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On a Differential Superordination Defined By Aouf et al Operator

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Abstract

The object of the present work is to obtain a certain superordination for certain new classes of analytic and univalent functions using Aouf et al derivative operator.

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1 Introduction

Recently, precisely in 1999, Kanas and Ronning [4] introduced and studied a new concept of analytic and univalent functions, a subset of the well known classes of analytic and univalent functions denoted by $A(\omega)$ and $S(\omega)$ respectively. The new class of analytic functions denoted by $A(\omega)$ is of the form

$$f(z) = (z-\omega) + \sum_{k=2}^{\infty} a_k (z-\omega)^k, \tag{1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$ and normalized by $f(\omega) = f'(\omega) - 1 = 0$ and ω is a fixed point in U, and $S(\omega) \subset A(\omega)$ denote the class of univalent functions.
Using (1), the authors in [4] introduced and studied the classes of ω-starlike and ω-convex functions, and many interesting results were obtained. Several other authors such as Acu and Owa [1] also introduced and studied the class of ω-close-to-convex functions and they obtained some coefficient inequalities and subordination results. Oladipo [5] extends the result in [4] using Ruscheweyh derivative operator and in [6] the author used the concept of (1) to define certain classes of Bazilevic functions and obtained some interesting results. Also, these classes of ω-starlike and ω-convex functions preserves Alexander relation that is, uniformly convergency is endowed in them [1,4,5,6].
Let $P(\omega) \subset P$(the class of Caratheodory functions) and let $p_\omega \in P(\omega)$ be analytic and of the form

$$p_\omega(z) = 1 + \sum_{k=1}^{\infty} B_k (z-\omega)^k \tag{2}$$

where

$$|B_k| \leq \frac{2}{(1+d)(1-d)^k}, d = |\omega|, k \geq 1$$

that are regular in U and satisfying $p_\omega(\omega) = 1$ and $Rep_{\omega)}(z) > 0$ [4,5,6,9], and ω is an arbitrary fixed point in U. We note here that$A(0) \equiv A$ and $P(0) \equiv P$.
The object of the present paper is to use the newly introduced concepts of analytic and univalent functions to determine properties of functions p_ω that satisfies the differential superordination

$$\left\{\phi_\omega(p_\omega(z), (z-\omega)p'_\omega(z), (z-\omega)^2 p''_\omega(z); z) : z \in U\right\} \subset W.[3, 7],$$

provided that, p_ω satisfy the differential subordination (which is the analogue by extension of the one in [8])

$$W_\omega \subset \left\{\phi(p_\omega(z), (z-\omega)p'_\omega(z), (z-\omega)^2 p''_\omega(z); z) : z \in U\right\}.$$

Here we shall denote by $H_\omega(U)$ the set of analytic functions in the unit disk, that is, $U = \{z : |z| < 1\}$. For $a \in C$ and $n \in N_0$ we denote by

$$H_\omega[a, n] = \{f \in H_\omega(U) : f(z) = a + a_n(z-\omega)^n + ...\}$$

and

$$A_n(\omega)\left\{f \in H_\omega(U) : f(z) = (z-\omega) + a_{n+1}(z-\omega)^{n+1} + ...\right\}.$$

For the purpose of this work the following definition shall be necessary.
Definition A. [1,2,7] Let $\phi_\omega : C^2 \times U \to C$ and h_ω be analytic in U.
If p_ω and $\phi_\omega(p_\omega(z), (z-\omega)p'_\omega(z); z)$ are univalent in U and satisfy the first order differential superordination

$$h_\omega(z) \prec \phi_\omega(p_\omega(z), (z-\omega)p'_\omega(z); z) \tag{3}$$

then, p_ω is called a solution of the differential superordination.
An analytic function q_ω is called a subordinant of the solutions of the differential superordination, or simply a subordinant if $q_\omega \prec p_\omega$ for all p_ω satisfying (3). A univalent subordinant $\overline{q}_\omega$ that satisfies $q_\omega \prec \overline{q}_\omega$ for all subordinants q_ω of (3) is said to be the best subordinant. The best subordinant is unique up to rotation of U.
For W_ω a set in C with ϕ_ω and p_ω as given in definition A, suppose (3) is replaced by

