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Cesáro and De La Vallée Poussin Means of Functions of Bounded Turning

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Abstract

In the present paper, we prove that the nth Cesáro means of first, second and third order and the nth De La Vallée Poussin mean of functions of bounded turning are also of bounded turning.

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1 Introduction

Let $\mathcal{A}$ be the class of functions f, analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Therefore, the Taylor's series expansion of any member f of the class $\mathcal{A}$ takes the following form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

For all z in $\mathbb{E}$, if $f \in \mathcal{A}$ satisfies the condition $\Re\ f'(z) > \alpha,\ 0 \leq \alpha < 1$ in $\mathbb{E}$, then f is called a function of bounded turning. Let $\mathcal{B}(\alpha)$ denote the class of functions of bounded turning. It is well-known that every close-to-convex function is univalent. In 1934/35, Noshiro [4] and Warchawski [8] obtained a simple but interesting criterion for close-to-convexity of analytic functions. They proved that if an analytic function f satisfies the condition $\Re\ f'(z) > 0$

for all z in $\mathbb{E}$, then f is close-to-convex and hence univalent in $\mathbb{E}$. It is clear that the functions in the class $\mathcal{B}(\alpha)$ are univalent close-to-convex in $\mathbb{E}$.

For all z in $\mathbb{E}$, $f \in \mathcal{A}$ and $\mathbb{N} = \{1, 2, 3, \cdots\}$. Let $\sigma_n^1(z, f)$, $\sigma_n^2(z, f)$, $\sigma_n^3(z, f)$ and $V_n(z, f)$ denote, respectively, the nth Cesáro means of first, second and third order and De La Vallée Poussin Mean of $f \in \mathcal{A}$ and define as under:

$$\sigma_n^1(z, f) = z + \sum_{k=2}^{n} \frac{n+1-k}{n} \, a_k z^k,$$

$$\sigma_n^2(z, f) = z + \sum_{k=2}^{n} \frac{(n+1-k)(n+2-k)}{n(n+1)} \, a_k z^k,$$

$$\sigma_n^3(z, f) = z + \sum_{k=2}^{n} \frac{(n+1-k)(n+2-k)(n+3-k)}{n(n+1)(n+2)} \, a_k z^k,$$

and

$$V_n(z, f) = \frac{n}{n+1} z + \sum_{k=2}^{n} \frac{n(n-1)\cdots(n+1-k)}{(n+1)(n+2)\cdots(n+k)} \, a_k z^k.$$

If f and g are two analytic functions having power series expansions

$$f(z) = \sum_{k=1}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k.$$

Then their convolution is denoted as $(f * g)(z)$ and defined by

$$(f * g)(z) = \sum_{k=1}^{\infty} a_k b_k z^k.$$

For $f \in \mathcal{A}$, the integral operator F defined as

$$F(z) = \frac{2}{z} \int_0^z f(\zeta) \, d\zeta = z + \sum_{k=2}^{\infty} \frac{2}{k+1} \, a_k z^k, \ z \in \mathbb{E},$$

is known as Libera integral operator and its nth partial sums F_n are given by

$$F_n(z) = z + \sum_{k=2}^{n} \frac{2}{k+1} \, a_k z^k.$$

Jahangiri and Farahmand [2], proved that the nth partial sums of the Libera integral operator of functions of bounded turning are also of bounded turning. In fact they proved that if $f \in \mathcal{B}(\alpha)$, then $F_n \in \mathcal{B}((4\alpha - 1)/3)$ for $1/4 \leq \alpha < 1$. Notice that the partial sums F_n do not remain with bounded turning for $\alpha < 1/4$. It is worthwhile to replace the Libera integral operator by the operators where the result holds for $0 \leq \alpha < 1$. In the present paper, we show that the Libera integral operator when replaced by the Alexander integral operator, the result holds for $0 \leq \alpha < 1$. We also show that the nth Cesáro means of first, second and third order and the nth De La Vallée Poussin mean of functions of bounded turning are also of bounded turning.

2 Preliminaries

To prove our results, we shall make use of the following lemmas.

