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Some preserving subordination and superordination results of certain integral operator

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Abstract

In this paper, we obtain some subordination and superordination-preserving results of certain integral operator. Sandwich-type result is also obtained.

Keywords: *Analytic function, integral operator, differential subordination, superordination.*

1 Introduction

Let $H(U)$ be the class of functions analytic in $U = \{z \in \mathbb{C} : |z| < 1\}$ and $H[a, n]$ be the subclass of $H(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$, with $H_0 = H[0, 1]$ and $H = H[1, 1]$. Let $A(p)$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, 3, ...\}; z \in U) \tag{1.1}$$

and let $A(1) = A$. Let f and F be members of $H(U)$. The function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $\omega(z)$ analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1 (z \in U)$, such that $f(z) = F(\omega(z))$. In such a case we write $f(z) \prec F(z)$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$ (see [5] and [6]).

Let $\phi : \mathbb{C}^2 \times U \rightarrow \mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the first order differential subordination:

$$\phi\left(p(z), zp^{'}(z); z\right) \prec h(z), \tag{1.2}$$

then $p(z)$ is a solution of the differential subordination (1.2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.2) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.2) is called the best dominant. If $p(z)$ and $\phi\left(p(z), zp^{'}(z); z\right)$ are univalent in U and if $p(z)$ satisfies first order differential superordination:

$$h(z) \prec \phi\left(p(z), zp^{'}(z); z\right), \tag{1.3}$$

then $p(z)$ is a solution of the differential superordination (1.3). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.3) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (1.3) is called the best subordinant (see [5] and [6]).

Motivated essentially by Jung et al. [2], Shams et al. [8] introduced the integral operator $I_p^{\alpha} : A(p) \rightarrow A(p)$ as follows (see also Aouf et al. [1]):

$$I_p^{\alpha} f(z) = \frac{(p+1)^{\alpha}}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)\, dt, \qquad (\alpha > 0; p \in \mathbb{N}), \tag{1.4}$$

and

$$I_p^0 f(z) = f(z), \quad (\alpha = 0; p \in \mathbb{N}). \tag{1.5}$$

For $f \in A(p)$ given by (1.1), then from (1.4) ,we deduce that

$$I_p^{\alpha} f(z) = z^p + \sum_{n=k}^{\infty} \left(\frac{p+1}{n+p+1}\right)^{\alpha} a_{p+n} z^{p+n}, \quad (\alpha \geq 0; p \in \mathbb{N}). \tag{1.6}$$

Using the above relation, it is easy to verify the identity:

$$z\left(I_p^{\alpha} f(z)\right)^{'} = (p+1)\, I_p^{\alpha-1} f(z) - I_p^{\alpha} f(z). \tag{1.7}$$

We note that the one-parameter family of integral operator $I_1^{\alpha} = I^{\alpha}$ was defined by Jung et al. [2].

To prove our results, we need the following definitions and lemmas.

Definition 1 [5]. Denote by F the set of all functions $q(z)$ that are analytic and injective on $\bar{U} \backslash E(q)$ where

$$E(q) = \left\{\zeta \in \partial U : \lim_{z \rightarrow \zeta} q(z) = \infty\right\},$$

and are such that $q^{'}(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(q)$. Further let the subclass of $\digamma$ for which $q(0) = a$ be denoted by $\digamma(a)$, $\digamma(0) \equiv \digamma_0$ and $\digamma(1) \equiv \digamma_1$.

Definition 2 [6]. A function $L(z,t)$ $(z \in U, t \geq 0)$ is said to be a subordination chain if $L(0,t)$ is analytic and univalent in U for all $t \geq 0, L(z,0)$ is continuously differentiable on $[0;1)$ for all $z \in U$ and $L(z,t_1) \prec L(z,t_2)$ for all $0 \leq t_1 \leq t_2$.

