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Some starlike and convexity properties associated with p-valent hypergeometric functions

S. P. Goyal, S. K. Bansal, and Pranay Goswami

Department of Mathematics, University of Rajasthan, Jaipur, India
e-mail: somprg@gmail.com

Department of Mathematics, Bansal School of Engg. and Tech. Jaipur, India
e-mail: bansalindian@gmail.com

Department of Mathematics, Amity University, Rajasthan, Jaipur, India
e-mail: pranaygoswami83@gmail.com

Abstract

In the present paper we obtain some conditions on a, b and c to verify that $z^p\ {}_2F_1(a,b;c;z)$ to be in various subclasses of starlike and convex functions. we also examine an integral operator related to the p-valent hypergeometric function.

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1 Introduction

Let T_p be the class of functions of the form:

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n, \quad (a_n \geq 0) \quad (p \in N = 1, 2, ...) \tag{1.1}$$

which are analytic and p-valent in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let $S_p^*(A, B, \beta)$ be the subclass of T_p consisting of functions which satisfy the condition:

$$\left|\frac{\frac{zf'(z)}{pf(z)}-1}{A-\frac{B}{p}\frac{zf'(z)}{f(z)}}\right|<\beta,(z\in\mathbb{U},-1\leq B<A\leq 1,0<\beta\leq 1\ and\ p\in N) \tag{1.2}$$

and $C_p^*(A,B,\beta)$ be the subclass of T_p consisting of functions which satisfy the condition:

$$\left|\frac{\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)-1}{A-\frac{B}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)}\right|<\beta,(z\in\mathbb{U},-1\leq B<A\leq 1,0<\beta\leq 1\ and\ p\in N) \tag{1.3}$$

From (1.2) and (1.3), we have

$$f(z)\in C_p^*(A,B,\beta)\Leftrightarrow\frac{zf'(z)}{p}\in S_p^*(A,B,\beta). \tag{1.4}$$

We also note that for $0\leq\alpha<1$,
(i) $S_1^*(1-2\alpha,-1,\beta)=S^*(\alpha,\beta)$ and
(ii) $C_1^*(1-2\alpha,-1,\beta)=C^*(\alpha,\beta)$,
(iii) $S^*(\alpha,1)=S^*(\alpha)$ and (iv) $C^*(\alpha,1)=C(\alpha)$
The subclasses of class of $S^*(\alpha,\beta)$ and $C^*(\alpha,\beta)$ were introduced and studied by Gupta and Jain([4], see also [6]) while the subclasses $S^*(\alpha)$ and $C(\alpha)$ were studied by Silverman [10].
Let ${}_2F_1(a,b;c;z)$ be the(Gaussian)hypergeometric function defined by

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n, \tag{1.5}$$

where $c\neq 0,-1,-2,...,$ and $(\alpha)_n$ is Pochhammer symbol defned by

$$(\alpha)_n=\begin{cases}1 & for\ \ n=0\\ \alpha(\alpha+1)...(\alpha+n-1) & for\ n\in\mathbb{N}\end{cases}$$

We note that ${}_2F_1(a,b;c;1)$ converges for $Re(c-a-b)>0$ and is related to the Gamma functions (see [7], p. 49) by

$${}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{1.6}$$

Silverman [9] gave necessary and sufficient conditions for $z_2F_1(a,b;c;z)$ to be in $S^*(\alpha)$ and $C(\alpha)$ and also examined a linear operator acting on hypergeometric functions. For the other interesting developments for univalent and multivalent hypergeometric function the reader can refer the works of Carlson and Shaffer [1], Merkes and Scott [5], Ruscheweyh and Singh [8], Cho et al. [2], Mostafa [6] and El-Ashwah et al. [3]