$$W_\omega \subset \{\phi_\omega(p_\omega(z), (z-\omega)p'_\omega(z); z) : z \in U\} \tag{4}$$

which is a differential containment, the condition (4) will be reffered to as a differential superordination, and the definitions of solution, subordinant and the best subordinant as given above can be extended to this generalization.
Let us denote by χ_ω the set of functions f that are analytic and injective on $\overline{U} \cap E(f)$, where

$$E(f) = \{\eta \in \partial U : lim_{z\to\eta} f(z) = \infty\}$$

and are such that $f'(\eta) \neq 0$ for $\eta \in \partial U \cap E(f)$. The subclass of χ_ω for which $f(\omega) = a$ is denoted by $\chi_\omega(a)$.
The following lemma shall be helpful in our present investigation.
Lemma A.[1,6] Let h_ω be ω-convex in U, with $h_\omega(\omega) = a, \gamma \neq 0$ with $Re\gamma \geq 0$, and $p_\omega \in H_\omega[a, n] \cap \chi_\omega$. If

$$p_\omega(z) + \frac{(z-\omega)p'_\omega(z)}{\gamma}$$

is univalent in U and,

$$h_\omega(z) \prec p_\omega(z) + \frac{(z-\omega)p'_\omega(z)}{\gamma}$$

then $q_\omega(z) \prec p_\omega(z)$, where

$$q_\omega(z) = \frac{\gamma}{n(z-\omega)^{\frac{\gamma}{n}}} \int_\omega^z (t-\omega)^{\frac{\gamma-n}{n}} h_\omega(t)dt$$

The function q_ω is ω-convex and is the best dominant.
Lemma B. [1,6] Let q_ω be ω-convex in U and let h_ω be defined by

$$h_\omega(z) = q_\omega(z) + \frac{(z-\omega)q'_\omega(z)}{\gamma},\ z \in U$$

with $Re\gamma \geq 0$, if $p_\omega \in H_\omega[a,n] \cap \chi_\omega$,

$$p_\omega(z) + \frac{(z-\omega)p'_\omega(z)}{\gamma}$$

is univalent in U, and

$$q_\omega(z) + \frac{(z-\omega)q'_\omega(z)}{\gamma} \prec p_\omega(z) + \frac{(z-\omega)p'_\omega(z)}{\gamma},\ z \in U$$

then $q_\omega(z) \prec p_\omega(z)$ where

$$q_\omega(z) = \frac{\gamma}{n(z-\omega)^{\frac{\gamma}{n}}} \int_\omega^z (t-\omega)^{\frac{\gamma-n}{n}} h_\omega(t)dt.$$

The function q_ω is the best subordinant and ω is a fixed point in U.
We shall also make use of the following derivative operator credited to Aouf et al [2] and it is defined as follows

$$I^0_\omega(\lambda,l)f(z) = f(z) \tag{5}$$

$$I^1_\omega(\lambda,l)f(z) = I_\omega(\lambda,l)f(z) = I^0_\omega(\lambda,l)f(z)\frac{1-\lambda+l}{1+l} + (I^0_\omega(\lambda,l)f(z))'\frac{\lambda(z-\omega)}{1+l}, \tag{6}$$

and in general

$$I^m_\omega(\lambda,l)f(z) = I_\omega(\lambda,l)(I^{m-1}_\omega(\lambda,l)f(z) \tag{7}$$

$$= (z-\omega) + \sum_{k=n+1}^{\infty} \left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k(z-\omega)^k$$

where $\lambda \geq 0, l \geq 0, m \in N_0$, and ω is a fixed point in U, for every $f \in A_n(\omega)$.

2 MAIN RESULT

In this section we state and prove the following
Theorem A. Let $h_\omega \in H_\omega(U)$ be ω-convex in U, with $h_\omega(\omega) = 1$ and ω is a fixed point in U.