Lemma 1 [7]**.** The function $L(z,t) : U\times\ [0;1) \longrightarrow \mathbb{C}$ of the form

$$L(z,t) = a_1(t)\, z + a_2(t)\, z^2 + ... \quad (a_1(t) \neq 0; t \geq 0),$$

and $\lim_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if

$$Re\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} > 0 \quad (z \in U, t \geq 0).$$

Lemma 2 [3]. Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition

$$Re\{H(is;t)\} \leq 0$$

for all real s and for all $t \leq -n(1+s^2)/2$, $n \in \mathbb{N}$.If the function $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + ...$ is analytic in U and

$$Re\left\{H\left(p(z); zp^{'}(z)\right)\right\} > 0 \quad (z \in U),$$

then $Re\{p(z)\} > 0$ for $z \in U$.

Lemma 3 [4]. Let $\kappa, \gamma \in \mathbb{C}$ with $\kappa \neq 0$ and let $h \in H(U)$ with $h(0) = c$. If $Re\{\kappa h(z) + \gamma\} > 0\,(z \in U)$, then the solution of the following differential equation:

$$q(z) + \frac{zq^{'}(z)}{\kappa q(z) + \gamma} = h(z) \quad (z \in U; q(0) = c)$$

is analytic in U and satisfies $Re\{\kappa h(z) + \gamma\} > 0$ for $z \in U$.

Lemma 4 [5]. Let $p \in \digamma(a)$ and let $q(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$be analytic in U with $q(z) \neq a$ and $n \geq 1$. If q is not subordinate to p, then there exists two points $z_0 = r_0 e^{i\theta} \in U$ and $\zeta_0 \in \partial U \backslash E(q)$ such that

$$q(U_{r_0}) \subset p(U); \quad q(z_0) = p(\zeta_0) \quad \text{and} \quad z_0 p^{'}(z_0) = m\zeta_0 p(\zeta_0) \quad (m \geq n).$$

Lemma 5 [6]. Let $q \in H[a;1]$ and $\varphi : \mathbb{C}^2 \to \mathbb{C}$. Also set $\varphi\left(q(z), zq^{'}(z)\right) = h(z)$. If $L(z,t) = \varphi\left(q(z), tzq^{'}(z)\right)$ is a subordination chain and $q \in H[a;1] \cap \digamma(a)$, then

$$h(z) \prec \varphi\left(q(z), zq^{'}(z)\right)$$

implies that $q(z) \prec p(z)$. Furthermore, if $\varphi\left(q(z), zq^{'}(z)\right) = h(z)$ has a univalent solution $q \in \digamma(a)$, then q is the best subordinant.

In the present paper, we aim at proving some subordination-preserving and superordination-preserving properties associated with the integral operator I_p^{α}. Sandwich-type result involving this operator is also derived.

2 Subordination, superordination and sandwich results involving the operator I_p^α

Unless otherwise mentioned, we assume throughout this section that $\alpha \geq 1, p \in \mathbb{N}$ and $z \in \mathbb{U}$.

Theorem 1. *Let* $f, g \in A(p)$ and let

$$Re\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \qquad \left(\phi(z)=\frac{I_p^{\alpha-1}g(z)}{z^{p-1}}; z\in U\right), \tag{2.1}$$

where δ is given by

$$\delta=\frac{1+(p+1)^2-\left|1-(p+1)^2\right|}{4(p+1)}. \tag{2.2}$$

Then the subordination condition

$$\frac{I_p^{\alpha-1}f(z)}{z^p} \prec \frac{I_p^{\alpha-1}g(z)}{z^p}$$

implies that

$$\frac{I_p^{\alpha}f(z)}{z^p} \prec \frac{I_p^{\alpha}g(z)}{z^p}$$

and the function $\dfrac{I_p^{\alpha}g(z)}{z^p}$ *is the best dominant.*

Proof. Let us define the functions $F(z)$ and $G(z)$ in U by

$$F(z)=\frac{I_p^{\alpha}f(z)}{z^p} \quad \text{and} \quad G(z)=\frac{I_p^{\alpha}g(z)}{z^p} \quad (z\in U), \tag{2.3}$$

we assume here, without loss of generality, that $G(z)$ is analytic and univalent on $\bar{U}$ and

$$G'(\zeta)\neq 0 \qquad (|\zeta|=1).$$

If not, then we replase $F(z)$ and $G(z)$ by $F(\rho z)$ and $G(\rho z)$, respectively, with $0<\rho<1$. These new functions have the desired properties on $\bar{U}$, and we can use them in the proof of our result. Therefore, the results would follow by letting $\rho \to 1$.

