$.
In order to prove our main results, we need:
In [5], Pascu gave the following univalence criterion:

Theorem 1.1 [5] *Let $\beta \in \mathbb{C}$ with $\mathrm{Re}\beta > 0$. If $f \in \mathcal{A}$ satisfies*

$$\frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

for all $z \in \mathcal{U}$, then the integral operator

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta-1} f'(t)dt\right)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Further, Pescar [6] gave the following theorem:

Theorem 1.2 [6] *Let $\beta \in \mathbb{C}$ with $\mathrm{Re}\beta > 0$, $c \in \mathbb{C}$ with $|c| \leq 1$, $c \neq -1$. If $f \in \mathcal{A}$ satisfies*

$$\left|c\,|z|^{2\beta} + \left(1-|z|^{2\beta}\right)\frac{zf''(z)}{\beta f'(z)}\right| \leq 1,$$

for all $z \in \mathcal{U}$ then the integral operator

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta-1} f'(t)dt\right)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Definition 1.3 *A function $f \in \mathcal{A}$ is said to be a member of the class $\mathcal{B}(\mu,\alpha)$, $0 \leq \alpha < 1$, $\mu \geq 0$ if and only if*

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < 1-\alpha \qquad (z \in \mathcal{U}). \tag{2}$$

The family $\mathcal{B}(\mu,\alpha)$ was introduced and studied by Frasin and Jahangiri (see [3]).
This family is a comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones, such as $\mathcal{B}(1,\alpha) = \mathcal{S}^*(\alpha)$. Frasin and Darus in [2] (see also [1]) have been introduced another interesting subclass $\mathcal{B}(2,\alpha) = \mathcal{B}(\alpha)$.
Now, we define the following new integral operator.

Definition 1.4 *Let* $\gamma_i, \delta_i \in \mathbb{C}$ *for all* $i \in \{1, 2, ..., n\}$, $\beta \in \mathbb{C}$ *with* $\mathrm{Re}\beta > 0$. *We let*

$$I_\beta\left(f_1, ..., f_n; g_1, ..., g_n\right) : \mathcal{A}^n \to \mathcal{A}$$

be the integral operator given by

$$I_\beta\left(f_1, ..., f_n; g_1, ..., g_n\right)(z) = \left(\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\delta_i} \left(e^{g_i(t)}\right)^{\gamma_i} dt\right)^{\frac{1}{\beta}}. \quad (3)$$

In the present paper, we study the univalence conditions for the integral operator $I_\beta\left(f_1, ..., f_n; g_1, ..., g_n\right)(z)$ defined by (3) to be univalent in $\mathcal{U}$, when $f_i, g_i \in \mathcal{B}\left(\mu_i, \alpha_i\right)$, $0 \leq \alpha_i < 1$, $\mu_i \geq 0$ for all $i \in \{1, 2, ..., n\}$.
For this purpose, we need the familiar Schwarz Lemma (see, for details, [4])

Lemma 1.5 (**General Schwarz Lemma**) [4]. *Let the function* f *be regular in the disk* $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$, *with* $|f(z)| < M$ *for fixed* M. *If* f *has one zero with multiplicity order bigger than* m *for* $z = 0$, *then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m \qquad (z \in \mathcal{U}_R).$$

The equality can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ *is constant.*

2 Main Results

Theorem 2.1 *Let the functions* $f_i, g_i \in \mathcal{A}$ *for all* $i \in \{1, 2, ..., n\}$, $\beta, \delta_i, \gamma_i$ *be complex numbers,* $\mathrm{Re}\beta > 0$ *and* $M_i \geq 1, N_i \geq 1$ *with*

$$\mathrm{Re}\beta \geq \sum_{\mathrm{i}=1}^{\mathrm{n}} [|\delta_\mathrm{i}| \left((2 - \alpha_\mathrm{i}) \mathrm{M}_\mathrm{i}^{\mu_\mathrm{i}-1} + 1\right) + |\gamma_\mathrm{i}| (2 - \alpha_\mathrm{i}) \mathrm{N}_\mathrm{i}^{\mu_\mathrm{i}}] \quad (4)$$

