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Weighted Value Sharing Of Certain Non-Linear Differential Polynomials

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Abstract

We use the notion of weighted sharing of values to study the uniqueness of meromorphic functions when certain non-linear differential polynomials share the same 1***-points. Our results improve and supplement and at the same time generalised the results of Lahiri-Sarkar [14], Meng [18] and Zhang-Lin [23]. At the last section we pose some open problems which are still unsolved in connection to the context of the paper. .***

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1 Introduction Definitions and Background

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$

and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

We adopt the standard notations of value distribution theory (see [7]). We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure.

In 1999 at the time of studying the problem of uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points Lahiri [8] raised the following question regarding the nonlinear differential polynomials.

What can be said if two nonlinear differential polynomials generated by two meromorphic functions share 1 *CM?*

During the last couple of years a large number of research papers investigating the shared value problems of different nonlinear differential polynomials and the uniqueness of their corresponding generating meromorphic functions were published {see [2]-[6], [12]-[19]}.

In 2001 Fang and Hong [6] proved the following result.

Theorem A. *Let f and g be two transcendental entire functions and $n(\geq 11)$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share* 1 *CM, then $f \equiv g$.*

So far to the knowledge of the authors the above is the first result related to the value sharing of nonlinear differential polynomials. Naturally it generates an increasing interest among the researchers to explore the value sharing of more generalised polynomials under weaker hypothesis.

Improving *Theorem A* Fang and Fang [5] obtained the following theorem.

Theorem B. *Let f and g be two non-constant entire functions and $n(\geq 8)$ be an integer . If $f^n(f-1)f'$ and $g^n(g-1)g'$ share* 1 *CM, then $f \equiv g$.*

In 2004 Lin and Yi [17] further improved and supplement *Theorem B* as follows.

Theorem C. *Let f and g be two transcendental entire functions and $n(\geq 7)$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share* 1 *CM, then $f \equiv g$.*

Theorem D. *Let f and g be two non-constant meromorphic functions and $n(\geq 13)$ be an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share* 1 *CM, then $f \equiv g$.*

In 2001 an idea of gradation of sharing of values was introduced in {[6], [7]} which measures how close a shared value is to being share CM or to being shared IM. This notion is known as weighted sharing and is defined as follows.

Definition 1.1 *[6, 7] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g

with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k), then f, g share (a,p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

In the mean time to investigate the uniqueness of meromorphic functions, Lahiri and Sarkar [14] considered two different types of nonlinear differential polynomials than those discussed earlier and proved the following.

Theorem E. *Let f and g be two non-constant meromorphic functions such that $f^n(f^2-1)f^{'}$ and $g^n(g^2-1)g^{'}$ share $(1,2)$, where $n(\geq 13)$ is an integer then either $f \equiv g$ or $f \equiv -g$. If n is an even integer then the possibility of $f \equiv -g$ does not arise.*

In 2009 C. Meng [18], also considered the value sharing of a nonlinear differential polynomial whose form is analogous to those considered by Lahiri-Sarkar. C. Meng obtained the following results.

Theorem F. *Let f and g be two non-constant meromorphic functions such that $f^n(f^3-1)f^{'}$ and $g^n(g^3-1)g^{'}$ share $(1,l)$, where n be a positive integer such that $n+1$ is not divisible by 3. If*

(i) $l = 2$ *and* $n > 14$*;*

(ii) $l = 1$ *and* $n \geq 17$*;*

(iii) $l = 0$ *and* $n \geq 35$*.*

then $f \equiv g$.

Recently Zhang and Lin [23] considered the sharing value problem of more generalised differential polynomials namely the kth derivative of a linear expression but confined their investigation for entire functions only. Zhang and Lin [23] obtained the following result.

Theorem G. *Let f and g be two non-constant entire functions and n, m and k be three positive integers with $n > 2k+m+4$. Suppose for two non zero constants a and b $[f^n(af^m+b)]^{(k)}$ and $[g^n(ag^m+b)]^{(k)}$ share $(1,\infty)$. Then $f \equiv g$.*

Remark 1.2 *The conclusion of the Theorem G is partially correct. Since in the proof of the theorem the possibilities other than $f \equiv g$ has not been considered.*

From the context of the above discussions the following questions are inevitable.

Question 1. When $n+1$ is divisible by 3 in *Theorem F* what can be the possible relationships between f and g ?

Question 2. In *Theorems E-F* if the sharing value problems of differential polynomials are replaced by more general one as considered in *Theorem G* then can the same theorems be obtained as a corollary of the main results so that *Theorem G* will also be rectified ?

2 Main Results

In the paper we are taking this aspect as background and improve extend and generalize all results stated above.

Following theorem is the main result of the paper.

