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Some Starlike and Convexity Properties of Sakaguchi Classes for Hypergeometric Functions

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Abstract

The objective of this paper is to give some characterizations for a (Gaussian) hypergeometric function to be in a subclass of Sakaguchi type functions. Two subclasses $S(\alpha, t)$ and $T(\alpha, t)$ of Sakaguchi type functions in the open unit disc U are also discussed.

Keywords: *Starlike, Convex, Hypergeometric function, Sakaguchi classes.*

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1 Introduction

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open unit disk $\mathrm{U} = \{z \in C :\mid z \mid < 1\}$. A function $f(z) \in A$ is said to be in the class $S(\alpha, t)$ if it satisfies

$$\mathrm{Re}\left\{\frac{(1-t)zf'(z)}{f(z)-f(tz)}\right\} > \alpha, \qquad |t| \leq 1, \quad t \neq 1 \tag{1.2}$$

for some $\alpha(0 \leq \alpha < 1)$and for all $z \in \mathrm{U}$. The class $S(0, -1)$was introduced by Sakaguchi [9]. Therefore, a function $f(z) \in S(\alpha, -1)$ is called Sakaguchi function of order α. We also denote by $T(\alpha, t)$the subclass of A consisting of all functions $f(z)$ such that $zf'(z) \in S(\alpha, t)$
We note that $S(\alpha, 0) = S^*(\alpha)$, the class of starlike functions of order $\alpha(0 \leq \alpha < 1)$ and $T(\alpha, 0) = T^*(\alpha)$, the class of convex functions of order $\alpha(0 \leq \alpha < 1)$ (see Silverman [10]).
However,for this paper, we consider a subclass T of A where T denotes the class consisting of the functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0) \tag{1.3}$$

clearly $T \subset A$. Let $F(a, b; c; z)$ be the (Gaussian) hypergeometric function defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \tag{1.4}$$

where $c \neq 0, -1, -2, ...,$ and $(\theta)_n = \theta(\theta+1)......(\theta+n-1)$,
$n \in N = \{1, 2, ...\}$ and $(\theta)_0 = 1$
We note that $F(a, b; c; z)$ converges for $\mathrm{Re}(c-a-b) > 0$ and is related to the gamma function [7]

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{1.5}$$

Silverman [11] gave necessary and sufficient conditions for $zF(a, b; c; z)$ to be in $S^*(\alpha)$ and $T^*(\alpha)$. For other interesting developments on $zF(a, b; c; z)$in connections with various subclasses of univalent and p-valent functions, the reader can refer to the works of Carlson and Shaffer [1], Cho et al. [2], Goyal et al. [3], Merkes and Scott[4], Mostafa [5], Owa et al. [6], Ruscheweyh and Singh [8] etc .

2 MAIN RESULTS

To establish our main results, we need the following lemma[6]
Lemma 2.1
(i) A function $f(z)$ defined by (1.3) is in the class $S(\alpha,t)$ if

$$\sum_{n=2}^{\infty}(|n-u_n|+(1-\alpha)|u_n|)a_n \leq 1-\alpha \tag{2.1}$$

(where $u_n = 1+t+t^2+.......+t^{n-1}$)
for some $\alpha(0\leq\alpha<1)$
(ii) A function $f(z)$ defined by (1.3) is in the class $T(\alpha,t)$ if

$$\sum_{n=2}^{\infty}n(|n-u_n|+(1-\alpha)|u_n|)a_n \leq 1-\alpha \tag{2.2}$$

for some $\alpha(0\leq\alpha<1)$
Theorem 1
(i) If $a,b>-1$, $c>0$ and $ab<0$, then $zF(a,b;c;z)$is in $S(\alpha,t)$ if

$$\frac{c-a-b-1}{ab} \leq \frac{\alpha(c-a-b-1)(1+|t|)}{ab(1-t)} \tag{2.3}$$

(ii) If $a,b>0$,$c>a+b+1$,then $F_1(a,b;c;z)=z[2-F(a,b;c;z)]$ is in $S(\alpha,t)$ if

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{ab}{(c-a-b)(1-\alpha)}+\frac{1}{1-\alpha}-\frac{\alpha(1+|t|)}{(1-t)}\right]\leq 2 \tag{2.4}$$

Proof.(i) Since

$$\begin{aligned} zF(a,b;c;z) &= z+\frac{ab}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n \\ &= z-\left|\frac{ab}{c}\right|\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n \end{aligned} \tag{2.5}$$

according to (i) of Lemma 2.1, we must show that

$$\sum_{n=2}^{\infty}(|n-u_n|+(1-\alpha)|u_n|)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \left|\frac{c}{ab}\right|(1-\alpha) \tag{2.6}$$

Note that left side of (2.6) diverges if $c<a+b+1$. Now using the result

$$\sum_{n=2}^{\infty}(|n-u_n|+(1-\alpha)|u_n|)a_n \geq \sum_{n=2}^{\infty}(n-|u_n|+|u_n|-\alpha|u_n|)a_n$$

in (2.6), we find that left- hand side of (2.6)

$$= \sum_{n=2}^{\infty}(n-1)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} + \sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

$$-\alpha\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}|u_n|$$

$$= \sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + \frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} - 1\right]$$

$$-\alpha\frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}|u_{n+1}| - u_1\right]$$

Hence (2.6) is equivalent to

$$\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + \frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\right] - \alpha\frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}|u_{n+1}|\right]$$

$$\leq (1-\alpha)\left[\left|\frac{c}{ab}\right| + \frac{c}{ab}\right] = 0 \qquad (2.7)$$