2 Main Results

In order to establish our main results we need following lemmas:
Lemma 1.*(i) A function $f(z)$ defined by (1.1) is in the class of $S_p^*(A,B,\beta)$ if and only if*

$$\sum_{n=p+1}^{\infty}[(n-p)+\beta(pA-Bn)]a_n \leq p\beta(A-B). \tag{2.1}$$

(ii) A function $f(z)$ defined by (1.1) is in the class of $C_p^*(A,B,\beta)$ if and only if

$$\sum_{n=p+1}^{\infty}n[(n-p)+\beta(pA-Bn)]a_n \leq p^2\beta(A-B). \tag{2.2}$$

Proof (i) We assume that the inequality (2.1) is true, Now

$$\left|\frac{\frac{zf'(z)}{pf(z)}-1}{A-\frac{B}{p}\frac{zf'(z)}{f(z)}}\right| \leq \left|\frac{\sum\limits_{n=p+1}^{\infty}(n-p)a_n z^{n-p}}{(A-B)p-\sum\limits_{n=p+1}^{\infty}(Ap-Bn)a_n z^{n-p}}\right| \tag{2.3}$$

$$\leq \frac{\sum\limits_{n=p+1}^{\infty}(n-p)a_n|z|^{n-p}}{(A-B)p-\sum\limits_{n=p+1}^{\infty}(Ap-Bn)a_n|z|^{n-p}}, \tag{2.4}$$

This shows that $\frac{\frac{zf'(z)}{pf(z)}-1}{A-\frac{B}{p}\frac{zf'(z)}{f(z)}}$ lie in the circle centered at origin whose radius is β. Hence $f(z) \in S_p^*(A,B,\beta)$.

Conversly, let $f(z) \in S_p^*(A,B,\beta)$, then

$$\left|\frac{\frac{zf'(z)}{pf(z)}-1}{A-\frac{B}{p}\frac{zf'(z)}{f(z)}}\right| < \beta,$$

$$\left|\frac{zf'(z)-pf(z)}{\beta(Apf(z)-Bzf'(z))}\right| < 1, \tag{2.5}$$

We note that

$$\left|\frac{zf'(z)-pf(z)}{\beta(Apf(z)-Bzf'(z))}\right|$$

$$= \left|\frac{\sum\limits_{n=p+1}^{\infty}(n-p)a_n z^n}{\beta\left(p(A-B)z^p-\sum\limits_{n=p+1}^{\infty}(pA-nB)a_n z^n\right)}\right|$$

$$\leq \frac{\sum_{n=p+1}^{\infty}(n-p)a_n|z|^{n-p}}{\beta\left(p(A-B)-\sum_{n=p+1}^{\infty}(pA-nB)a_n|z|^{n-p}\right)} \tag{2.6}$$

Letting $z \longrightarrow 1^-$ through real values, we have

$$\leq \frac{\sum_{n=p+1}^{\infty}(n-p)a_n}{\beta\left(p(A-B)-\sum_{n=p+1}^{\infty}(pA-nB)a_n\right)} \tag{2.7}$$

The extreme-right-side expression of the above inequality would remain bounded by 1 if

$$\sum_{n=p+1}^{\infty}(n-p)a_n \leq \beta\left(p(A-B)-\sum_{n=p+1}^{\infty}(pA-nB)a_n\right) \tag{2.8}$$

which leads to the desired inequality (2.1). This completes the proof.

Finally, the following function:

$$f(z) = z^p - \frac{\beta(A-B)p}{(n-p)+\beta(Ap-Bn)}z^{n+p}, \tag{2.9}$$

is an extremal function for the Lemma.
(ii) On replacing a_n by $\frac{na_n}{p}$ and using (1.4), we immediately get the part (ii) of the Lemma.