Theorem 2.1 *Let f and g be two transcendental meromorphic functions and n, $k(\geq 1)$, $m(\geq 2)$ be three positive integers. Suppose for two non zero constants a and b $[f^n(af^m+b)]^{(k)}$ and $[g^n(ag^m+b)]^{(k)}$ share $(1,l)$. Then $f \equiv tg$ for some constant t, satisfying $t^d \equiv 1$, where $d=(n+m,n)$ or $[f^n(af^m+b)]^{(k)}[g^n(ag^m+b)]^{(k)} \equiv 1$ provided one of the following holds.*

(i) $l \geq 2$ *and* $n > 3k+m+8-2\{\Theta(\infty;f)+\Theta(\infty;g)\}$ $-k\min\{\Theta(\infty;f),\Theta(\infty;g)\}$;

(ii) $l=1$ *and* $n > 4k+\frac{3m}{2}+9-\left(\frac{k}{2}+\frac{5}{2}\right)\{\Theta(\infty;f)+\Theta(\infty;g)\}$;

(iii) $l=0$ *and* $n > 9k+4m+14-(2k+3)\{\Theta(\infty;f)+\Theta(\infty;g)\}$ $-\min\{\Theta(\infty;f),\Theta(\infty;g)\}$.

When $k=1$ the possibility $[f^n(af^m+b)]^{(k)}[g^n(ag^m+b)]^{(k)} \equiv 1$ does not occur.

Putting $n=s+1$, $m=2$ $a=\frac{1}{s+3}$ $b=-\frac{1}{s+1}$ and $k=1$ in the above theorem and noting that here $d=(s+3,s+1)$ we can immediately deduce the following corollary.

Corollary 2.2 *Let f and g be two non-constant meromorphic functions and s be a positive integer. Suppose $f^s(f^2-1)f'$ and $g^s(g^2-1)g'$ share $(1,l)$. Then $f \equiv g$ or $f \equiv -g$ provided one of the following holds.*

(i) $l \geq 2$ *and* $s > 12-2\{\Theta(\infty;f)+\Theta(\infty;g)\}-\min\{\Theta(\infty;f),\Theta(\infty;g)\}$;

(ii) $l=1$ *and* $s > 15-3\{\Theta(\infty;f)+\Theta(\infty;g)\}$;

(iii) $l=0$ *and* $s > 30-5\{\Theta(\infty;f)+\Theta(\infty;g)\}-\min\{\Theta(\infty;f),\Theta(\infty;g)\}$.

If s is an even integer then the possibility of $f \equiv -g$ does not arise.

Putting $n=s+1$, $m=3$ $a=\frac{1}{s+3}$ $b=-\frac{1}{s+1}$ and $k=1$ in the above theorem and noting that here $d=(s+4,s+1)$ we can immediately deduce the following corollary.

Corollary 2.3 *Let f and g be two non-constant meromorphic functions and s be a positive integer such that $s+1$ is not divisible by 3. Suppose $f^s(f^3-1)f'$ and $g^s(g^3-1)g'$ share $(1,l)$. Then $f \equiv g$ provided one of the following holds.*

(i) $l \geq 2$ *and* $s > 13 - 2\{\Theta(\infty; f) + \Theta(\infty; g)\} - \min\{\Theta(\infty; f), \Theta(\infty; g)\}$;

(ii) $l = 1$ *and* $s > 16.5 - 3\{\Theta(\infty; f) + \Theta(\infty; g)\}$;

(iii) $l = 0$ *and* $s > 34 - 5\{\Theta(\infty; f) + \Theta(\infty; g)\} - \min\{\Theta(\infty; f), \Theta(\infty; g)\}$.

Remark 2.4 *Since* Theorems E-F *can be obtained as a special case of* Theorem 2.1*, clearly* Theorem 2.1 *improves and supplements* Theorems E-F.

Theorem 2.5 *Let f and g be two non-constant entire functions and n, $k(\geq 1)$, $m(\geq 2)$ be three positive integers. Suppose for two non zero constants a and b $[f^n(af^m+b)]^{(k)}$ and $[g^n(ag^m+b)]^{(k)}$ share $(1,l)$. Then $f \equiv tg$ for some constant t, satisfying $t^d \equiv 1$, where $d = (n+m, n)$ provided one of the following holds.*

(i) $l \geq 2$ *and* $n > 2k + m + 4$;

(ii) $l = 1$ *and* $n > 3k + \frac{3m}{2} + 4$;

(iii) $l = 0$ *and* $n > 5k + 4m + 7$.

Also the possibility $f \equiv -g$ does not arise if n and m are both odd or if n is odd and m is even or if n is even and m is odd.

3 More Definitions

We now explain some definitions and notations which are used in the paper.