Let $f \in A_n(\omega), n \in N$ and suppose that $[I_\omega^{m+1}(\lambda,l)f(z)]'$ is univalent and $[I_\omega^m(\lambda,l)f(z)]' \in H_\omega[1,n] \cap \chi_\omega$ if

$$h_\omega(z) \prec [I_\omega^{m+1}(\lambda,l)f(z)]' \tag{8}$$

then

$$q_\omega(z) \prec [I_\omega^m(\lambda,l)f(z)]' \tag{9}$$

where

$$q_\omega(z) = \frac{1+l}{n\lambda(z-\omega)^{\frac{1+l}{n\lambda}}} \int_\omega^z (t-\omega)^{\frac{1-n\lambda}{n\lambda}} h(t)dt. \tag{10}$$

The function q_ω is ω-convex and is the best subordinant.
Proof. Let $f \in A_n(\omega)$. By using the properties of the operator $I_\omega^m(\lambda,l)$ we have

$$I_\omega^{m+1}(\lambda,l)f(z) = I_\omega(I_\omega^m(\lambda,l)f(z)) = \frac{1-\lambda+l}{1+l} I_\omega^m(\lambda,l)f(z) + \frac{\lambda(z-\omega)}{1+l}(I_\omega^m(\lambda,l)f(z))'. \tag{11}$$

Differentiating (11) we have

$$[I_\omega^{m+1}(\lambda,l)f(z)]' = [I_\omega^m(\lambda,l)f(z)]' + \frac{\lambda(z-\omega)}{1+l}(I_\omega^m(\lambda,l)f(z))''. \tag{12}$$

If we let

$$p_\omega(z) = [I_\omega^m(\lambda,l)f(z)]'$$

then (12) becomes

$$[I_\omega^{m+1}(\lambda,l)f(z)]' = p_\omega(z) + \frac{\lambda(z-\omega)}{1+l} p'_\omega(z). \tag{13}$$

By using Lemma A, for $\gamma = \frac{1+l}{\lambda}$, we have

$$q_\omega(z) = \frac{1+l}{n\lambda(z-\omega)^{\frac{1+l}{n\lambda}}} \int_\omega^z (t-\omega)^{\frac{1+l}{n\lambda}-1} h_\omega(t)dt. \tag{14}$$

The function $q_\omega(z)$ is the best dominant
Theorem B. Let $h_\omega \in H_\omega(U)$ be ω-convex in U, with $h_\omega(\omega) = 1$ and let $f \in A_n(\omega)$, $[I_\omega^m(\lambda,l)f(z)]'$ is univalent and

$$\frac{I_\omega^m(\lambda,l)f(z)}{(z-\omega)} \in H_\omega[1,n] \cap \chi_\omega,$$

if

$$h_\omega(z) \prec [I_\omega^m(\lambda,l)f(z)]', \tag{15}$$

then

$$q_\omega(z) \prec \frac{I_\omega^m(\lambda,l)f(z)}{(z-\omega)}, \tag{16}$$

where

$$q_\omega(z) = \frac{1}{n(z-\omega)^{\frac{1}{n}}} \int_\omega^z (t-\omega)^{\frac{1-n}{n}} h_\omega(t)dt.$$

Proof. Let us set

$$p_\omega(z) = \frac{I_\omega^m(\lambda, l)f(z)}{(z-\omega)} \tag{17}$$

and we have

$$I_\omega^m(\lambda, l)f(z) = (z-\omega)p_\omega(z). \tag{18}$$

By differentiating (18) we obtain

$$[I_\omega^m(\lambda, l)f(z)]' = p_\omega(z) + (z-\omega)p'_\omega(z)$$

and (15) becomes

$$h_\omega(z) \prec p_\omega(z) + (z-\omega)p'_\omega.$$

By Lemma A we get

$$q_\omega(z) \prec p_\omega(z) = \frac{I_\omega^m(\lambda, l)f(z)}{(z-\omega)},$$

where

$$q_\omega(z) = \frac{1}{n(z-\omega)^{\frac{1}{n}}} \int_\omega^z (t-\omega)^{\frac{1-n}{n}} h_\omega(t)dt.$$

The function $q_\omega(z)$ is ω-convex and is the best dominant.
Theorem C. Let q_ω be ω-convex in U and let h_ω be defined by

$$h_\omega(z) = q_\omega(z) + \frac{\lambda}{1+l}(z-\omega)q'_\omega(z),\ z \in U,\ \lambda > 0. \tag{19}$$