We first show that, if

$$q(z)=1+\frac{zG''(z)}{G'(z)} \qquad (z\in U), \tag{2.4}$$

then

$$Re\{q(z)\}>0 \qquad (z\in U).$$

From (1.6) and the definition of the functions G, ϕ, we obtain that

$$\phi(z) = G(z) + \frac{zG'(z)}{p+1}. \tag{2.5}$$

Differentiating both side of (2.5) with respect to z yields

$$\phi'(z) = \left(1 + \frac{1}{p+1}\right) G'(z) + \frac{zG''(z)}{p+1}. \tag{2.6}$$

Combining (2.4) and (2.6), we easily get

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z)+p+1} = h(z) \quad (z \in U) \tag{2.7}$$

It follows from (2.1) and (2.7) that

$$Re\{h(z) + p + 1\} > 0 \quad (z \in U). \tag{2.8}$$

Moreover, by using Lemma 2, we conclude that the differential equation (2.7) has a solution $q(z) \in H(U)$ with $h(0) = q(0) = 1$. Let

$$H(u, v) = u + \frac{v}{u+p+1} + \delta$$

where δ is given by (2.2). From (2.7) and (2.8), we obtain

$$Re\left\{H\left(q(z); zq'(z)\right)\right\} > 0 \quad (z \in U).$$

To verify the condition that

$$Re\{H(is;t)\} \leq 0 \quad \left(s \in \mathbb{R}; t \leq -\frac{1+s^2}{2}\right), \tag{2.9}$$

we proceed it as follows:

$$\begin{aligned} Re\{H(is;t)\} &= Re\left\{is + \frac{t}{is+p+1} + \delta\right\} = \frac{t(p+1)}{s^2 + (p+1)^2} + \delta \\ &\leq -\frac{\Psi_p(\delta, s)}{2\left[s^2 + (p+1)^2\right]}, \end{aligned}$$

where

$$\Psi_p(\delta, s) = [(p+1) - 2\delta] s^2 - 2\delta (p+1)^2 + (p+1). \tag{2.10}$$

For δ given by (2.2), we observe that the expression $\Psi_p(\delta, s)$ in (2.10) is a postive, which implies that (2.9) holds. Thus, by using Lemma 2, we conclude that

$$Re\{q(z)\} > 0 \quad (z \in U).$$

By the definition of $q(z)$, we know that G is convex. To prove $F \prec G$, let the function $L(z,t)$ be defined by

$$L(z,t) = G(z) + \frac{(1+t)\, zG'(z)}{p+1} \quad (0 \leq t < \infty; z \in U). \tag{2.11}$$

Since G is convex, then

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left(1 + \frac{1+t}{p+1}\right) \neq 0 \quad (0 \leq t < \infty; z \in U)$$

and

$$Re\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = Re\left\{(p+1) + (1+t)\, q(z)\right\} > 0 \quad (0 \leq t < \infty; z \in U).$$

Therefore, by using Lemma 1, we deduce that $L(z,t)$ is a subordination chain. It follows from the definition of subordination chain that

$$\phi(z) = G(z) + \frac{zG'(z)}{p+1} = L(z,0),$$

and

$$L(z,0) \prec L(z,t) \quad (0 \leq t < \infty),$$

which implies that

$$L(\zeta,t) \notin L(U,t) = \phi(U) \quad (0 \leq t < \infty; \zeta \in \partial U). \tag{2.12}$$

If F is not subordinate to G, by using Lemma 4, we know that there exist two points $z_0 \in U$ and $\zeta_0 \in \partial U$ such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F'(z_0) = (1+t)\, \zeta_0 G'(\zeta_0) \quad (0 \leq t < \infty). \tag{2.13}$$

Hence, by virtue of (1.6) and (2.13), we have

$$L(\zeta_0,t) = G(\zeta_0) + \frac{(1+t)\, zG'(\zeta_0)}{p+1} = F(z_0) + \frac{z_0 F'(z_0)}{p+1} = \frac{I_p^{\alpha} f(z_0)}{z^p} \in \phi(U).$$

This contradicts to (2.12). Thus, we deduce that $F \prec G$. Considering $F = G$, we see that the function G is the best dominant. This completes the proof of Theorem 1. □

We now derive the following superordination result.