Definition 3.1 *[14] Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.*

(i) $N(r, a; f \mid\geq p)$ $(\overline{N}(r, a; f \mid\geq p))$ *denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.*

(ii) $N(r, a; f \mid\leq p)$ $(\overline{N}(r, a; f \mid\leq p))$ *denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.*

Definition 3.2 *{11, cf.[21]} For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \ldots \overline{N}(r, a; f \mid\geq p)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.*

Definition 3.3 *Let $a, b \in \mathbb{C} \cup \{\infty\}$. Let p be a positive integer. We denote by $\overline{N}(r, a; f \mid \geq p \mid g = b)$ ($\overline{N}(r, a; f \mid \geq p \mid g \neq b)$) the reduced counting function of those a-points of f with multiplicities $\geq p$, which are the b-points (not the b-points) of g.*

Definition 3.4 *$\{$cf.[1], 2$\}$ Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r, 1; f)$ the counting function of those 1-points of f and g where $p > q$, by $N_E^{1)}(r, 1; f)$ the counting function of those 1-points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2}(r, 1; f)$ the counting function of those 1-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r, 1; g)$, $N_E^{1)}(r, 1; g)$, $\overline{N}_E^{(2}(r, 1; g)$.*

Definition 3.5 *$\{$cf.[1], 2$\}$ Let k be a positive integer. Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_{f>k}(r, 1; g)$ the reduced counting function of those 1-points of f and g such that $p > q = k$. $\overline{N}_{g>k}(r, 1; f)$ is defined analogously.*

Definition 3.6 *[9, 10] Let f, g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*

Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

4 Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function.

$$H = \left(\frac{F^{(k+2)}}{F^{(k+1)}} - \frac{2F^{(k+1)}}{F^{(k)} - 1}\right) - \left(\frac{G^{(k+2)}}{G^{(k+1)}} - \frac{2G^{(k+1)}}{G^{(k)} - 1}\right). \tag{1}$$

Lemma 4.1 *[7]Let f be a non-constant meromorphic function, k a positive integer and let c be a non-zero finite complex number. Then*

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, \infty; f) + N(r, 0; f) + N\left(r, c; f^{(k)}\right) - N\left(r, 0; f^{(k+1)}\right) + S(r, f) \\ &\leq \overline{N}(r, \infty; f) + N_{k+1}(r, 0; f) + \overline{N}\left(r, c; f^{(k)}\right) - N_0\left(r, 0; f^{(k+1)}\right) + S(r, f), \end{aligned}$$

where $N_0\left(r, 0; f^{(k+1)}\right)$ is the counting function of the zeros of $f^{(k+1)}$ which are not the zeros of $f(f^{(k)} - c)$

Lemma 4.2 *[22] Let f be a non-constant meromorphic function and p, k be positive integers, then*

$$N_p(r,0;f^{(k)}) \leq N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 4.3 *[1] If f, g be two non-constant meromorphic functions such that they share $(1,1)$. Then*

$$\begin{aligned} & 2\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + \overline{N}_E^{(2}(r,1;f) - \overline{N}_{f>2}(r,1;g) \\ \leq\ & N(r,1;g) - \overline{N}(r,1;g). \end{aligned}$$

Lemma 4.4 *[2] Let f, g share $(1,1)$. Then*

$$\overline{N}_{f>2}(r,1;g) \leq \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) - \frac{1}{2}N_{\oslash}(r,0;f^{'}) + S(r,f),$$

where $N_{\oslash}(r,0;f^{'})$ is the counting function of those zeros of $f^{'}$ which are not the zeros of $f(f-1)$.

Lemma 4.5 *[2] Let f and g be two non-constant meromorphic functions sharing $(1,0)$. Then*

$$\begin{aligned} & \overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + \overline{N}_E^{(2}(r,1;f) - \overline{N}_{f>1}(r,1;g) - \overline{N}_{g>1}(r,1;f) \\ \leq\ & N(r,1;g) - \overline{N}(r,1;g). \end{aligned}$$

Lemma 4.6 *[2] Let f, g share $(1,0)$. Then*

$$\overline{N}_L(r,1;f) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f)$$

Lemma 4.7 *[2] Let f, g share $(1,0)$. Then*

$$\begin{aligned} (i) \quad & \overline{N}_{f>1}(r,1;g) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_{\oslash}(r,0;f^{'}) + S(r,f) \\ (ii) \quad & \overline{N}_{g>1}(r,1;f) \leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) - N_{\oslash}(r,0;g^{'}) + S(r,g). \end{aligned}$$

Lemma 4.8 *[20] Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, where $a_0, a_1, a_2 \ldots, a_n$ are constants and $a_n \neq 0$. Then $T(r,P(f)) = nT(r,f) + O(1)$.*

Lemma 4.9 *Let f and g be two non-constant meromorphic functions and a, b be two non zero constants. Then*

$$[f^n(af^m+b)]^{(k)}[g^n(ag^m+b)]^{(k)} \not\equiv 1,$$

where n, $m \geq 2$, $k = 1$ be three positive integers and $n(\geq m+3)$.