Let $f \in A_n(\omega)$ and suppose that $[I_\omega^{m+1}(\lambda, l)f(z)]'$ is univalent in U, $[I_\omega^m(\lambda, l)f(z)]' \in H_\omega[1, n] \cap \chi_\omega$ and

$$h_\omega(z) = q_\omega(z) + \frac{\lambda}{1+l}(z-\omega)q'_\omega(z) \prec [I_\omega^{m+1}(\lambda, l)f(z)]' \tag{20}$$

then

$$q_\omega(z) \prec [I_\omega^m(\lambda, l)f(z)]' \tag{21}$$

where

$$q_\omega(z) = \frac{1+l}{n\lambda(z-\omega)^{\frac{1+l}{n\lambda}}} \int_\omega^z (t-\omega)^{\frac{1+l}{n\lambda}-1} h_\omega(t)dt.$$

The function q_ω is the best dominant.
Proof. Let $f \in A_n(\omega)$. By using the properties of the operator $I_\omega^m(\lambda, l)$ we have

$$[I_\omega^{m+1}(\lambda, l)f(z)]' = [I_\omega^m(\lambda, l)f(z)]' + \frac{\lambda}{1+l}(z-\omega)[I_\omega^m(\lambda, l)f(z)]'$$

and by denoting

$$p_\omega(z) = [I_\omega^m(\lambda, l)f(z)]'$$

then we obtain

$$[I_\omega^{m+1}(\lambda,l)f(z)]' = p_\omega(z) + \frac{\lambda}{1+l}(z-\omega)p'_\omega(z).$$

By using Lemma B we have

$$q_\omega(z) \prec [I_\omega^m(\lambda,l)f(z)]'$$

where

$$q_\omega(z) = \frac{1+l}{n\lambda(z-\omega)^{\frac{1+l}{n\lambda}}}\int_\omega^z (t-\omega)^{\frac{1+l}{n\lambda}-1}h_\omega(t)dt.$$

Theorem D. Let q_ω be ω-convex in U and h_ω be defined by

$$h_\omega(z) = q_\omega(z) + (z-\omega)q'_\omega(z) \tag{22}$$

Let $f \in A_n(\omega)$ and suppose that $[I_\omega^m(\lambda,l)f(z)]'$ is univalent in U,

$$\frac{I_\omega^m(\lambda,l)f(z)}{(z-\omega)} \in H_\omega[1,n] \cap \chi_\omega$$

and

$$h_\omega(z) = q_\omega(z) + (z-\omega)q'_\omega(z) \prec [I_\omega^m(\lambda,l)f(z)]' \tag{23}$$

then

$$q_\omega(z) \prec \frac{I_\omega^m(\lambda,l)f(z)}{(z-\omega)}, \tag{24}$$

where

$$q_\omega(z) = \frac{1}{n(z-\omega)^{\frac{1}{n}}}\int_\omega^z (t-\omega)^{\frac{1-n}{n}}h_\omega(t)dt.$$

The function q_ω is the best subordinant.
Proof. Let us put

$$p_\omega(z) = \frac{I_\omega^m(\lambda,l)f(z)}{(z-\omega)},$$

we obtain

$$[I_\omega^m(\lambda,l)f(z)]' = p_\omega(z) + (z-\omega)p'_\omega(z).$$

Then (24) becomes

$$q_\omega(z) + (z-\omega)q'_\omega(z) \prec p_\omega(z) + (z-\omega)p'_\omega(z)$$

By Lemma B, we have

$$q_\omega(z) = \frac{I_\omega^m(\lambda,l)f(z)}{(z-\omega)}$$

where

$$q_\omega(z) = \frac{1}{n(z-\omega)^{\frac{1}{n}}}\int_\omega^z (t-\omega)^{\frac{1-n}{n}}h_\omega(t)dt.$$

For various choices of our parameters similar result could be obtained.
For example for $l = 0, \lambda = 1$ we have the function

$$h_\omega(z) = \frac{1 + (2\alpha - 1)(z - \omega)}{1 + (z - \omega)}$$

and at $\omega = 0$ we have

$$h_0(z) = \frac{1 + (2\alpha - 1)z}{1 + z}$$

3 Open problem

Conclusively, From Definition A and relation (7) the author wish to say here here that the ω-modified Ruscheweyh operator and ω-modified Dziok-Srivastava operator can also be used to study this form of differential superordination. A new set of results shall be obtained which could be compared with our earlier results and the ones in [8].

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