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On a subclass of p-valent Prestarlike Functions with Negative Coefficient Defined by Dziok-Srivastava Linear operator

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Abstract

In this manuscript, a new subclass of multivalent prestarlike functions with negative coefficients defined by Dziok-Srivastava linear operator is introduced. Coefficient estimate, distortion Theorems associated with fractional derivative operator are investigated for this class. Further class preserving integral operator, extreme points, radii of p-valently convexity and other interesting properties for the said class have been determined.

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1 Introduction

Let S_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in N = 1, 2, 3, ...), \tag{1}$$

which are analytic and multivalent in the unit disk $U = \{z : |z| < 1\}$. Also denote by T_p the class of functions of the form

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \ \ (z \in U) \ \ (a_{k+p} \geq 0), \tag{2}$$

which are analytic and multivalent in U.
A function $f(z) \in S_p$ is said to be starlike of order α $(0 \leq \alpha < p)$, denoted by $S_p(\alpha)$, if and only if

$$Re\{\frac{zf'(z)}{f(z)}\} > \alpha \ \ (z \in U), \tag{3}$$

and it is called convex of order α $(0 \leq \alpha < p)$, denoted by $K_p(\alpha)$, if and only if

$$Re\{1+\frac{zf''(z)}{f'(z)}\} > \alpha \quad (z \in U). \tag{4}$$

The classes $S_p(\alpha)$ and $K_p(\alpha)$ were introduced by Patil and Thakare [5].
The function

$$S_\gamma(z) = z^p(1-z)^{-2(p-\gamma)} \quad (0 \leq \gamma < p, p = 1,2,3,...), \tag{5}$$

is the familiar extremal function for the class $S_p(\gamma)$.
Setting

$$C(\gamma,k) = \frac{\prod_{i=2}^{k+1} (2(p-\gamma)+i-2)}{k!} \qquad k \geq 1, \tag{6}$$

then $S_\gamma(z)$ can be written in the form

$$S_\gamma(z) = z^p+\sum_{k=1}^{\infty} C(\gamma,k)\ z^{p+k}. \tag{7}$$

We note that $C(\gamma,k)$ is a decreasing function in γ and that

$$\lim_{k\to\infty} C(\gamma,k) = \begin{cases} \infty\ , & \gamma < (2p-1)/2 \\ 1\ , & \gamma = (2p-1)/2 \\ 0\ , & \gamma > (2p-1)/2 \end{cases} . \tag{8}$$

For functions

$$f_j(z) = z^p-\sum_{k=1}^{\infty} a_{k+p,j}\ z^{k+p}\ (a_{n,j} \geq 0)\ (j=1,2), \tag{9}$$

in the class T_p, the modified Hadamard product $f_1 * f_2(z)$ of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=1}^{\infty} a_{k+p,1}\ a_{k+p,2}\ z^{k+p}. \tag{10}$$

A function $f(z) \in S_p$ is said to be p-valent $\gamma-$ pre-starlike function of order $\alpha(0 \leq \alpha < p; 0 \leq \gamma < p)$, if

$$(f * S_\gamma)(z) \in S_p(\alpha) \tag{11}$$

We denote by $R_p(\gamma,\alpha)$ the class of all p-valent $\gamma-$pre-starlike functions of order α . The class $R_p(\gamma,\alpha)$ was studied by Aouf and Siverman [1] while the class

$R_p(\alpha,\alpha) = R_p(\alpha)$ was studied by Silverman and Silvia [8], Shenan, Salim and Marouf [7] and others. We note that $R_p(\alpha) = S_p(\alpha)$ for $\alpha = 1/2$.
Let $S_p^*(\alpha) = S_p(\alpha)\bigcap T_p$, $K_p^*(\alpha) = K_p(\alpha)\bigcap T_p$ and $R_p^*(\gamma,\alpha) = R_p(\gamma,\alpha)\bigcap T_p$.