Proof. We note that when $k = 1$, according to the statement of the lemma we have to prove

$$[f^{n-1}\left(a(n+m)f^m + bn\right)f^{'}]\ \ [g^{n-1}\left(a(n+m)g^m + bn\right)g^{'}] \not\equiv 1.$$

If possible let us suppose that

$$[f^{n-1}\left(a(n+m)f^m + bn\right)f^{'}]\ \ [g^{n-1}\left(a(n+m)g^m + bn\right)g^{'}] \equiv 1. \tag{2}$$

Let z_0 be a zero of f with multiplicity $p(\geq 1)$. So from (2) we get z_0 be a pole of g with multiplicity $q(\geq 1)$ such that

$$np - 1 = (n+m)q + 1, \tag{3}$$

i.e.

$$mq = n(p-q) - 2 \geq n - 2.$$

Again from (3) we get

$$\begin{aligned} np &= (n+m)q + 2 \\ &\geq (n+m)\frac{n-2}{m} + 2, \end{aligned}$$

i.e.,

$$p \geq \frac{n+m-2}{m}.$$

Therefore

$$\Theta(0; f) \geq 1 - \frac{m}{n+m-2}.$$

Suppose $a(n+m)f^m + bn = a(n+m)\ (f-\alpha_1)(f-\alpha_2)\ldots(f-\alpha_m)$. Let z_1 be a zero of $(f-\alpha_i)$ $i = 1, 2, \ldots, m$ with multiplicity p. Then from (2) we have z_1 be a pole of g with multiplicity $q(\geq 1)$ such that

$$2p - 1 = (n+m)q + 1$$

i.e.,

$$p \geq \frac{n+m+2}{2}.$$

Hence

$$\Theta(\alpha_i; f) \geq 1 - \frac{2}{n+m+2}.$$

Since

$$\Theta(0; f) + \sum_{i=1}^{m} \Theta(\alpha_i; f) \leq 2,$$

it follows that

$$\frac{2m}{n+m+2} + \frac{m}{n+m-2} \geq m - 1,$$

which is a contradiction.

Lemma 4.10 *Let f and g be two non-constant entire functions. Then*

$$[f^n(af^m+b)]^{(k)}[g^n(ag^m+b)]^{(k)} \not\equiv 1,$$

where a and b are nonzero complex numbers; n, m, k be three positive integers and $n(>2k+m+4)$.

Proof. We omit the proof since the proof can be found in the proof of *Theorem 1* in [23].

Lemma 4.11 *Let f and g be two nonconstant meromorphic functions and $n(\geq 2)$, $m(\geq 2)$ be two distinct integers satisfying $n+m \geq d+7$. Then for two nonzero constants a, b,*

$$f^n\left(af^m+b\right) \equiv g^n\left(ag^m+b\right)$$

implies $f \equiv tg$, for some constant t, satisfying $t^d \equiv 1$, where $d=(n+m,n)$.

Proof. Suppose $F = f^n\left(af^m+b\right)$ and $G = g^n\left(ag^m+b\right)$. Let $f \not\equiv tg$ for a constant t satisfying $t^d=1$. We put $h=\frac{f}{g}$. Then $h^d \not\equiv 1$. First suppose that h is constant. Also $F \equiv G$ implies

$$g^m = -\,\frac{b}{a}\,\frac{h^n-1}{h^{n+m}-1}.$$

We note that the numerator and the denominator has d common factors namely $h-v_k$, $k=0,1,2,\ldots,d-1$, where $v_k = exp\left(\frac{2k\pi i}{d}\right)$. Since $(h-v_1)(h-v_2)\ldots(h-v_k) \neq 0$, it follows that g is a constant, which is impossible. So h is nonconstant. We observe that since a nonconstant meromorphic function can not have more than two Picard exceptional values h can take at least $n+m-d-2$ values among $u_j = exp\left(\frac{2j\pi i}{n+m}\right)$, where $j=0,1,2,\ldots,n+m-1$. Since f^m has no simple pole $h-u_j$ has no simple zero for at least $n+m-d-2$ values of u_j, for $j=0,1,2,\ldots,n+m-1$ and for these values of j we have $\Theta(u_j;h) \geq \frac{1}{2}$, which leads to a contradiction. This proves the lemma.

5 Proofs of the theorems

Proof of Theorem 2.1 Let $F=f^n(af^m+b)$ and $G=g^n(ag^m+b)$. It follows that $F^{(k)}$ and $G^{(k)}$ share $(1,l)$.
Case 1 Let $H \not\equiv 0$.
Subcase 1.1 $l \geq 1$
From (1) we get

$$\begin{aligned} N(r,\infty;H) \;\leq\; & \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_*\left(r,1;F^{(k)},G^{(k)}\right) \\ & +\overline{N}\left(r,0;F^{(k)}\mid\geq 2\right)+\overline{N}\left(r,0;G^{(k)}\mid\geq 2\right) \\ & +\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)+\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right), \end{aligned} \tag{4}$$

where $\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)$ is the reduced counting function of those zeros of $F^{(k+1)}$ which are not the zeros of $F^{(k)}\left(F^{(k)}-1\right)$ and $\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)$ is similarly defined.

