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Univalence Condition for an Integral Operator

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Abstract

In this paper for analytic functions we introduce a new integral operator and we prove the univalence condition for this operator.

Keywords: *Analytic, univalent, unit disk, regular.*

1 Introduction and definitions

Let $\mathcal{U} = \{z : |z| < 1\}$ the open unit disk and $\mathcal{A}$ the class of all functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in $\mathcal{U}$. By $\mathcal{S}$ we denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.
To prove our main result we need Becker univalence criterion and Schwarz Lemma.

Theorem 1.1. *[1] If the function f is regular in the unit disk $\mathcal{U}$, $f(z) = z + a_2 z^2 + \dots$ and*

$$(1 - |z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

for all $z \in \mathcal{U}$, then the function f is univalent in $\mathcal{U}$.

Lemma 1.2. *[2] (Schwarz lemma) Let the function g be regular in the unit disk $\mathcal{U}$ and $g(0) = 0$. If $|g(z)| \leq 1, \forall z \in \mathcal{U}$, then*

$$|g(z)| \leq |z|, \forall z \in \mathcal{U}$$

and equality can hold only if $g(z) = \varepsilon z$, where $|\varepsilon| = 1$.

We introduce a new integral operator defined by

$$I(f,g)(z) = \int_0^z f(t)^\alpha g(t)^\beta dt \tag{1}$$

for all $z \in \mathcal{U}$ and for the analytical functions f and g.

2 The univalence condition

Theorem 2.1. *Let $f, g \in \mathcal{A}$ and α, β positive real numbers. If $|f(z)| \leq 1, |g(z)| \leq 1$ and*

$$\left|\frac{f'(z)}{f(z)}\right| \leq M, \quad \left|\frac{g'(z)}{g(z)}\right| \leq N \tag{2}$$

and

$$\alpha M^{\alpha-1} + \beta N^{\beta-1} \leq \frac{1}{4} \tag{3}$$

for $M, N (M, N > 0)$ positive real numbers, then the operator $I(f,g)(z)$ defined by (1) is in the univalent function class $\mathcal{S}$.

Proof. From (1) we obtain that

$$\frac{zI''(f,g)(z)}{I'(f,g)(z)} = \frac{z\alpha f'(z)^{\alpha-1}}{f(z)^\alpha} + \frac{z\beta g'(z)^{\beta-1}}{g(z)^\beta}$$

From Becker univalence criterion we have

$$\begin{aligned}(1-|z|^2)\left|\frac{zI''(f,g)(z)}{I'(f,g)(z)}\right| &= (1-|z|^2)\left|\frac{z\alpha f'(z)^{\alpha-1}}{f(z)^\alpha} + \frac{z\beta g'(z)^{\beta-1}}{g(z)^\beta}\right| \\ &\leq (1-|z|^2)|z|\left(\alpha\left|\frac{f'(z)}{f(z)}\right|^{\alpha-1} \cdot |f(z)| + \beta\left|\frac{g'(z)}{g(z)}\right|^{\beta-1} \cdot |g(z)|\right)\end{aligned}$$

From hypothesis we have that $|f(z)| \leq 1$ and $|g(z)| \leq 1$. Using Schwarz lemma we obtain that $|f(z)| \leq |z|$ and $|g(z)| \leq |z|$. Hence and using (2) we obtain

$$\begin{aligned}(1-|z|^2)\left|\frac{zI''(f,g)(z)}{I'(f,g)(z)}\right| &\leq (1-|z|^2)|z|(\alpha M^{\alpha-1}|z| + \beta N^{\beta-1}|z|) \\ &\leq (1-|z|^2)|z|^2(\alpha M^{\alpha-1} + \beta N^{\beta-1}).\end{aligned} \tag{4}$$

We define the function $F : [0,1] \to \mathbb{R}, F(x) = (1-x^2)x^2, x = |z|$. This function has a maximum at a point $x = \frac{\sqrt{2}}{2}$ and hence results that $F(x) \leq \frac{1}{4}$.
Hence, from (4) using (3) we obtain that

$$(1-|z|^2)\left|\frac{zI''(f,g)(z)}{I'(f,g)(z)}\right| \leq 1,$$

so the operator is univalent. □

For $M, N = 1$ we obtain

Corollary 2.2. *Let $f, g \in \mathcal{A}$ and α, β positive real numbers. If $|f(z)| \leq 1, |g(z)| \leq 1$ and*

$$\left|\frac{f'(z)}{f(z)}\right| \leq 1, \quad \left|\frac{g'(z)}{g(z)}\right| \leq 1$$

and

$$\alpha + \beta \leq \frac{1}{4},$$

then the operator $I(f, g)(z)$ defined by (1) is in the univalent function class $\mathcal{S}$.

3 Open Problem

The open problem is to study the convexity order and the starlikeness for the integral operator $I(f, g)(z)$ defined by (1) in the first part of this paper.

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