The generalized hypergeometric function is defined by

$$ {}_lF_m(\alpha_1,...,\alpha_l;\beta_1,...,\beta_m) = \sum_{k=0}^{\infty}\frac{(\alpha_1)_k...(\alpha_l)_k}{(\beta_1)_k...(\beta_m)_k}.\frac{z^k}{k!}\ , \tag{12}$$

$$(l \le m+1; m \in N_0 = \{0,1,2...\})\ ,$$

where $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1; & \text{k=0} \\ a(a+1)(a+2)...(a+k+1), & k \in N = 1,2,... \end{cases} \tag{13}$$

Corresponding to the function $h_p(\alpha_1,...,\alpha_l;\beta_1,...,\beta_m;z) = z^p\ {}_lF_m(\alpha_1,...,\alpha_l;\beta_1,...,\beta_m)$ the Dziok-Srivastava operator [3], $H_p^{l,m}(\alpha_1,...,\alpha_l;\beta_1,...,\beta_m)$ is defined by

$$H_p^{L,m}(\alpha_1,...,\alpha_l;\beta_1,...,\beta_m)f(z) = h_p(\alpha_1,...,\alpha_l;\beta_1,...,\beta_m;z)*f(z) \tag{14}$$

$$= z^p + \sum_{k=1}^{\infty}\frac{(\alpha_1)_k...(\alpha_l)_k}{(\beta_1)_k...(\beta_m)_k}\ a_{k+p}\ \frac{z^{k+p}}{(k)!}.$$

It is well known [3] that

$$\alpha_1 H_p^{L,m}(\alpha_1+1,...,\alpha_l;\beta_1,...,\beta_m)f(z) = z[H_p^{L,m}(\alpha_1,...,\alpha_l;\beta_1,...,\beta_m)f(z)]'$$

$$+(\alpha_1-p)H_p^{L,m}(\alpha_1,...,\alpha_l;\beta_1,...,\beta_m)f(z) \tag{15}$$

To make the notation simple, we write,

$$H_p^{L,m}[\alpha_1]f(z) = H_p^{L,m}(\alpha_1,...,\alpha_l;\beta_1,...,\beta_m)f(z)\ .$$

We note that special cases of the Dziok-Srivastava operator $H_p^{L,m}[\alpha_1]$ include the Hohlov linear operator [4], the Carlson-Shafer operator[2], the Ruschweyh derivative operator[6], the Srivastava-Owa fractional operators [10], and many others.
We note that $H_p^{1,0}[1]f(z) = f(z)$ and $H_p^{2,1}[1+p,1;p]f(z) = \frac{zf'(z)}{p}$

Now using $H_p^{L,m}[\alpha_1]$ we define the following subclass of analytic function.

Definition 1. For $\alpha_i \in C\ (i = 1,2,3...,l)$, $\beta_j \in C-\{0,-1,-2,..\}\ (j = 1,2,3,...,m)$, and $-1 \le B < A \le 1$, $0 \le \alpha < p$, $0 \le \gamma < p$, we let $S_p^{l,m}(\alpha_l,\beta_m,A,B,\gamma,\alpha)$ be the subclass of T_p consisting of functions $f(z)$ of the form (2) and satisfying the following condition,

$$\frac{z(H_p^{L,m}[\alpha_1](f*S_\gamma)(z))'}{H_p^{L,m}[\alpha_1](f*S_\gamma)(z)} \prec \frac{p+[pB+(A-B)(p-\alpha)]z}{1+Bz}\quad (z \in U), \tag{16}$$

where $\prec$ denote the subordination . From the definition, it follows that $f(z) \in T_p$ belongs to the class $S_p^{l,m}(\alpha_l, \beta_m, A, B, \gamma, \alpha)$ if there exists a function $w(z)$ regular in U and satisfying $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$ such that

$$\frac{z(H_p^{L,m}[\alpha_1](f * S_\gamma)(z))'}{H_p^{L,m}[\alpha_1](f * S_\gamma)(z)} = \frac{p + [pB + (A - B)(p - \alpha)]w(z)}{1 + Bw(z)} \quad (z \in U). \qquad (17)$$