Let z_0 be a simple zero of $F^{(k)}-1$. Then z_0 is a simple zero of $G^{(k)}-1$ and a zero of H. So

$$N\left(r,1;F^{(k)}\mid=1\right)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,F)+S(r,G) \tag{5}$$

While $l\geq 2$, using (4) and (5) we get

$$\begin{aligned}
& \overline{N}\left(r,1;F^{(k)}\right) && (6)\\
\leq\ & N\left(r,1;F^{(k)}\mid=1\right)+\overline{N}\left(r,1;F^{(k)}\mid\geq 2\right)\\
\leq\ & \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r.0;F^{(k)}\mid\geq 2)+\overline{N}\left(r.0;G^{(k)}\mid\geq 2\right)\\
& +\overline{N}_{*}\left(r,1;F^{(k)},G^{(k)}\right)+\overline{N}\left(r,1;F^{(k)}\mid\geq 2\right)+\overline{N}_{\otimes}(r,0;F^{(k+1)})\\
& +\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)+S(r,F)+S(r,G).
\end{aligned}$$

So from *Lemmas 4.1* and *4.8* we have

$$\begin{aligned}
& T(r,F)+T(r,G) && (7)\\
\leq\ & 2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+N_{k+1}(r,0;F)+N_{k+1}(r,0;G)\\
& +\overline{N}\left(r,0;F^{(k)}\mid\geq 2\right)+\overline{N}\left(r,0;G^{(k)}\mid\geq 2\right)+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)\\
& +\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)+\overline{N}\left(r,1;G^{(k)}\right)+\overline{N}\left(r,1;F^{(k)}\mid\geq 2\right)\\
& +\overline{N}_{*}\left(r,1;F^{(k)},G^{(k)}\right)-N_0\left(r,0;F^{(k+1)}\right)-N_0\left(r,0;G^{(k+1)}\right)\\
& +S(r,F)+S(r,G).
\end{aligned}$$

We note that

$$\begin{aligned}
& N_{k+1}(r,0;F)+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\right)+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right) && (8)\\
\leq\ & N_{k+1}(r,0;F)+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\mid F=0\right)\\
& +\overline{N}\left(r,0;F^{(k)}\mid\geq 2\mid F\neq 0\right)+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)\\
\leq\ & N_{k+1}(r,0;F)+\overline{N}\left(r,0;F\mid\geq k+2\right)+\overline{N}_0\left(r,0;F^{(k+1)}\right)\\
\leq\ & N_{k+2}(r,0;F)+\overline{N}_0\left(r,0;F^{(k+1)}\right).
\end{aligned}$$

Clearly similar expression holds for G. Also

$$\begin{aligned}
& \overline{N}\left(r,1;F^{(k)}\mid\geq 2\right)+\overline{N}_{*}\left(r,1;F^{(k)},G^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right) && (9)\\
\leq\ & \overline{N}\left(r,1;G^{(k)}\mid=2\right)+2\overline{N}_L\left(r,1;F^{(k)}\right)+2\overline{N}_L\left(r,1;G^{(k)}\right)\\
& +\overline{N}_E^{(3}\left(r,1;G^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right)\\
\leq\ & N\left(r,1;G^{(k)}\right)\\
\leq\ & T\left(r,G^{(k)}\right)+O(1)\\
\leq\ & T(r,G)+k\overline{N}(r,\infty;G)+S(r,G).
\end{aligned}$$

Using *Lemma 4.8*, (8) and (9) in (7) we obtain for $\varepsilon > 0$

$$\begin{aligned}
& (n+m)T(r,f) && (10)\\
= \;& T(r,F)+O(1)\\
\leq \;& N_{k+2}(r,0;F)+N_{k+2}(r,0;G)+2\overline{N}(r,\infty;F)\\
& +(k+2)\overline{N}(r,\infty;G)+S(r,F)+S(r,G)\\
\leq \;& N_{k+2}\left(r,0;f^{n}\right)+N_{k+2}(r,0;af^{m}+b)+N_{k+2}\left(r,0;g^{n}\right)\\
& +N_{k+2}(r,0;ag^{m}+b)+2\overline{N}(r,\infty;f)+(k+2)\overline{N}(r,\infty;g)\\
& +S(r,f)+S(r,g)\\
\leq \;& (4+m+k-2\Theta(\infty;f)+\varepsilon)\,T(r,f)+(4+m+2k-(2+k)\Theta(\infty;g)\\
& +\varepsilon)\,T(r,g)+S(r,f)+S(r,g)\\
\leq \;& (8+2m+3k-2\Theta(\infty;f)-2\Theta(\infty;g)-k\min\{\Theta(\infty;f),\Theta(\infty;g)\}\\
& +2\varepsilon)\,T(r)+S(r,f)+S(r,g).
\end{aligned}$$

In a similar way we can obtain

$$\begin{aligned}
& (n+m)T(r,g) && (11)\\
\leq \;& (8+2m+3k-2\Theta(\infty;f)-2\Theta(\infty;g)-k\min\{\Theta(\infty;f),\Theta(\infty;g)\}\\
& +2\varepsilon)\,T(r)+S(r,f)+S(r,g).
\end{aligned}$$

So from (10) and (11) we get

$$\begin{aligned}
& (n-m-3k-8+2\Theta(\infty;f)+2\Theta(\infty;g)+k\min\{\Theta(\infty;f),\Theta(\infty;g)\} && (12)\\
& -2\varepsilon)\,T(r)\leq S(r).
\end{aligned}$$

Since $\varepsilon > 0$ be arbitrary, (12) gives a contradiction.