for all $i \in \{1, 2, ..., n\}$.
If $f_i, g_i \in \mathcal{B}\left(\mu_i, \alpha_i\right)$, $0 \leq \alpha_i < 1$, $\mu_i \geq 0$ *satisfy the conditions*

$$|f_i(z)| \leq M_i \qquad (z \in \mathcal{U}), \qquad |g_i(z)| \leq N_i \qquad (z \in \mathcal{U})$$

for all $i \in \{1, 2, ..., n\}$, *then the integral operator* $I_\beta\left(f_1, ..., f_n; g_1, ..., g_n\right)(z)$ *defined by (3) is in the class* $\mathcal{S}$.

Proof. We begin by setting

$$h(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\delta_i} \left(e^{g_i(t)}\right)^{\gamma_i} dt \tag{5}$$

and then we calculate for $h(z)$ the derivates of the first and second orders. From (5) we obtain

$$h'(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{z}\right)^{\delta_i} \left(e^{g_i(z)}\right)^{\gamma_i}$$

and

$$h''(z) = \sum_{i=1}^{n} \left[\delta_i \left(\frac{f_i(z)}{z}\right)^{\delta_i - 1} \left(\frac{zf_i'(z) - f_i(z)}{z^2}\right) \left(e^{g_i(z)}\right)^{\gamma_i}\right] \prod_{\substack{k=1\\k\neq i}}^{n} \left(\frac{f_k(z)}{z}\right)^{\delta_k} \left(e^{g_k(z)}\right)^{\gamma_k}$$

$$+\left[\left(\frac{f_i(z)}{z}\right)^{\delta_i} \gamma_i \left(e^{g_i(z)}\right)^{\gamma_i - 1} g_i'(z) e^{g_i(z)}\right] \prod_{\substack{k=1\\k\neq i}}^{n} \left(\frac{f_k(z)}{z}\right)^{\delta_k} \left(e^{g_k(z)}\right)^{\gamma_k}.$$

After the calculus we obtain that

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \left[\delta_i \left(\frac{zf_i'(z)}{f_i(z)} - 1\right) + \gamma_i z g_i'(z)\right]. \tag{6}$$

Thus we have

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^{n} \left[|\delta_i| \left(\left|\frac{zf_i'(z)}{f_i(z)}\right| + 1\right) + |\gamma_i| \, |zg_i'(z)|\right]$$

so

$$\frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \left|\frac{zh''(z)}{h'(z)}\right|$$

$$\begin{aligned}
&\leq \frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \sum_{i=1}^{n} \left[|\delta_i| \left(\left|\frac{zf_i'(z)}{f_i(z)}\right| + 1\right) + |\gamma_i| \, |zg_i'(z)|\right] \\
&\leq \frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \sum_{i=1}^{n} \Bigg[|\delta_i| \left(\left|f_i'(z) \left(\frac{z}{f_i(z)}\right)^{\mu_i}\right| \left|\frac{f_i(z)}{z}\right|^{\mu_i - 1} + 1\right) \\
&\qquad + |\gamma_i| \left|g_i'(z) \left(\frac{z}{g_i(z)}\right)^{\mu_i}\right| \left|\frac{g_i(z)}{z}\right|^{\mu_i} |z|\Bigg].
\end{aligned}$$