The condition (17) is equivalent to

$$\left| \frac{\frac{z\left(H_p^{L,m}[\alpha_1](f*S_\gamma)(z)\right)'}{H_p^{L,m}[\alpha_1](f*S_\gamma)(z)} - p}{pB+(A-B)(p-\alpha)-B\frac{z\left(H_p^{L,m}[\alpha_1](f*S_\gamma)(z)\right)'}{H_p^{L,m}[\alpha_1](f*S_\gamma)(z)}} \right| < 1 \, , \, z \in U. \qquad (18)$$

It may be noted that the class $S_p^{l,m}(\alpha_l, \beta_m, A, B, \gamma, \alpha)$ extends the classes of starlike and convex functions for suitable choice of $l, m, \alpha_i, \beta_j, A, B,\ \gamma$ and α. For example
i) For $A = l = \alpha_1 = -B = 1$ and $m = 0$ the class $S_p^{l,m}(\alpha_l, \beta_m, A, B, \gamma, \alpha)$ reduces to the class of γ-prestarlike functions of order α.
i) For $A = l = \alpha_1 = -B = 1$, $m = 0$ and $\gamma = \frac{2p-1}{2}$ the class $S_p^{l,m}(\alpha_l, \beta_m, A, B, \gamma, \alpha)$ reduces to the class of starlike functions of order α.
(ii) For $A = m = -B = \alpha_2 = 1$, $l = 2$, $\alpha_1 = 1 + p$ and $\beta_1 = p$ we obtain the class of convex function of order α.

among several interesting definitions of fractional derivative and fractional integral given in literature we find it to be convenient to to restrict ourselves to the following definition used by Owa [9].

Definition 2. The fractional integral of order λ for a function $f(z)$ is defined by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}}\, dt \ , \qquad (19)$$

where $\lambda > 0, f(z)$ is analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z - t)^{\lambda-1}$ is removed by requiring $log(z - t)$ to be real when $(z - t) > 0$.

Definition 3. The fractional derivative of order λ for a function $f(z)$ is defined by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\lambda}}\, dt \ , \qquad (20)$$

where $0 \leq \lambda < 1,\ f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z - t)^{-\lambda}$ is removed by requiring $log(z - t)$ to be real when $(z - t) > 0$.

Definition 4. Under the hypothesis of Definition 2, the fractional derivative of order $n+\lambda$ of $f(z)$ is defined by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z), \quad (0 \le \lambda < 1, \ n \in N_0) \tag{21}$$

Following Owa and Srivastava [10], we introduce the linear operator $U_z^{(\lambda,p)}$ defined by

$$U_z^{(\lambda,p)} f(z) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda} f(z) \ , \tag{22}$$

where $D_z^{\lambda} f(z)$ is the fractional derivative of f, of order λ $(0 \le \lambda < 1)$, while

$$U_z^{(0,p)} f(z) = f(z) \quad ; \quad U_z^{(1,p)} f(z) = \frac{z f'(z)}{p}$$

Lemma 1. [9]. If $0 \le \lambda < 1$ then,

$$D_z^{\lambda} z^p = \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)} \ z^{p-\lambda} \tag{23}$$

Applying Lemma 1 for the function $f(z)$ defined by (2) we have from (17)

$$U_z^{(\lambda,p)} f(z) = z^p - \sum_{k=1}^{\infty} a_{k+p} \ \frac{(p+1)_k}{(p+1-\lambda)_k} \ z^{k+p}, \ (k \ge 1) \tag{24}$$