While $l = 1$, using *Lemmas 4.2, 4.3* and *4.4*, (4) and (5) we get

$$\begin{aligned}
& \overline{N}\left(r,1;F^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right) && (13)\\
\leq \;& N\left(r,1;F^{(k)}\mid=1\right)+\overline{N}_{L}\left(r,1;F^{(k)}\right)\\
& +\overline{N}_{L}\left(r,1;G^{(k)}\right)+\overline{N}^{(2}_{E}\left(r,1;G^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right)\\
\leq \;& N\left(r,1;F^{(k)}\mid=1\right)+N\left(r,1;G^{(k)}\right)-\overline{N}_{L}\left(r,1;F^{(k)}\right)\\
& -\overline{N}_{L}\left(r,1;G^{(k)}\right)+\overline{N}_{F^{(k)}>2}\left(r,1;G^{(k)}\right)\\
\leq \;& \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}\left(r.0;F^{(k)}\mid\geq 2\right)+\overline{N}\left(r.0;G^{(k)}\mid\geq 2\right)\\
& +\overline{N}_{*}\left(r,1;F^{(k)},G^{(k)}\right)-\overline{N}_{L}\left(r,1;F^{(k)}\right)-\overline{N}_{L}\left(r,1;G^{(k)}\right)\\
& +\frac{1}{2}\overline{N}\left(r,0;F^{(k)}\right)+\frac{1}{2}\overline{N}\left(r,\infty;F^{(k)}\right)+T\left(r,G^{(k)}\right)+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)\\
& +\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)+S(r,F)+S(r,G)\\
\leq \;& \left(\frac{k}{2}+\frac{3}{2}\right)\overline{N}(r,\infty;F)+(k+1)\overline{N}(r,\infty;G)+\overline{N}\left(r.0;F^{(k)}\mid\geq 2\right)
\end{aligned}$$

$$+\overline{N}\left(r.0;G^{(k)}\mid\geq 2\right)+\frac{1}{2}N_{k+1}(r,0;F)+T(r,G)+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)$$
$$+\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)+S(r,F)+S(r,G).$$

So in view of *Lemmas 4.1*, *4.8*, (8) and (13) we get for $\varepsilon>0$

$$\begin{aligned}
&(n+m)T(r,f) \qquad (14)\\
=\;& T(r,F)+O(1)\\
\leq\;& \left(\frac{k}{2}+\frac{5}{2}\right)\overline{N}(r,\infty;F)+(k+2)\overline{N}(r,\infty;G)+\frac{1}{2}N_{k+1}(r,0;F)\\
&+N_{k+2}(r,0;F)+N_{k+2}(r,0;G)+S(r,F)+S(r,G)\\
\leq\;& \left(2k+5+\frac{3m}{2}-\left(\frac{k}{2}+2\right)\Theta(\infty;f)-\frac{1}{2}\Theta(\infty;f)+\varepsilon\right)T(r,f)\\
&+\left(2k+4+m-\left(\frac{k}{2}+2\right)\Theta(\infty;g)-\frac{k}{2}\Theta(\infty;g)+\varepsilon\right)T(r,g)\\
&+S(r,f)+S(r,g)\\
\leq\;& \left(4k+9+\frac{5m}{2}-\left(\frac{k}{2}+\frac{5}{2}\right)(\Theta(\infty;f)+\Theta(\infty;g))+2\varepsilon\right)T(r)\\
&+S(r).
\end{aligned}$$

In a similar manner we can get

$$\begin{aligned}
&(n+m)T(r,g) \qquad (15)\\
\leq\;& \left(4k+9+\frac{5m}{2}-\left(\frac{k}{2}+\frac{5}{2}\right)(\Theta(\infty;f)+\Theta(\infty;g))+2\varepsilon\right)T(r)+S(r).
\end{aligned}$$

Combining (14) and (15) we get

$$\begin{aligned}
&\left(n-4k-9-\frac{3m}{2}+\left(\frac{k}{2}+\frac{5}{2}\right)(\Theta(\infty;f)+\Theta(\infty;g))-2\varepsilon\right)T(r) \qquad (16)\\
\leq\;& S(r).
\end{aligned}$$

Since $\varepsilon>0$ be arbitrary, (16) implies a contradiction.
Subcase 1.2 $l=0$. Here (5) changes to

$$N_E^{1)}\left(r,1;F^{(k)}\mid=1\right)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,F)+S(r,G) \qquad (17)$$