From the hypothesis of Theorem 2.1, we have

$$|f_i(z)| \leq M_i \qquad (z \in \mathcal{U}), \qquad |g_i(z)| \leq N_i \qquad (z \in \mathcal{U}),$$

for all $i \in \{1, 2, ..., n\}$, then by General Schwarz Lemma for the functions f_i, g_i, we obtain

$$|f_i(z)| \leq M_i |z| \qquad (z \in \mathcal{U}), \qquad |g_i(z)| \leq N_i |z| \qquad (z \in \mathcal{U}) \tag{7}$$

for all $i \in \{1, 2, ..., n\}$.
Therefore, by using the inequalities (2) and (7), we arrive at the following inequality

$$\begin{aligned}
&\frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \left| \frac{zh''(z)}{h'(z)} \right| \\
&\leq \frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \sum_{i=1}^{n} \left[|\delta_i| \left(\left(\left| f_i'(z) \left(\frac{z}{f_i(z)} \right)^{\mu_i} - 1 \right| + 1 \right) M_i^{\mu_i - 1} + 1 \right) \right] \\
&\qquad + \frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \sum_{i=1}^{n} \left[|\gamma_i| \left(\left| g_i'(z) \left(\frac{z}{g_i(z)} \right)^{\mu_i} - 1 \right| + 1 \right) N_i^{\mu_i} \right] \\
&\leq \frac{1}{\mathrm{Re}\beta} \sum_{i=1}^{n} [|\delta_i| \left((2 - \alpha_i) M_i^{\mu_i - 1} + 1 \right) + |\gamma_i| (2 - \alpha_i) N_i^{\mu_i}]
\end{aligned}$$

$z \in \mathcal{U}$, which in the lights of the hypothesis (4) of Theorem 2.1, we obtain

$$\frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \left| \frac{zh''(z)}{h'(z)} \right| \leq 1 \qquad (z \in \mathcal{U}).$$

Applying Theorem 1.1 for the functions $h(z)$ we obtain that the integral operator $I_\beta (f_1, ..., f_n; g_1, ..., g_n)(z)$ defined by (3) is in the class $\mathcal{S}$.

Corollary 2.2 *Let the functions $f_i, g_i \in \mathcal{A}$ for all $i \in \{1, 2, ..., n\}$, $\beta, \delta_i, \gamma_i$ be complex numbers,* $\mathrm{Re}\beta > 0$ *and*

$$\mathrm{Re}\beta \geq \sum_{\mathrm{i}=1}^{\mathrm{n}} [|\delta_{\mathrm{i}}| (3 - \alpha_{\mathrm{i}}) + |\gamma_{\mathrm{i}}| (2 - \alpha_{\mathrm{i}})]$$

for all $i \in \{1, 2, ..., n\}$.
If $f_i, g_i \in \mathcal{S}^* (\alpha_i)$, $0 \leq \alpha_i < 1$ satisfy the conditions

$$|f_i(z)| \leq 1 \qquad (z \in \mathcal{U}), \qquad |g_i(z)| \leq 1 \qquad (z \in \mathcal{U})$$

for all $i \in \{1, 2, ..., n\}$, then the integral operator $I_\beta (f_1, ..., f_n; g_1, ..., g_n)(z)$ defined by (3) is in the class $\mathcal{S}$.

Proof. In Theorem 2.1, we consider $\mu_i = 1$ for all $i \in \{1, 2, ..., n\}$, $M_1 = M_2 = ... = M_n = 1$ and $N_1 = N_2 = ... = N_n = 1$.

Corollary 2.3 *Let the functions $f, g \in \mathcal{A}$, β, δ, γ be complex numbers,* $\mathrm{Re}\beta > 0$ *and $M \geq 1, N \geq 1$ with*

$$\mathrm{Re}\beta \geq |\delta| \left((2-\alpha)\, \mathrm{M}^{\mu-1} + 1\right) + |\gamma| \,(2-\alpha)\, \mathrm{N}^{\mu}.$$

If $f, g \in \mathcal{B}(\mu, \alpha)$, $0 \leq \alpha < 1$, $\mu \geq 0$ and

$$|f(z)| \leq M \qquad (z \in \mathcal{U}), \qquad |g(z)| \leq N \qquad (z \in \mathcal{U})$$

then the integral operator

$$I_\beta\,(f; g)\,(z) = \left(\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t}\right)^\delta \left(e^{g(t)}\right)^\gamma dt\right)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Proof. In Theorem 2.1, we consider $n = 1$.

