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Certain Classes of Univalent Functions With Negative Coefficient

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Abstract

In this paper, the authors introduce and study the classes $S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$, $S^*_{c,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$ and $S^*_{sc,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$ consisting of analytic functions with negative coefficients defined by using Aouf et al derivative operator.These classes are respectively, n-ω-λ-l-starlike with respect to symmetric points, n-ω-λ-l-starlike with respect to conjugate points and n-ω-λ-l-starlike with respect to symmetric conjugate points. Properties such as coefficient estimates , distortion theorem , extreme points , radius theorem , e.t.c and the consequences of the parametrics involved are discussed.

Keywords: *analytic functions, univalent functions, coefficient bounds, growth theorem, distortion theorems, radii of starlikeness,convexity, close-to-convexity, Auof operator.*

1 Introduction

Let $A(\omega)$ denote the class of function of the form

$$f(z) = (z-\omega) + \sum_{k=2}^{\infty} a_k (z-\omega)^k \quad a_k \geq 0 \tag{1}$$

which are analytic and univalent in the open unit disc $U = \{z : |z| < 1\}$, and normalized with $f(\omega) = 0$ and $f'(\omega) - 1 = 0$, where ω is a fixed point in U. $S(\omega) \subset A(\omega)$ denote the class of analytic and univalent functions. [1,2,4,5].

Definition 1.1 (5). *Let $f(z)$ be defined by (1) and ω is a fixed point in U, for a function $f(z) \in S(\omega)$ we define $I_\omega^n(\lambda, l) : A(\omega) \to A(\omega)$ as follows*

$$\begin{aligned}
& I_\omega^0(\lambda,l)f(z) = f(z) \\
& I_\omega^1(\lambda,l)f(z) = I_\omega(\lambda,l)f(z) \\
& = I_\omega^0(\lambda,l)f(z)\tfrac{1-\lambda+l}{1+l} + (I_\omega^0(\lambda,l)f(z))'\tfrac{\lambda(z-\omega)}{1+l} \\
& = (z-\omega) + \textstyle\sum_{k=2}^{\infty}\left(\tfrac{1+\lambda(k-1)+l}{1+l}\right) a_k(z-\omega)^k, \\
I_\omega^2(\lambda,l)f(z) \quad & = I_\omega^1(\lambda,l)f(z)\tfrac{1-\lambda+l}{1+l} + (I_\omega^1(\lambda,l)f(z))'\tfrac{\lambda(z-\omega)}{1+l} \\
& = (z-\omega) + \textstyle\sum_{k=2}^{\infty}\left(\tfrac{1+\lambda(k-1)+l}{1+l}\right)^2 a_k(z-\omega)^k
\end{aligned}$$

and in general

$$\begin{aligned}
& I_\omega^n(\lambda,l)f(z) = I_\omega(I_\omega^{n-1}(\lambda,l)f(z)) \\
& = (z-\omega) + \textstyle\sum_{k=2}^{\infty}\left(\tfrac{1+\lambda(k-1)+l}{1+l}\right)^n a_k(z-\omega)^k \quad n \in N_0, \lambda \geq 0, l \geq 0.
\end{aligned} \tag{2}$$

Definition 1.2. *Let the function $f(z)$ be defined by (1) , then $f(z) \in S^*_{n,l,\lambda}(\omega)$ if and only if*

$$Re\frac{I_\omega^{n+1}(\lambda,l)f(z)}{I_\omega^n(\lambda,l)f(z)} > 0 \quad n \in N_0, \lambda \geq 0, l \geq 0, (z \in U) \tag{3}$$

and $S^*_{n,\lambda,l}(\omega)$ denote the class of ω-n-λ-l-starlike function.

Definition 1.3. *Let the function $f(z)$ be defined by (1) and $S^*_{s,n,\lambda,l}(\omega)$ be the subclass of $A(\omega)$ consisting of functions of the form (1) satisfying*

$$Re\frac{I_\omega^{n+1}(\lambda,l)f(z)}{I_\omega^n(\lambda,l)f(z) - I_\omega^n(\lambda,l)f(-z)} > 0 \quad n \in N_0, \lambda \geq 0, l \geq 0, (z \in U). \tag{4}$$

These classes of functions are called starlike with respect to symmetric points, and ω is a fixed point in U.