Using *Lemmas 4.2*, *4.5*, *4.6*, *4.7* and (4) and (17) we get

$$\begin{aligned}
&\overline{N}\left(r,1;F^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right) \qquad (18)\\
\leq\;& N_E^{1)}\left(r,1;F^{(k)}\right)+\overline{N}_L\left(r,1;F^{(k)}\right)+\overline{N}_L\left(r,1;G^{(k)}\right)\\
&+\overline{N}_E^{(2}\left(r,1;F^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right)
\end{aligned}$$

$$\begin{aligned}
&\leq N_E^{1)}\left(r,1;F^{(k)}\right)+N\left(r,1;G^{(k)}\right)-\overline{N}_L\left(r,1;G^{(k)}\right)\\
&\quad+\overline{N}_{F^{(k)}>1}\left(r,1;G^{(k)}\right)+\overline{N}_{G^{(k)}>1}\left(r,1;F^{(k)}\right)\\
&\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\right)+\overline{N}\left(r,0;G^{(k)}\mid\geq 2\right)\\
&\quad+\overline{N}_*\left(r,1;F^{(k)},G^{(k)}\right)+T\left(r,G^{(k)}\right)-\overline{N}_L\left(r,1;G^{(k)}\right)\\
&\quad+\overline{N}_{F^{(k)}>1}\left(r,1;G^{(k)}\right)+\overline{N}_{G^{(k)}>1}\left(r,1;F^{(k)}\right)\\
&\quad+\overline{N}_\otimes\left(r,0;F^{(k+1)}\right)+\overline{N}_\otimes\left(r,0;G^{(k+1)}\right)+S(r,F)+S(r,G)\\
&\leq (2k+3)\overline{N}(r,\infty;F)+(2k+2)\overline{N}(r,\infty;G)+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\right)\\
&\quad+\overline{N}\left(r,0;G^{(k)}\mid\geq 2\right)+2N_{k+1}(r,0;F)+N_{k+1}(r,0;G)+T(r,G)\\
&\quad+\overline{N}_\otimes\left(r,0;F^{(k+1)}\right)+\overline{N}_\otimes\left(r,0;G^{(k+1)}\right)+S(r,F)+S(r,G).
\end{aligned}$$

So in view of *Lemmas 4.1*, *4.8*, (8) and (18) we get for $\varepsilon>0$

$$\begin{aligned}
&\quad (n+m)T(r,f) \qquad (19)\\
&= T(r,F)+O(1)\\
&\leq (2k+4)\overline{N}(r,\infty;f)+(2k+3)\overline{N}(r,\infty;g)+2N_{k+1}(r,0;F)\\
&\quad+N_{k+1}(r,0;G)+N_{k+2}(r,0;F)+N_{k+2}(r,0;G)+S(r,f)+S(r,g)\\
&\leq (9k+14+5m-(2k+3)\Theta(\infty;f)-(2k+3)\Theta(\infty;g)\\
&\quad-\min\{\Theta(\infty;f),\Theta(\infty;g)\}+2\varepsilon)\,T(r)+S(r).
\end{aligned}$$

Similarly we can obtain

$$\begin{aligned}
&\quad (n+m)T(r,g) \qquad (20)\\
&= T(r,G)+O(1)\\
&\leq (9k+14+5m-(2k+3)\Theta(\infty;f)-(2k+3)\Theta(\infty;g)\\
&\quad-\min\{\Theta(\infty;f),\Theta(\infty;g)\}+2\varepsilon)\,T(r)+S(r).
\end{aligned}$$

Combining (19) and (20) we get

$$\begin{aligned}
&(n-9k-14+4m+(2k+3)\Theta(\infty;f)+(2k+3)\Theta(\infty;g) \qquad (21)\\
&+\min\{\Theta(\infty;f),\Theta(\infty;g)\}-2\varepsilon)\,T(r)\leq S(r).
\end{aligned}$$

(21) implies a contradiction for $\varepsilon>0$.
Case 2 Next we suppose that $H\equiv 0$. Then by integration we get from (1)

$$\frac{1}{F^{(k)}-1}\equiv\frac{bG^{(k)}+a-b}{G^{(k)}-1}, \tag{22}$$

where a, b are constants and $a\neq 0$. From (22) it is clear that $F^{(k)}$ and $G^{(k)}$ share $(1,\infty)$ and hence they share $(1,2)$. So in this case always $n>3k+m+8-2\{\Theta(\infty;f)+\Theta(\infty;g)\}-k\min\{\Theta(\infty;f),\Theta(\infty;g)\}$. We now

consider the following subcases.
Subcase 2.1 Let $b \neq 0$ and $a \neq b$.
If $b = -1$, then from (22) we have

$$F^{(k)} = \frac{-a}{G^{(k)} - a - 1}.$$

Therefore

$$\overline{N}\left(r, a+1; G^{(k)}\right) = \overline{N}\left(r, \infty; F^{(k)}\right) = \overline{N}(r, \infty; f).$$