Lemma 2.1 *If p is analytic in $\mathbb{E}$ with $p(0) = 1$ and $\Re\, p(z) > 1/2$ in $\mathbb{E}$. Then for a function q analytic in $\mathbb{E}$, the convolution function $p * q$ takes values in the convex hull of the image of $\mathbb{E}$ under q.*

Lemma 2.2 *(Rogosinski and Szegö [6]). For all θ, $0 \leq \theta \leq \pi$,*

$$\frac{1}{2} + \sum_{k=1}^{n} \frac{\cos k\theta}{k+1} \geq 0.$$

Lemma 2.3 *For all z in $\mathbb{E}$, we have*

$$\Re\left(1 + \sum_{k=2}^{n} \frac{1}{k} z^{k-1}\right) > \frac{1}{2}.$$

Proof. In view of the minimum principle for harmonic functions and Lemma 2.2, by writing $z = re^{i\theta}$, $0 \leq r \leq 1$, $0 \leq \theta \leq \pi$, we get:

$$\begin{aligned}
\Re\left(1 + \sum_{k=2}^{n} \frac{1}{k} z^{k-1}\right) &= \Re\left(1 + \sum_{k=1}^{n-1} \frac{1}{k+1} z^{k}\right) \\
&= 1 + \sum_{k=1}^{n-1} \frac{r^k \cos k\theta}{k+1} \\
&> \min_{0 \leq \theta \leq \pi} \left(1 + \sum_{k=1}^{n-1} \frac{\cos k\theta}{k+1}\right) \geq \frac{1}{2}.
\end{aligned}$$

Hence the result follows.

Lemma 2.4 *(Dhaliwal [1]). If f is a starlike function of order $1/2$ in $\mathbb{E}$, then for $n \in \mathbb{N}$,*

$$\Re\frac{\sigma_n^i(z,f)}{z} > \frac{1}{2},\ z \in \mathbb{E},\ for\ \ i = 1, 2, 3.$$

Lemma 2.5 *For a convex function f in $\mathbb{E}$, we have*

$$\Re\frac{V_n(z,f)}{z} > \frac{1}{2},\ z \in \mathbb{E}.$$

Proof. Pólya and Schoenberg [5], proved that if f is convex in $\mathbb{E}$, then $V_n(z, f)$ is convex in $\mathbb{E}$ and Marx [3] and Strohhäcker [7] proved that if f is convex in $\mathbb{E}$, then $\Re\frac{f(z)}{z} > \frac{1}{2},\ z \in \mathbb{E}$. Hence $\Re\frac{V_n(z,f)}{z} > \frac{1}{2},\ z \in \mathbb{E}$.

3 Main Results

Theorem 3.1 *If $f \in \mathcal{B}(\alpha)$ where $0 \leq \alpha < 1$, then $\sigma_n^i(z, f) \in \mathcal{B}(\alpha)$ for $i = 1, 2, 3$.*

Proof. Since $f \in \mathcal{B}(\alpha)$, therefore

$$\Re\left(1 + \sum_{k=2}^{\infty} ka_k z^{k-1}\right) > \alpha. \tag{1}$$

By definition of nth Cesáro mean of first order of $f \in \mathcal{A}$, we have

$$\sigma_n^1(z, f) = z + \sum_{k=2}^{n} \frac{n+1-k}{n}\ a_k z^k,\ z \in \mathbb{E}. \tag{2}$$

Differentiate (2) w.r.t. z, we get

$$\begin{aligned} \sigma_n^{1'}(z,f) &= 1 + \sum_{k=2}^{n} \frac{k(n+1-k)}{n}\ a_k z^{k-1} \\ &= \left(1 + \sum_{k=2}^{n} \frac{n+1-k}{n}\ z^{k-1}\right) * \left(1 + \sum_{k=2}^{\infty} ka_k z^{k-1}\right) \\ &= \frac{\sigma_n^1(z, \frac{z}{1-z})}{z} * \left(1 + \sum_{k=2}^{\infty} ka_k z^{k-1}\right) \end{aligned} \tag{3}$$

Using Lemma 2.1, Lemma 2.4 and the condition (1), from (3), we obtain:

$$\Re\, \sigma_n^{1'}(z,f) > \alpha \text{ and hence } \sigma_n^1(z,f) \in \mathcal{B}(\alpha).$$