Theorem 1. *Let* $z \in \mathbb{U}, -1 \leq B < A \leq 1, 0 \leq \beta < 1$ *and* $p \in \mathbb{N}$
(i) If $a, b > -1, c > 0$ *and* $ab < 0$*, then* $z^p\ {}_2F_1(a,b;c;z)$ *is in* $S_p^*(A,B,\beta)$ *if and only if*

$$c \geq a+b+1-\frac{(1-\beta B)ab}{p\beta(A-B)} \tag{2.10}$$

(ii) If $a, b > 0$ *and* $c > a+b+1$*, then* $h_p(a,b;c;z) = z^p\left(2 - {}_2F_1(a,b;c;z)\right)$ *is in* $S_p^*(A,B,\beta)$*, if and only if*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)(c-b)}\left[1+\frac{ab(1-\beta B)}{p\beta(A-B)(c-a-b-1)}\right] \leq 2 \tag{2.11}$$

Proof. (i) Since

$$z^p\ {}_2F_1(a,b;c;z) = z^p + \frac{ab}{c}\sum_{n=p+1}^{\infty}\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}}z^n,$$

$$= z^p - \left|\frac{ab}{c}\right|\sum_{n=p+1}^{\infty}\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}}z^n, \tag{2.12}$$

According to the Lemma 1(i), we must show that

$$\sum_{n=p+1}^{\infty}[(n-p)+\beta(pA-Bn)]\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}}\leq\beta p(A-B)\left|\frac{c}{ab}\right|. \tag{2.13}$$

Note that the left side of (2.13) diverges if $c\leq a+b+1$. Now

$$\sum_{n=p+1}^{\infty}[(n-p)+\beta(pA-Bn)]\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}}$$

$$=(1-\beta B)\sum_{n=p+1}^{\infty}n\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}}$$

$$+p(A\beta-1)\sum_{n=p+1}^{\infty}\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}}.$$

$$=(1-\beta B)\sum_{n=0}^{\infty}(n+p+1)\frac{(a+1)_{n}(b+1)_{n}}{(c+1)_{n}(1)_{n+1}}+p(A\beta-1)\sum_{n=0}^{\infty}\frac{(a+1)_{n}(b+1)_{n}}{(c+1)_{n}(1)_{n+1}}.$$

$$=(1-\beta B)\sum_{n=0}^{\infty}\frac{(a+1)_{n}(b+1)_{n}}{(c+1)_{n}(1)_{n}}+p\beta(A-B)\sum_{n=0}^{\infty}\frac{(a+1)_{n}(b+1)_{n}}{(c+1)_{n}(1)_{n+1}}.$$

$$=(1-\beta B)\sum_{n=0}^{\infty}\frac{(a+1)_{n}(b+1)_{n}}{(c+1)_{n}(1)_{n}}+\frac{p\beta(A-B)c}{ab}\sum_{n=1}^{\infty}\frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}}.$$

$$=(1-\beta B)\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)(c-b)}+\frac{p\beta(A-B)c}{ab}\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)(c-b)}-1\right]$$

Hence, (2.13) is equivalent to

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)(c-b)}\left[(1-\beta B)+\frac{p\beta(A-B)(c-a-b-1)}{ab}\right]$$

$$\leq p\beta(A-B)\left[\left|\frac{c}{ab}\right|+\frac{c}{ab}\right]=0 \tag{2.14}$$

Thus (2.14) is valid if and only if $(1-\beta B)+\frac{p\beta(A-B)(c-a-b-1)}{ab}\leq 0$, or equivalently,
$c\geq a+b+1-\frac{(1-\beta B)ab}{p\beta(A-B)}$.
(ii) Since

$$h_p(a,b;c;z)=z^p-\sum_{n=p+1}^{\infty}\frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}}z^n, \tag{2.15}$$

then according to Lemma 1 (i), we only need to show that

$$\sum_{n=p+1}^{\infty}[n(1-\beta B)+p(\beta A-1)]\frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}}\leq p\beta(A-B). \tag{2.16}$$

Now L.H.S. of (2.16)

$$=(1-\beta B)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n-1}}+p\beta(A-B)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}. \tag{2.17}$$

Using $(\alpha)_n=\alpha(\alpha+1)_{n-1}$ in (2.17), we have

$$L.H.S.of(2.15)=\frac{ab}{c}(1-\beta B)\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}}+p\beta(A-B)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}.$$

$$=\frac{ab}{c}(1-\beta B)\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}$$

$$+p\beta(A-B)\left[\frac{\Gamma(c)\Gamma(c-a-b))}{\Gamma(c-a)\Gamma(c-b)}-1\right]. \quad (by(1.6))$$

$$=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[p\beta(A-B)+\frac{ab(1-\beta B)}{c-a-b-1}\right]-p\beta(A-B).$$

But this last expression is bounded above by $p\beta(A-B)$ if and only if (2.11)holds.