Making use of Theorem 1.2, we prove

Theorem 2.4 *Let the functions $f_i, g_i \in \mathcal{A}$ for all $i \in \{1, 2, ..., n\}$, $\beta, \delta_i, \gamma_i$ be complex numbers,* $\mathrm{Re}\beta > 0$ *and $M_i \geq 1, N_i \geq 1$ with*

$$\mathrm{Re}\beta \geq \sum_{\mathrm{i}=1}^{\mathrm{n}} [|\delta_\mathrm{i}| \left((2-\alpha_\mathrm{i})\, \mathrm{M_i}^{\mu_\mathrm{i}-1} + 1\right) + |\gamma_\mathrm{i}| \,(2-\alpha_\mathrm{i})\, \mathrm{N_i}^{\mu_\mathrm{i}}]$$

for all $i \in \{1, 2, ..., n\}$.
If $f_i, g_i \in \mathcal{B}\,(\mu_i, \alpha_i)\,,\ 0 \leq \alpha_i < 1, \mu_i \geq 0$ satisfy the conditions

$$|f_i(z)| \leq M_i \qquad (z \in \mathcal{U}), \qquad |g_i(z)| \leq N_i \qquad (z \in \mathcal{U})$$

and

$$|c| \leq 1 - \frac{1}{\mathrm{Re}\beta} \sum_{i=1}^{n} [|\delta_i| \left((2-\alpha_i)\, M_i^{\mu_i-1} + 1\right) + |\gamma_i| \,(2-\alpha_i)\, N_i^{\mu_i}] \tag{8}$$

for all $i \in \{1, 2, ..., n\}$, then the integral operator $I_\beta\,(f_1, ..., f_n; g_1, ..., g_n)\,(z)$ defined by (3) is in the class $\mathcal{S}$.

Proof. From the proof of Theorem 2.1, we have

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \left[\delta_i \left(\frac{zf_i'(z)}{f_i(z)} - 1\right) + \gamma_i z g_i'(z)\right].$$

Thus, we have

$$\left|c\,|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right|$$

$$=\left|c\,|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{1}{\beta}\sum_{i=1}^{n}\left[\delta_i\left(\frac{zf_i'(z)}{f_i(z)}-1\right)+\gamma_i z g_i'(z)\right]\right|$$

$$\leq |c|+\frac{1}{|\beta|}\sum_{i=1}^{n}\left[|\delta_i|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right)+|\gamma_i|\,|zg_i'(z)|\right]$$

$$\leq |c|+\frac{1}{|\beta|}\sum_{i=1}^{n}\left[|\delta_i|\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu_i}\right|\left|\frac{f_i(z)}{z}\right|^{\mu_i-1}+1\right)\right]$$

$$+\frac{1}{|\beta|}\sum_{i=1}^{n}\left[|\gamma_i|\left|g_i'(z)\left(\frac{z}{g_i(z)}\right)^{\mu_i}\right|\left|\frac{g_i(z)}{z}\right|^{\mu_i}|z|\right].$$

From the hypothesis of Theorem 2.4, we have

$$|f_i(z)|\leq M_i \quad (z\in\mathcal{U}) \quad ,|g_i(z)|\leq N_i \quad (z\in\mathcal{U})$$

for all $i\in\{1,2,...,n\}$, then by General Schwarz Lemma for the functions f_i, g_i we obtain

$$|f_i(z)|\leq M_i\,|z| \quad (z\in\mathcal{U}), \quad |g_i(z)|\leq N_i\,|z| \quad (z\in\mathcal{U}) \tag{9}$$

for $i\in\{1,2,...,n\}$.
Therefore, by using the inequalities (2) and (9), we arrive at the following inequality

$$\left|c\,|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right|$$

$$\leq |c|+\frac{1}{|\beta|}\sum_{i=1}^{n}\left[|\delta_i|\left(\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu_i}-1\right|+1\right)M_i^{\mu_i-1}+1\right)\right]$$

$$+\frac{1}{|\beta|}\sum_{i=1}^{n}\left[|\gamma_i|\left(\left|g_i'(z)\left(\frac{z}{g_i(z)}\right)^{\mu_i}-1\right|+1\right)N_i^{\mu_i}\,|z|\right]$$

$$\leq |c|+\frac{1}{\mathrm{Re}\beta}\sum_{i=1}^{n}[|\delta_i|\left((2-\alpha_i)\,M_i^{\mu_i-1}+1\right)+|\gamma_i|\,(2-\alpha_i)\,N_i^{\mu_i}]$$