2 Coefficient Estimates

Theorem 1. A function $f(z)$ defined by (2) is in the class $S_p^{l,m}(\alpha_l, \beta_m, A, B, \gamma, \alpha)$, $\alpha_i \in C \, (i = 1,2,3...,l)$, $\beta_j \in C - \{0,-1,-2,..\}$ $(j = 1,2,3,...,m)$, $-1 \le B < A \le 1$, $0 \le \gamma < p$ and $0 \le \alpha < p$ if and only if

$$\sum_{k=1}^{\infty} \{k(1-B) + (A-B)(p-\alpha)\} \, \phi(p,k) C(\gamma,k) \, a_{k+p} \le (A-B)(p-\alpha) \ , \tag{25}$$

where $C(\gamma,k)$ is given by (6) and

$$\phi(p,k) = \frac{(\alpha_1)_k ... (\alpha_l)_k}{(1)_k (\beta_1)_k ... (\beta_m)_k} \ . \tag{26}$$

and the result is sharp.

Proof.
Assuming that (25) holds and $|z| = 1$, we have

$|z(H_p^{L,m}[\alpha_1](f * S_\gamma)(z))' - pH_p^{L,m}[\alpha_1](f * S_\gamma)(z)|$

$$-|\{(pB+(A-B)(p-\alpha))H_p^{L,m}[\alpha_1](f*S_\gamma)(z)\}-Bz(H_p^{L,m}[\alpha_1](f*S_\gamma)(z))'|$$

$$=|\sum_{k=1}^{\infty}(k)\phi(p,k)C(\gamma,k)\ a_{k+p}\ z^{k+p}|-|(A-B)(p-\alpha)z^p$$

$$+\sum_{k=1}^{\infty}\{kB-(A-B)(p-\alpha)\}\,\phi(p,k)C(\gamma,k)\ a_{k+p}\ z^{k+p}|$$

$$\leq\sum_{k=1}^{\infty}\{k(1-B)+(A-B)(p-\alpha)\}\,\phi(p,k)C(\gamma,k)\ a_{k+p}$$

$$-(A-B)(p-\alpha)$$

$$\leq 0.$$

Hence by maximum modulus principle $f(z)\in S_p^{l,m}(\alpha_l,\beta_m,A,B,\gamma,\alpha)$.
Conversely, assume that $f(z)$ is in the class $S_p^{l,m}(\alpha_l,\beta_m,A,B,\gamma,\alpha)$. Then

$$\left|\frac{\frac{z(H_p^{L,m}[\alpha_1](f*S_\gamma)(z))'}{H_p^{L,m}[\alpha_1](f*S_\gamma)(z)}-p}{pB+(A-B)(p-\alpha)-B\frac{z(H_p^{L,m}[\alpha_1](f*S_\gamma)(z))'}{H_p^{L,m}[\alpha_1](f*S_\gamma)(z)}}\right|<1\ ,\ z\in U. \tag{27}$$

$$=\frac{\left|\sum_{k=1}^{\infty}k\phi(p,k)C(\gamma,k)\ a_{k+p}\ z^{k+p}\right|}{\left|(A-B)(p-\alpha)z^p+\sum_{k=1}^{\infty}\{Bk-(A-B)(p-\alpha)\}\,\phi(p,k)C(\gamma,k)\ a_{k+p}\ z^{k+p}\right|}$$

Since $|Re(z)|\leq|z|$ for any z,we find from (27) that

$$Re\left\{\frac{\sum_{k=1}^{\infty}k\phi(p,k)C(\gamma,k)\ a_{k+p}\ z^{k+p}}{(A-B)(p-\alpha)z^p+\sum_{k=1}^{\infty}\{kB-(A-B)(p-\alpha)\}\phi(p,k)C(\gamma,k)\ a_{k+p}\ z^{k+p}}\right\}<1 \tag{28}$$

Now choosing, the value of z on the real axis so that $\frac{z(H_p^{L,m}[\alpha_1](f*S_\gamma)(z))'}{H_p^{L,m}[\alpha_1](f*S_\gamma)(z)}$ is real, then upon clearing the denominator in (28) and letting $z\to 1$ through real values, we have

$$\sum_{k=1}^{\infty}k\phi(p,k)C(\gamma,k)\ a_{k+p}\leq(A-B)(p-\alpha)$$

$$+\sum_{k=1}^{\infty}\{kB-(A-B)(p-\alpha)\}\,\phi(p,k)C(\gamma,k)\ a_{k+p}$$

which gives the desired assertion(25).