Theorem 2. *Let $z\in\mathbb{U}, -1\leq B<A\leq 1, 0\leq\beta<1$ and $p\in\mathbb{N}$*

(i) If $a,b>-1, ab<0$ and $c>a+b+2$, then a necessary and sufficient condition for $z^p\ {}_2F_1(a,b;c;z)$ to be in $C_p^*(A,B,\beta)$ is that

$$(1-\beta b)ab(a+1)(b+1)+\{p(1+\beta(A-2B))+1-\beta B\}\,ab(c-a-b-2)$$

$$+p^2\beta(A-B)(c-a-b-2)(c-a-b-1)\geq 0. \tag{2.18}$$

(ii)If $a,b>0$ and $c>a+b+2$, then a necessary and sufficient condtion for $h_p(a,b;c;z)$ to be in $C_p^*(A,B,\beta)$ is that

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{\{p(1+\beta(A-2B))+1-\beta B\}\,ab}{(c-a-b-1)\beta p^2(A-B)}\right.$$

$$\left.+\frac{(1-\beta B)ab(a+1)(b+1)}{(c-a-b-1)(c-a-b-2)\beta p^2(A-B)}\right]\leq 2 \tag{2.19}$$

Proof. (i) Since $z^p\ {}_2F_1(a,b;c;z)$ has the form (2.12), using (ii) of Lemma 1, our conclusion can be written as

$$\sum_{n=p+1}^{\infty}n\left[(n-p)+\beta(pA-Bn)\right]\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}}\leq\left|\frac{c}{ab}\right|p^2\beta(A-B). \tag{2.20}$$

Noting that for $c > a + b + 2$, the left hand side of (2.20) converges. Now

$$L.H.S.of(2.20) = \sum_{n=0}^{\infty}(n+p+1)\left[(n+p+1)(1-\beta B) + p(A\beta - 1)\right] \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$= (1-\beta B)\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + p\left\{1+\beta(A-2B)\right\}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}$$

$$+\beta(A-B)p^2\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$= (1-\beta B)\frac{(a+1)(b+1)}{(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n} + \left[p\left\{1+\beta\left(A-2B\right)\right\}+1-\beta B\right]$$

$$\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + \frac{c}{ab}\beta(A-B)p^2\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$= \frac{\Gamma(c+1)\,\Gamma(c-a-b-2)}{\Gamma(c-a)\,\Gamma(c-b)}\Big[(a+1)(b+1)(1-\beta B) + \left[p\left\{1+\beta\left(A-2B\right)\right\}\right.$$

$$\left. +1-\beta B\right] + (c-a-b-2) + p^2\frac{\beta(A-B)}{ab}(c-a-b-2)(c-a-b-1)\Big]$$

$$-p^2\frac{\beta(A-B)c}{ab}$$

This last expression is bounded above by $\left|\frac{c}{ab}\right| p^2\beta(A-B)$ if and only if

$$(a+1)(b+1)(1-\beta B) + \left[p\left\{1+\beta\left(A-2B\right)\right\}+1-\beta B\right](c-a-b-2)$$

$$+p^2\frac{\beta(A-B)}{ab}(c-a-b-2)(c-a-b-1) \leq 0,$$

which is equivalent to (2.18).