Definition 1.4. *Let $f(z)$ be defined by (1). Then the class $S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$ of functions $f(z)\in S(\omega)$ if it satisfies the following condition :*

$$\left|\frac{I^{n+1}_\omega(\lambda,l)f(z)}{I^n_\omega(\lambda,l)f(z)-I^n_\omega f(\lambda,l)(-z)}-1\right|<\beta\left|\frac{\alpha I^{n+1}_\omega(\lambda,l)f(z)}{I^n_\omega(\lambda,l)f(z)-I^n_\omega(\lambda,l)f(-z)}+1\right|, \quad (5)$$

for some $0\le\alpha\le1$,$0<\beta\le1$,$l\ge0$,$\lambda\ge0$ and $z\in U$, and ω is a fixed point in U. We denote the class of n-λ-l-starlike with respect to symmetric points by $S^*_{s,n,\lambda,l}T(\omega,\alpha,\beta)$.
Let $T(\omega)$ denote the subclass of $S(\omega)$ consisting of functions of the form:

$$f(z)=(z-\omega)-\sum_{k=2}^{\infty}a_k(z-\omega)^k \quad (a_k\ge0). \quad (6)$$

Definition 1.5. *Let the function $f(z)$ be defined by (6). Then $f(z)$ is said to be ω-n-λ-l-starlike with respect to symmetric points if it satisfies the following condition:*

$$\left|\frac{I^{n+1}_\omega(\lambda,l)f(z)}{I^n_\omega(\lambda,l)f(z)-I^n_\omega(\lambda,l)f(-z)}-1\right|<\beta\left|\frac{\alpha I^{n+1}_\omega(\lambda,l)f(z)}{I^n_\omega(\lambda,l)f(z)-I^n_\omega(\lambda,l)f(-z)}+1\right|, \quad (7)$$

where $n\in N_0=N\bigcup\{0\}$,$0\le\alpha\le1$, $0<\beta\le1$,$l\ge0$,$\lambda\ge0$,$0\le\frac{2(1-\beta)}{1+\alpha\beta}<1$,and $z\in U$. We denote the class of ω-n-λ-l-starlike with respect to symmetric points by $S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$, ω is a fixed point in U.

Definition 1.6. *Let the function $f(z)$ be defined by (6). Then $f(z)$ is said to be ω-n-λ-l-starlike with respect to conjugate points if it satisfies the following condition:*

$$\left|\frac{I^{n+1}_\omega(\lambda,l)f(z)}{I^n_\omega(\lambda,l)f(z)+\overline{I^n_\omega(\lambda,l)f(\overline{z})}}-1\right|<\beta\left|\frac{\alpha I^{n+1}_\omega(\lambda,l)f(z)}{I^n_\omega(\lambda,l)f(z)+\overline{I^n_\omega(\lambda,l)f(\overline{z})}}+1\right|, \quad (8)$$

where $n\in N_0=N\bigcup\{0\}$,$0\le\alpha\le1$, $0<\beta\le1$,$l\ge0$,$\lambda\ge0$,$0\le\frac{2(1-\beta)}{1+\alpha\beta}<1$, and $z\in U$,and ω is a fixed point in U. We denote the class of ω-n-λ-l-starlike with respect to conjugate points by $S^*_{c,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$.

Definition 1.7. *Let the function $f(z)$ be defined by (6). Then $f(z)$ is said to be ω-n-λ-l-starlike with respect to symmetric conjugate points if it satisfies the following condition :*

$$\left|\frac{I^{n+1}_\omega(\lambda,l)f(z)}{I^n_\omega(\lambda,l)f(z)-\overline{I^n_\omega(\lambda,l)f(\overline{-z})}}-1\right|<\beta\left|\frac{\alpha I^{n+1}_\omega(\lambda,l)f(z)}{I^n_\omega(\lambda,l)f(z)-\overline{I^n_\omega(\lambda,l)f(\overline{-z})}}+1\right|, \quad (9)$$

where $n\in N_0=N\bigcup\{0\}$,$0\le\alpha\le1$, $0<\beta\le1$,$l\ge0$,$\lambda\ge0$,$0\le\frac{2(1-\beta)}{1+\alpha\beta}<1$,and $z\in U$,and ω is a fixed point in U. We denote the class of ω-n-λ-l-starlike with respect to symmetric conjugate points by $S^*_{sc,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$.