Since

$$|u_{n+1}| = |1+t+\ldots\ldots+t^n| = \left|\frac{1-t^{n+1}}{1-t}\right| \leq \frac{(1+|t|^{n+1})}{1-t} \leq \frac{(1+|t|)}{1-t}$$

as $(|t|^n \leq 1)$
Therefore (2.7) is valid if

$$\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + \frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\right]$$

$$-\alpha\frac{c}{ab(1-t)}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} + (1+|t|)\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}1\right] \leq 0$$

or equivalently,

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + \frac{c}{ab}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

$$-\alpha\frac{c}{ab}\frac{(1+|t|)}{(1-t)}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \leq 0$$

or

$$1 + \frac{c-a-b-1}{ab} \leq \frac{\alpha(c-a-b-1)(1+|t|)}{ab(1-t)}$$

which is the required inequality (2.3).
(ii) Since

$$F_1(a,b;c;z)=z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n \tag{2.8}$$

by (i) of Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty}(|n-u_n|+(1-\alpha)\,|u_{_n}|)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\leq(1-\alpha) \tag{2.9}$$

Now left side of (2.9)

$$=\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-2}}+\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}-\alpha\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\,|u_n|$$

Noting that $(\theta)_n=\theta(\theta+1)_{n-1}$ then left side of (2.9)

$$=\frac{ab}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-2}}+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-\alpha\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\,|u_{n+1}|$$

$$=\frac{ab}{c}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-1\right]-\alpha\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\,|u_{n+1}|-|u_1|\right]$$

But this last expression is bounded above by $1-\alpha$ if and only if (2.4) holds.

Theorem 2

(i) If $a,b>-1$,$ab<0$ and $c>a+b+2$, then $zF(a,b;c;z)$is in $T(\alpha,t)$ if

$$ab(a+1)(b+1)+3ab(c-a-b-2)+(c-a-b-2)_2$$

$$-\alpha\frac{(1+|t|^2)}{(1-t)}(c-a-b-2)ab-\frac{\alpha(1+|t|)}{(1-t)}(c-a-b-2)_2\geq 0 \tag{2.10}$$

(ii) If $a,b>0,c>a+b+2$, then $F_1(a,b;c;z)=z[2-F(a,b;c;z)]$ is in $T(\alpha,t)$ if

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{(a)(b)(a+1)(b+1)}{(c-a-b-2)_2}+\frac{3ab}{c-a-b-1}\right.$$

$$\left.-\alpha\frac{(1+|t|^2)ab}{(1-t)(c-a-b-1)}-\frac{\alpha(1+|t|)}{(1-t)}\right]\leq 2(1-\alpha) \tag{2.11}$$

Proof.(i) Since $zF(a,b;c;z)$ has the form (2.5), we see from (ii) of **Lemma 2.1**, that our conclusion is equivalent to

$$\sum_{n=2}^{\infty}n(|n-u_n|+(1-\alpha)\,|u_{_n}|)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\leq\left|\frac{c}{ab}\right|(1-\alpha) \tag{2.12}$$

Note that for $c > a+b+2$, the left side of (2.12) converges.
Now left side of (2.12)

$$= \sum_{n=2}^{\infty} (n^2 - \alpha n\,|u_n|)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

$$= \sum_{n=2}^{\infty} \left[(n-1)^2 + 1 + 2(n-1)\right]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

$$-\alpha\sum_{n=2}^{\infty}(n-1)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\,|u_n| - \alpha\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\,|u_n|$$

$$= \sum_{n=2}^{\infty}(n-1)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-2}} + \sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

$$+2\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-2}} - \alpha\sum_{n=2}^{\infty}(n-1)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\,|u_n|$$

$$-\alpha\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\,|u_n|$$

$$= \frac{(a+1)(b+1)}{(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n} + \frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} - 1\right]$$

$$+3\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} - \alpha\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}\left|\frac{1-t^{n+2}}{1-t}\right|$$

$$-\alpha\frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\left|\frac{1-t^{n+1}}{1-t}\right| - |u_1|\right]$$

$$= \frac{(a+1)(b+1)}{(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n} + \frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} - 1\right]$$

$$+3\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} - \frac{\alpha}{1-t}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}(1+|t|^{n+2})$$

$$-\frac{\alpha c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\frac{(1+|t|^{n+1})}{(1-t)} - |u_1|\right]$$

This last expression is bounded above by $\left|\frac{c}{ab}\right|(1-\alpha)$ if and only if (2.10) holds.
(ii) In view of (ii) of **Lemma 2.1** we need to show that

$$\sum_{n=2}^{\infty} n(|n-u_n| + (1-\alpha)\,|u_n|)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \le (1-\alpha) \qquad (2.13)$$

Now left side of (2.13)

$$= \sum_{n=2}^{\infty} (n^2 - \alpha n \, |u_n|) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

$$= \sum_{n=0}^{\infty} (n+2)^2 \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} - \alpha \sum_{n=0}^{\infty} (n+2) \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \, |u_{n+2}|$$

$$= \sum_{n=1}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n-1}} + 3 \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$-\alpha \frac{ab}{c} \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} \, |u_{n+2}| - \alpha \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} \, |u_{n+1}|$$

Now by the same techniques used in the proof of **Theorem 2** (i) above expression is bounded above by $(1-\alpha)$ if and only if (2.11) holds.

3 REMARK

Putting $t = 0$ in the above results, we obtain the results of Silverman [11].

4 OPEN PROBLEM

We define an integral operator $G(a, b; c; z)$(or similar to it) acting on $F(a, b; c; z)$ as follows:

$$G(a, b; c; z) = \int_0^z F(a, b; c; x)dx \tag{2.14}$$

What are conditions to be imposed on the parameters a, b, c, so that the operator $G(a, b; c; z)$ defined by (2.14) is in $S(\alpha, t)$ and $T(\alpha, t)$?

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