A little calculation yields

$$\sigma_n^{2'}(z,f) = \frac{\sigma_n^2(z,\frac{z}{1-z})}{z} * \left(1 + \sum_{k=2}^{\infty} k a_k z^{k-1}\right)$$

and

$$\sigma_n^{3'}(z,f) = \frac{\sigma_n^3(z,\frac{z}{1-z})}{z} * \left(1 + \sum_{k=2}^{\infty} k a_k z^{k-1}\right)$$

Similar to the above case, the result in case of nth Cesáro means of second and third order follows from the use of Lemma 2.1, Lemma 2.4 and the condition (1).

Theorem 3.2 *If $f \in \mathcal{B}(\alpha)$ where $0 \leq \alpha < 1$, then $\Re\, V_n'(z,f) > \alpha$.*

Proof. By definition of De La Vallée Poussin mean of $f \in \mathcal{A}$, we have

$$V_n(z,f) = \frac{n}{n+1} z + \sum_{k=2}^{n} \frac{n(n-1)\cdots(n+1-k)}{(n+1)(n+2)\cdots(n+k)}\, a_k z^k. \tag{4}$$

Differentiate (4) w.r.t. z, we get

$$\begin{aligned} V_n'(z,f) &= \frac{n}{n+1} + \sum_{k=2}^{n} \frac{n(n-1)\cdots(n+1-k)}{(n+1)(n+2)\cdots(n+k)}\, k a_k z^{k-1} \\ &= \left(\frac{n}{n+1} + \sum_{k=2}^{n} \frac{n(n-1)\cdots(n+1-k)}{(n+1)(n+2)\cdots(n+k)}\, z^{k-1}\right) \\ &\quad * \left(1 + \sum_{k=2}^{\infty} k a_k z^{k-1}\right) \\ &= \frac{V_n(z,\frac{z}{1-z})}{z} * \left(1 + \sum_{k=2}^{\infty} k a_k z^{k-1}\right) \end{aligned} \tag{5}$$

Using Lemma 2.1, Lemma 2.5 and the condition (1), from (5), we get:

$$\Re\, V_n'(z,f) > \alpha. \text{ This completes the proof.}$$

For all z in $\mathbb{E}$ and $f \in \mathcal{A}$, the integral operator G defined as

$$G(z) = \int_0^z \frac{f(\zeta)}{\zeta}\, d\zeta = z + \sum_{k=2}^{\infty} \frac{1}{k}\, a_k z^k,$$

is called the Alexander integral operator and its nth partial sums G_n are given by

$$G_n(z) = z + \sum_{k=2}^{n} \frac{1}{k}\, a_k z^k. \tag{6}$$

Theorem 3.3 *If $f \in \mathcal{B}(\alpha)$, then $G_n(z) \in \mathcal{B}(\alpha)$ where $0 \le \alpha < 1$.*

Proof. Differentiate (6) w.r.t. z, we get

$$\begin{aligned} G_n'(z) &= 1 + \sum_{k=2}^{n} a_k z^{k-1} \\ &= \left(1 + \frac{1}{2(1-\alpha)} \sum_{k=2}^{\infty} k a_k z^{k-1}\right) * \left(1 + 2(1-\alpha) \sum_{k=2}^{n} \frac{1}{k} z^{k-1}\right) \end{aligned} \tag{7}$$

Since the condition (1) can be rewritten as

$$\Re\left(1 + \frac{1}{2(1-\alpha)} \sum_{k=2}^{\infty} k a_k z^{k-1}\right) > \frac{1}{2}. \tag{8}$$

In view of Lemma 2.3, we obtain

$$\Re\left(1 + 2(1-\alpha) \sum_{k=2}^{n} \frac{1}{k} z^{k-1}\right) > \alpha. \tag{9}$$

Using the conditions (8) and (9) in the light of Lemma 2.1, from (7), we get:

$$\Re\, G_n'(z) > \alpha \text{ and hence } G_n(z) \in \mathcal{B}(\alpha).$$

4 Open Problem

We, here, prove that the partial sums of the Alexander integral operator of functions of bounded turning are also of bounded turning. It is worthwhile to prove this result for more general integral operators.

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