Finally ,we note that equality in (25) holds for the function

$$f(z) = z - \frac{(A-B)(p-\alpha)}{\{k(1-B)) + (A-B)(p-\alpha)\}\, \phi(p,k)C(\gamma,k)\ a_{k+p}}\, z^{k+p}. \tag{29}$$

3 Distortion Theorem

Theorem 2. Let the function $f(z)$, defined by (2), be in the class $S_p^{l,m}(\alpha_l, \beta_m, A, B, \gamma, \alpha)$. Then

$$|D_z^{\lambda} f(z)| \geq \frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)} |z|^{p-\lambda}$$
$$\times \left\{ 1 - \frac{2(p-\gamma)(A-B)(p-\alpha)(p+1) \prod_{j=1}^{m} \beta_j}{[(1-B)+(A-B)(p-\alpha)](p+1-\lambda) \prod_{i=1}^{l} \alpha_i} |z| \right\} \tag{30}$$

and

$$|D_z^{\lambda} f(z)| \leq \frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)} |z|^{p-\lambda}$$
$$\times \left\{ 1 + \frac{2(p-\gamma)(A-B)(p-\alpha)(p+1) \prod_{j=1}^{m} \beta_j}{[(1-B)+(A-B)(p-\alpha)](p+1-\lambda) \prod_{i=1}^{l} \alpha_i} |z| \right\} \tag{31}$$

for $z \in U$ *and* $0 < \lambda \leq 1$.
Equalities in (30)and (31)are attained by the function given by (29).

Proof. In view of Theorem 1, we have

$$\sum_{k=1}^{\infty} a_{k+p} \leq \frac{2(p-\gamma)(A-B)(p-\alpha) \prod_{j=1}^{m} \beta_j}{[(1-B)+(A-B)(p-\alpha)] \prod_{i=1}^{l} \alpha_i} \tag{32}$$

we have from (22)

$$U_z^{(\lambda,p)} f(z) = z^p - \sum_{k=1}^{\infty} a_{k+p}\ \delta(k)\ z^{k+p},\ (k \geq 1) \tag{33}$$

where $\delta(k)$ is given by

$$\delta(k) = \frac{(p+1)_k}{(p+1-\lambda)_k} \tag{34}$$

It is easily seen that $\delta(k)$ is non-increasing , that is , it satisfies the inequality $\delta(k+1) \leq \delta(k)$ for all $k \geq 1$, and we have

$$0 < \delta(k) \leq \delta(p+1) = \frac{(p+1)}{(p+1-\lambda)}. \tag{35}$$

Consequently, we obtain

$$|U_z^{(\lambda,p)} f(z)| \geq |z|^p - |z|^{p+1} \sum_{k=1}^{\infty} a_{k+p}\delta(k)$$

$$\geq |z|^p - \delta(p+1)|z|^{p+1} \sum_{k=1}^{\infty} a_{k+p}$$

$$\geq |z|^p - \frac{2(p-\gamma)(A-B)(p-\alpha)(p+1)(\prod_{j=1}^{m} \beta_j)}{[(1-B)+(A-B)(p-\alpha)] \prod_{i=1}^{l} \alpha_i} |z|^{p+1}$$

which proves (30).
similarly (31) can be proved.

Corollary1 . Let the function $f(z)$, defined by (8), be in the class $S_p^{l,m}(\alpha_l, \beta_m, A, B, \gamma, \alpha)$. Then

$$|f(z)| \geq |z|^p \times \left\{ 1 - \frac{2(p-\gamma)(A-B)(p-\alpha) \prod_{j=1}^{m} \beta_j}{[(1-B)+(A-B)(p-\alpha)] \prod_{i=1}^{l} \alpha_i} |z| \right\} \tag{36}$$

and

$$|f(z)| \leq |z|^p \times \left\{ 1 + \frac{2(p-\gamma)(A-B)(p-\alpha) \prod_{j=1}^{m} \beta_j}{[(1-B)+(A-B)(p-\alpha)] \prod_{i=1}^{l} \alpha_i} |z| \right\}. \tag{37}$$

The result is sharp for the function defined in (29).
Proof. The proof follows readily from Theorem 2 in the special case when $\lambda = 0$.