$z\in\mathcal{U}$, which in the lights of the hypothesis (8) of Theorem 2.4, we obtain

$$\left|c\,|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right|\leq 1 \quad (z\in\mathcal{U}).$$

Applying Theorem 1.2 for the functions $h(z)$ we obtain that the integral operator $I_\beta\,(f_1,...,f_n;g_1,...,g_n)\,(z)$ defined by (3) is in the class $\mathcal{S}$.

Corollary 2.5 *Let the functions* $f_i, g_i \in \mathcal{A}$ *for all* $i \in \{1, 2, ..., n\}$, $\beta, \delta_i, \gamma_i$ *be complex numbers,* $\mathrm{Re}\beta > 0$ *and*

$$\mathrm{Re}\beta \geq \sum_{\mathrm{i}=1}^{\mathrm{n}} [|\delta_{\mathrm{i}}| (3 - \alpha_{\mathrm{i}}) + |\gamma_{\mathrm{i}}| (2 - \alpha_{\mathrm{i}})]$$

for all $i \in \{1, 2, ..., n\}$.
If $f_i, g_i \in \mathcal{S}^* (\alpha_i)$, $0 \leq \alpha_i < 1$ *satisfy the conditions*

$$|f_i(z)| \leq 1 \qquad (z \in \mathcal{U}), \qquad |g_i(z)| \leq 1 \qquad (z \in \mathcal{U})$$

and

$$|c| \leq 1 - \frac{1}{\mathrm{Re}\beta} \sum_{i=1}^{n} [|\delta_i| (3 - \alpha_i) + |\gamma_i| (2 - \alpha_i)]$$

for all $i \in \{1, 2, ..., n\}$*, then the integral operator* $I_\beta (f_1, ..., f_n; g_1, ..., g_n) (z)$ *defined by (3) is in the class* $\mathcal{S}$.

Proof. In Theorem 2.4, we consider $\mu_i = 1$ for all $i \in \{1, 2, ..., n\}$, $M_1 = M_2 = ... = M_n = 1$ and $N_1 = N_2 = ... = N_n = 1$.

Corollary 2.6 *Let the functions* $f, g \in \mathcal{A}$, β, δ, γ *be complex numbers,* $\mathrm{Re}\beta > 0$ *and* $M \geq 1, N \geq 1$ *with*

$$\mathrm{Re}\beta \geq [|\delta| \left((2 - \alpha) \mathrm{M}^{\mu - 1} + 1\right) + |\gamma| (2 - \alpha) \mathrm{N}^{\mu}].$$

If $f, g \in \mathcal{B}(\mu, \alpha)$, $0 \leq \alpha < 1, \mu \geq 0$ *satisfy the conditions*

$$|f(z)| \leq M \qquad (z \in \mathcal{U}), \qquad |g(z)| \leq N \qquad (z \in \mathcal{U})$$

and

$$|c| \leq 1 - \frac{1}{\mathrm{Re}\beta} [|\delta| \left((2 - \alpha) M^{\mu - 1} + 1\right) + |\gamma| (2 - \alpha) N^{\mu}]$$

then the integral operator

$$I_\beta (f; g) (z) = \left(\beta \int_0^z t^{\beta - 1} \left(\frac{f(t)}{t}\right)^\delta \left(e^{g(t)}\right)^\gamma\right)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Proof. In Theorem 2.4, we consider $n = 1$.

3 Open Problem

New results can be obtained by using the integral operator defined in Definition 1.4 for other classes of analytic functions.

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