Since $a \neq b = -1$, from *Lemma 4.1* we have

$$\begin{aligned}
& (n+m)T(r,g) \\
= \; & T(r,G) + O(1) \\
\leq \; & \overline{N}(r,\infty;G) + N_{k+1}(r,0;G) + \overline{N}(r,a+1;G^{(k)}) + S(r,G) \\
\leq \; & \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + N_{k+1}(r,0;G) + S(r,G) \\
\leq \; & (1-\Theta(\infty;f)+\varepsilon)T(r,f) + (k+2+m-\Theta(\infty;g)+\varepsilon)T(r,g) + S(r,g)
\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leq T(r,g)$ for $r \in I$.
So for $r \in I$ we have

$$(n-k-3+\Theta(\infty;f)+\Theta(\infty;g)-2\varepsilon)T(r,g) \leq S(r,g),$$

which is a contradiction for arbitrary $\varepsilon > 0$.
If $b \neq -1$, from (22) we obtain that

$$F^{(k)} - \left(1 + \frac{1}{b}\right) = \frac{-a}{b^2[G^{(k)} + (a-b)/b]}.$$

Therefore

$$\overline{N}\left(r, (b-a)/b; G^{(k)}\right) = \overline{N}\left(r, \infty; F^{(k)} - (1+1/b)\right) = \overline{N}(r, \infty; f)$$

Using *Lemma 4.1* and the same argument as used in the case when $b = -1$ we can get a contradiction.
Subcase 2.2 Let $b \neq 0$ and $a = b$.
If $b = -1$, then from (22) we have

$$F^{(k)}G^{(k)} \equiv 1,$$

that is

$$[f^n(af^m+b)]^{(k)}[g^n(ag^m+b)]^{(k)} \equiv 1,$$

which is impossible by *Lemma 4.9* when $k = 1$.
If $b \neq -1$, from (22) we have

$$\frac{1}{F^{(k)}} = \frac{bG^{(k)}}{(1+b)G^{(k)} - 1}.$$

Hence from *Lemma 4.2* we have

$$\begin{aligned}\overline{N}\left(r,1/(1+b);G^{(k)}\right) &= \overline{N}\left(r,0;F^{(k)}\right)\\ &\leq N_{k+1}(r,0;F)+k\overline{N}(r,\infty;f).\end{aligned}$$

From *Lemma 4.1* we have

$$\begin{aligned}&(n+m)T(r,g)+O(1)\\ =\;& T(r,G)\\ \leq\;& \overline{N}(r,\infty;G)+N_{k+1}(r,0;G)+\overline{N}\left(r,\frac{1}{b+1};G^{(k)}\right)+S(r,G)\\ \leq\;& k\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+N_{k+1}(r,0;F)+N_{k+1}(r,0;G)+S(r,G)\\ \leq\;& (2k+1+m-k\Theta(\infty;f)+\epsilon)T(r,f)\\ &+(k+2+m-\Theta(\infty;g)+\epsilon)T(r,g)+S(r,g)\end{aligned}$$

For $r\in I$ we have

$$(n-3k-3-m+k\Theta(\infty;f)+\Theta(\infty;g)-2\epsilon)T(r,g)\leq S(r,g),$$

which is a contradiction for $n\geq 3k+9$.
Subcase 2.3 Let $b=0$. From (22) we obtain

$$F^{(k)}=\frac{G^{(k)}+a-1}{a}. \tag{23}$$

If $a-1\neq 0$ then From (23) we obtain

$$\overline{N}\left(r,1-a;G^{(k)}\right)=\overline{N}\left(r,0;F^{(k)}\right).$$

We can similarly deduce a contradiction as in Subcase 2.2. Therefore $a=1$ and from (23) we obtain

$$F=G+p(z), \tag{24}$$

where $p(z)$ is a polynomial of degree at most $k-1$. We claim that $p(z)\equiv 0$. Otherwise noting that f is transcendental when $k\geq 2$, in view of *Lemma 4.8* we have

$$\begin{aligned}(n+m)T(r,f) &= T(r,F)+O(1) \qquad (25)\\ &\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;f)+\overline{N}(r,p;F)+S(r,F)\\ &\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;f)+\overline{N}(r,0;G)+S(r,F)\\ &\leq 3T(r,f)+2T(r,g)+S(r,f)\end{aligned}$$

Also from (24) we get

$$T(r,f)=T(r,g)+S(r,f),$$

which together with (25) implies a contradiction. So

$$F \equiv G.$$

So from *Lemma 4.11* we get the conclusion of the theorem.
Proof of Theorem 2.5 We omit the proof since instead of *Lemma 4.9* using *Lemma 4.10* and proceeding in the same way the proof of the theorem can be carried out in the line of proof of *Theorem 2.1* and *Theorem 1* of [23].

6 Open Problem

Theorems *2.1* and *2.5* are proved for transcendental meromorphic functions. Are both the theorems also true for non-constant meromorphic functions ? Keeping all other conditins intact can the second conclusion in Theorem *2.1* be removed when $k \geq 2$?

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