Corollary2 . Let the function $f(z)$, defined by (2), be in the class $S_p^{l,m}(\alpha_l, \beta_m, A, B, \gamma, \alpha)$. Then

$$|f'(z)| \geq p^2 |z|^p \times \left\{ 1 - \frac{2(p-\gamma)p(A-B)(p-\alpha) \prod_{j=1}^{m} \beta_j}{[(1-B)+(A-B)(p-\alpha)] \prod_{i=1}^{l} \alpha_i} |z| \right\} \tag{38}$$

and

$$|f'(z)| \leq p^2|z|^p \times \left\{ 1 + \frac{2(p-\gamma)p(A-B)(p-\alpha)\prod\limits_{j=1}^{m}\beta_j}{[(1-B)+(A-B)(p-\alpha)]\prod\limits_{i=1}^{l}\alpha_i}\,|z| \right\}. \tag{39}$$

The result is sharp for the function defined in (29).
Proof. The proof follows readily from Theorem 2 in the special case when $\lambda = 1$.

4 Radius of convexity

Theorem 3. Let the function $f(z)$, defined by (2), be in the class $S_p^{l,m}(\alpha_l, \beta_m, A, B, \gamma, \alpha)$, then $f(z)$ is convex in the disc $|z| < r_1$

$$r_1 = inf_{k\geq 1}\left(\frac{p^2[k(1-B)+(A-B)(p-\alpha)]\phi(p,k)C(\gamma,k)}{(k+p)^2(A-B)(p-\alpha)}\right)^{\frac{1}{k}}. \tag{40}$$

The result is sharp for the function $f(z)$ given by (29).
Proof. To establish the required result it is sufficient to show that

$$\left|1 + \frac{zf''(z)}{f'(z))} - p\right| \leq p,\ for|z| < r_1\ . \tag{41}$$

we have

$$\left|1 + \frac{zf''(z)}{f'(z))} - p\right| = \left|\frac{\sum\limits_{k=1}^{\infty} k(k+p)a_{k+p}z^k}{p - \sum\limits_{k=1}^{\infty}(k+p)a_{k+p}z^k}\right|$$

$$\leq \frac{\sum\limits_{k=1}^{\infty} k(k+p)a_{k+p}|z|^k}{p - \sum\limits_{k=1}^{\infty}(k+p)a_{k+p}|z|^k}$$

Therefore, if (41) is true, then

$$\sum_{k=1}^{\infty} k(k+p)a_{k+p}z^k \leq p\left\{p - \sum_{k=1}^{\infty}(k+p)a_{k+p}z^k\right\}$$

that is

$$\sum_{k=1}^{\infty}\frac{(k+p)^2}{p^2}a_{k+p}|z|^k \leq 1\ . \tag{42}$$

By virtue of Theorem 1, (42) is true if

$$|z| \leq \left(\frac{p^2[k(1-B)+(A-B)(p-\alpha)]\phi(p,k)C(\gamma,k)}{(k+p)^2(A-B)(p-\alpha)}\right)^{\frac{1}{k}}.$$

Thus $f(z)$ is p-valently convex in $|z| < r_1$, where r_1 is given by (40).