Theorem 2. Let $f, g \in A(p)$ and let

$$Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \quad \left(\phi(z) = \frac{I_p^{\alpha-1} g(z)}{z^p}; z \in U\right), \tag{2.14}$$

where δ is given by (2.2). If the function $\dfrac{I_p^{\alpha-1}g(z)}{z^p}$ is univalent in U and $\dfrac{I_p^{\alpha}g(z)}{z^p} \in \digamma$, then the superordination condition

$$\frac{I_p^{\alpha-1}g(z)}{z^p} \prec \frac{I_p^{\alpha-1}f(z)}{z^p}$$

implies that

$$\frac{I_p^{\alpha}g(z)}{z^p} \prec \frac{I_p^{\alpha}f(z)}{z^p}$$

and the function $\dfrac{I_p^{\alpha}g(z)}{z^p}$ *is the best subordinant.*

Proof. Suppose that the functions F, G and q are defined by (2.3) and (2.4), respectively. By applying the similar method as in the proof of Theorem 1, we get

$$Re\,\{q\,(z)\} > 0 \qquad (z \in U)\,.$$

Next, to arrive at our desired result, we show that $G \prec F$. For this, we suppose that the function $L\,(z,t)$ be defined by (2.11). Since G is convex, by applying a similar method as in Theorem 1, we deduce that $L\,(z,t)$ is subordination chain. Therefore, by using Lemma 5, we conclude that $G \prec F$. Moreover, since the differential equation

$$\phi\,(z) = G\,(z) + \frac{zG'\,(z)}{p+1} = \varphi\left(G\,(z)\,, zG'\,(z)\right)$$

has a univalent solution G, it is the best subordinant. This completes the proof of Theorem 2. $\square$

Combining the above-mentioned subordination and superordination results involving the operator I_p^{α}, the following "sandwich-type result" is derived.

Theorem 3. Let $f, g_j \in A\,(p)\,(j = 1,2)$ and let

$$Re\left\{1 + \frac{z\phi_j''\,(z)}{\phi_j'\,(z)}\right\} > -\delta \qquad \left(\phi_j\,(z) = \frac{I_p^{\alpha-1}g_j(z)}{z^p}\,(j = 1,2)\,; z \in U\right),$$

where δ is given by (2.2). If the function $\dfrac{I_p^{\alpha-1}g_1(z)}{z^p}$ is univalent in U and $\dfrac{I_p^{\alpha}g_1(z)}{z^p} \in \digamma$, then the condition

$$\frac{I_p^{\alpha-1}g_1(z)}{z^p} \prec \frac{I_p^{\alpha-1}f(z)}{z^p} \prec \frac{I_p^{\alpha-1}g_2(z)}{z^p}$$

implies that

$$\frac{I_p^\alpha g_1(z)}{z^p} \prec \frac{I_p^\alpha f(z)}{z^p} \prec \frac{I_p^\alpha g_2(z)}{z^p}$$

and the functions $\frac{I_p^\alpha g_1(z)}{z^p}$ and $\frac{I_p^\alpha g_2(z)}{z^p}$ *are* , respectively, the best subordinant and the best dominant.

3 Open Problem

Find sufficient conditions for normalized analytic functions $f, g_j \in A(p)$ $(j = 1, 2)$ and μ to satisfy the following sandwich-type result

$$\left(\frac{z^p}{I_p^\alpha g_1(z)}\right)^\mu \prec \left(\frac{z^p}{I_p^\alpha f(z)}\right)^\mu \prec \left(\frac{z^p}{I_p^\alpha g_2(z)}\right)^\mu.$$

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