(ii) In view of Lemma 1 (ii) and (2.15) we only need to show that

$$\sum_{n=p+1}^{\infty} n\left[n(1-\beta B)+p(\beta A-1)\right]\frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} \leq p^2\beta(A-B). \qquad (2.21)$$

Now

$$L.H.S.of(2.21) = \sum_{n=0}^{\infty}(n+p+1)\left[(n+p+1)(1-\beta B)+p(\beta A-1)\right]\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}.$$

$$= (1-\beta B)\sum_{n=0}^{\infty}(n+1)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}+p\left\{1+\beta(A-2B)\right\}\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$
$$+p^2\beta(A-B)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$
$$= (1-\beta B)\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+p\left\{1+\beta(A-2B)\right\}\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}$$
$$+p^2\beta(A-B)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$
$$= (1-\beta B)\sum_{n=0}^{\infty}\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_{n}}+\left[p\left\{1+\beta(A-2B)\right\}+1-\beta B\right]\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}$$
$$+p^2\beta(A-B)\sum_{n=1}^{\infty}\frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}} \tag{2.22}$$

Since $(a)_{n+k}=(a)_k(a+k)_n$, and using (1.6), we can write the equation (2.22) as
L.H.S. of (2.21)

$$= (1-\beta B)ab(a+1)(b+1)\frac{\Gamma(c)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}+\left[p(1+\beta(A-2B))+1-\beta B\right]$$
$$ab\frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}+p^2\beta(A-B)\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right]$$

Upon simplification, we find that the last expression is bounded above by $p^2\beta(A-B)$ if and only if (2.19) holds

Remark 2.1 *Putting $p=1, A=1-2\alpha, B=-1$ in Theorems 1 and 2, we get the results given recently by Mostafa [6], which contains the results due to Silverman [9].*

3 Integral Operator

Let

$$H_p(a,b;c;z)=z^{p-1}\int_0^z {}_2F_1(a,b;c;t)dt \tag{3.1}$$

be an integral opeartor acting on ${}_2F_1(a,b;c;z)$. If we evaluate the integral in (3.1), we find that

$$H_p(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}z^{n+p}$$

$$= z^p + \frac{ab}{c} \sum_{n=p+1}^{\infty} \frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p+1}} \tag{3.2}$$

Now let $a, b > -1$, s.t. $ab < 0$ and $c > 0$, then (3.2) reduces to

$$H_p(a, b; c; z) == z^p - \left|\frac{ab}{c}\right| \sum_{n=p+1}^{\infty} \frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p+1}} \tag{3.3}$$

Theorem 3. *Let $a, b > -1, ab < 0, c > max\{0, a+b\}, z \in \mathbb{U}, -1 \le B < A \le 1, 0 \le \beta < 1$. Then $H_p(a, b; c; z)$ defined by (3.1) is in $S_p^*(A, B, \beta)$ if and only if*

$$\frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[\frac{(1-\beta B)}{ab} + \frac{[p\beta(A-B)+B\beta-1](c-a-b)}{ab(a-1)(b-1)}\right]$$

$$-\frac{[p\beta(A-B)+B\beta-1]\, c(c-1)}{ab(a-1)(b-1)} \le 0 \tag{3.4}$$

Proof. In view of (3.3) and Lemma 1(i), we need only to show that

$$\sum_{n=p+1}^{\infty} [(n-p) + \beta(pA - Bn)] \frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p+1}}$$

$$\le \beta p(A-B) \left|\frac{c}{ab}\right| \tag{3.5}$$

Now

$$L.H.S.of(3.5) = (1-\beta B) \sum_{n=p+1}^{\infty} n \frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p+1}} + p(A\beta - 1)$$

$$\sum_{n=p+1}^{\infty} \frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p+1}}$$

$$= (1-\beta B) \sum_{n=0}^{\infty} (n+p+1) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}} + p(A\beta - 1) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}}$$

$$= (1-\beta B) \sum_{n=0}^{\infty} (n+2) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}} + [p\beta(A-B) + B\beta - 1]$$

$$\sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}}$$

$$= (1-\beta B) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} + [p\beta(A-B) + B\beta - 1]$$