5 Integral operators

Theorem 4. Let c be a real number such that $c > -p$ if $f(z) \in S_p^{l,m}(\alpha_l, \beta_m, A, B, \gamma, \alpha)$ then the function $F(z)$ defined by

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\, dt \tag{43}$$

also belongs to $S_p^{l,m}(\alpha_l, \beta_m, A, B, \gamma, \alpha)$

Proof. Let $f(z)$ defined by (2) be in the class $S_p^{l,m}(\alpha_l, \beta_m, A, B, \gamma, \alpha)$ then from the representation of $F(z)$

$$F(z) = z^p - \sum_{k=1}^{\infty} b_{p+k}\, z^{p+k} \tag{44}$$

where

$$b_{p+k} = \frac{c+p}{c+p+k}\, a_{p+k} < a_{p+k} \tag{45}$$

Therefore

$$\sum_{k=1}^{\infty}[(1-B)k + (A-B)(p-\alpha)]\phi(p,k)C(\gamma,k)b_{k+p}$$

$$< \sum_{k=1}^{\infty}[(1-B)k + (A-B)(p-\alpha)]\phi(p,k)C(\gamma,k)a_{k+p}$$

$$\le (A-B)(p-\alpha).$$

Since $f(z) \in S_p^{l,m}(\alpha_l, \beta_m, A, B, \gamma, \alpha)$ hence by Theorem 1, $F(z) \in S_p^{l,m}(\alpha_l, \beta_m, A, B, \gamma, \alpha)$.

6 Extreme points of the class $S_p^{l,m}(\alpha_l, \beta_m, A, B, \gamma, \alpha)$.

Theorem 5. Let $f_p(z) = z^p$ and

$$f_{p+k}(z) = z^p - \frac{(A-B)(p-\alpha)}{[(1-B)k + (A-B)(p-\alpha)]\phi(p,k)C(\gamma,k)} z^{k+p}, \ (k \ge 1) \tag{46}$$

Then $f(z) \in S_p^{l,m}(\alpha_l, \beta_m, A, B, \gamma, \alpha)$ if and only if it can be expressed in the form

$$f(z) = \lambda_p f_p(z) + \sum_{k=1}^{\infty} \lambda_{p+k} f_{p+k}(z), \tag{47}$$

where $\lambda_{p+k} \ge 0$ and $\sum\limits_{k=o}^{\infty} \lambda_{k+p} = 1$.

Proof. Let (47) holds, then by (46) we have

$$f(z) = z^p - \sum_{k=1}^{\infty} \frac{(A-B)(p-\alpha)}{[(1-B)k + (A-B)(p-\alpha)]\phi(p,k)C(\gamma,k)} \lambda_{p+k} z^{p+k}.$$

Now

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{\{k(1-B)+(A-B)(p-\alpha)\}\,\phi(p,k)C(\gamma,k)}{(A-B)(p-\alpha)}\\&\qquad\times\sum_{k=1}^{\infty}\frac{(A-B)(p-\alpha)}{[(1-B)k+(A-B)(p-\alpha)]\phi(p,k)C(\gamma,k)}\lambda_{p+k}\\=&\sum_{k=1}^{\infty}\lambda_{k+p}\\=&\ 1-\lambda_p\leq 1.\end{aligned}$$

hence by Theorem 1, $f(z)\in S_p^{l,m}(\alpha_l,\beta_m,A,B,\gamma,\alpha)$
Conversely, suppose $f(z)\in S_p^{l,m}(\alpha_l,\beta_m,A,B,\gamma,\alpha)$. Since

$$a_{p+k}\leq\frac{(A-B)(p-\alpha)}{[(1-B)k+(A-B)(p-\alpha)]\phi(p,k)C(\gamma,k)}\lambda_{p+k}\ ,\quad (k\geq 1)$$

setting $\lambda_{p+k}=\frac{\{k(1-B)+(A-B)(p-\alpha)\}\phi(p,k)C(\gamma,k)}{(A-B)(p-\alpha)}\ a_{p+k}$ and $\lambda_p=1-\sum\limits_{k=1}^{\infty}\lambda_{k+p}$, we get (47). This completes the proof of the Theorem.

7 Open Problem

One can define a new subclass of multivalent uniformly functions with negative coefficients instead of class of prestarlike functions using the same linear operator defined in this paper and hence new results can be obtained.

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