$$\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n+1}}$$

$$=(1-\beta B)\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n}}+[p\beta(A-B)+B\beta-1]\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}$$

$$=(1-\beta B)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}+[p\beta(A-B)+B\beta-1]\frac{c}{ab}\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n}$$

$$=(1-\beta B)\frac{c}{ab}\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right]+[p\beta(A-B)+B\beta-1]\frac{c(c-1)}{ab(a-1)(b-1)}$$

$$\sum_{n=2}^{\infty}\frac{(a-1)_n(b-1)_n}{(c-1)_n(1)_n}$$

$$=\frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{(1-\beta B)}{ab}+\frac{[p\beta(A-B)+B\beta-1](c-a-b)}{ab(a-1)(b-1)}\right]$$

$$+\frac{[p\beta(A-B)+B\beta-1]c(c-1)}{ab(a-1)(b-1)}-p\beta(A-B)\frac{c}{ab}$$

which is bounded above by $p\beta(A-B)\left|\frac{c}{ab}\right|$ if and only if (3.4) holds.

Theorem 4. *Let $a,b>-1, ab<0, c>a+b+2, z\in\mathbb{U}, -1\le B<A\le 1, 0\le\beta<1$. Then $H_p(a,b;c;z)$ defined by (3.1) is in $C_p^*(A,B,\beta)$ if and only if*

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{[p-1-B\beta(2p-1)+pA\beta](c-a-b-1)}{ab}\right.$$

$$\left.+1-\beta B+\frac{(p-1)[-B\beta(p-1)+pA\beta](c-a-b)(c-a-b-1)}{ab(a+1)(b+1)}\right]$$

$$-\left[(p-1)-B\beta(2p-1)-p^2\beta(A-B)+pA\beta\right]\frac{c}{ab}+(p-1)[B\beta-1$$

$$+p\beta(A-B)]-\frac{(p-1)(a-1)(b-1)[-\beta(p-1)+pA\beta]c(c+1)}{(c-1)ab(a+1)(b+1)}\le 0 \quad (4.1)$$

Proof. The proof of Theorem 4 can be developed on the lines similar to Theorem 3 and using Lemma 1(ii).

If we take $p=1$ in Theorem 3 and 4 and we arrive at the following results contained in

Corollary 1. *Let $a,b>-1, ab<0, c>max\{0,a+b\}, z\in\mathbb{U}, -1\le B<A\le 1, 0\le\beta<1$. Then*

$$H_1(a,b;c;z)=H(a,b;c;z)=\int_0^z {}_2F_1(a,b;c;t)dt \quad (4.2)$$

is in $S^*(A, B, \beta)$ *if and only if*

$$\frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{1-\beta B}{ab}-\frac{(1-AB)(c-a-b)}{ab(a-1)(b-1)}\right]-\frac{(1-A\beta)c(c-1)}{ab(a-1)(b-1)}\leq 0 \tag{4.3}$$

and

Corollary 2. *Let* $a, b > -1, ab < 0$ *and* $c > a+b+2, z \in \mathbb{U}, -1 \leq B < A \leq 1, 0 \leq \beta < 1$. *Then* $H(a, b; c; z)$ *defined by (4.2) is in* $C^*(A, B, \beta)$ *if and only if*

$$c > a+b+1-\frac{B\beta-1}{\beta(A-B)}ab \tag{4.4}$$

For $A = 1-2\alpha$ and $B = -1$ in Corollary 1 and 2, we easily arrive at the recent results due to Mostafa [6], which evidently contains the results due to Silverman [9] for $\beta = 1$.

4 Open Problem

In section 3, we introduced a new integral operator

$$H_p(a, b; c; z) = z^{p-1}\int_0^z {}_2F_1(a, b; c; t)dt$$

and obtained starlike and convex conditions for this operator. If we define a new integral operator

$$I_p(a, b; c; z) = \int_0^z \frac{{}_2F_1(a, b; c; t)}{t^p}dt.$$

then, what will be the conditions of starlikeness and convexity for this integral operator ?

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