2 Coefficient estimates

Let all the parameters remain as earlier defined except otherwise defined. Then we state and proof the following.

Theorem 2.1. *Let function $f(z)$ be defined by (6) and $I_\omega^n(\lambda,l)f(z)-I_\omega^n(\lambda,l)f(-z)\neq 0$ for $z\neq 0$. Then $f(z)\in S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$ if and only if*

$$\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n\left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)(1+\alpha\beta)+(\beta-1)[1-(-1)^k]\right] a_k(r+d)^{k-1}\leq\beta(\alpha+2)-1 \quad (10)$$

Proof: Using Definiton , (6) and (2) , that is

$$\left|\frac{I_\omega^{n+1}(\lambda,l)f(z)}{I_\omega^n(\lambda,l)f(z)-I_\omega^n(\lambda,l)f(-z)}-1\right|<\beta\left|\frac{\alpha I_\omega^{n+1}(\lambda,l)f(z)}{I_\omega^n(\lambda,l)f(z)-I_\omega^n(\lambda,l)f(-z)}+1\right|$$

$$f(z)=(z-\omega)-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n a_k(z-\omega)^k$$

and

$$f(z)=-(z-\omega)-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n a_k(-1)^k(z-\omega)^k$$

then we have

$$\begin{aligned}|(z-\omega)-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^{n+1} a_k(z-\omega)^k-(z-\omega)+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n \\ a_k(z-\omega)^k-(z-\omega)-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n(-1)^k a_k(z-\omega)^k| \\ <\beta|\alpha(z-\omega)-\sum_{k=2}^{\infty}\alpha\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^{n+1} a_k(z-\omega)^k+(z-\omega) \\ -\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n a_k(z-\omega)^k+(z-\omega) \\ +\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n(-1)^k a_k(z-\omega)^k|\end{aligned}$$

which readily gives

$$\left|-(z-\omega)-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)-1+(-1)^{k}\right]a_{k}(z-\omega)^{k}\right|$$
$$<\beta\left|(\alpha+2)(z-\omega)-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^{n}\left[\alpha\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+1-(-1)^{k}\right]\right|\cdot$$
$$\cdot\left|a_{k}(z-\omega)^{k}\right|$$

Also,

$$|-(z-\omega)|+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[\frac{1+\lambda(k-1)+l}{1+l}-1+(-1)^{k}\right]a_{k}\left|(z-\omega)^{k}\right|$$
$$\leq\beta\left|(\alpha+2)(z-\omega)\right|-$$
$$-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[\alpha\beta\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+\beta-\beta(-1)^{k}\right]a_{k}\left|(z-\omega)^{k}\right|$$

That is

$$|(z-\omega)|+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)-1+(-1)^{k}\right]a_{k}\left|(z-\omega)^{k}\right|$$
$$+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[\alpha\beta\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+\beta-\beta(-1)^{k}\right]a_{k}\left|(z-\omega)^{k}\right|$$
$$\leq\beta(\alpha+2)\left|(z-\omega)\right|$$

which gives

$$\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)(1+\alpha\beta)+(\beta-1)[1-(-1)^{k}]\right]$$
$$\cdot a_{k}\left|(z-\omega)^{k}\right|\leq\beta(\alpha+2)\left|(z-\omega)\right|-\left|(z-\omega)\right|$$

Let $|(z-\omega)|=r+d$. So we have

$$\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)(1+\alpha\beta)+(\beta-1)[1-(-1)^{k}]\right]$$
$$\cdot a_{k}(r+d)^{k-1}\leq\beta(\alpha+2)-1$$

Hence by the maximum modulus theorem, we have $f\in S^{*}_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$.

For the converse, we assume that

$$\left|\frac{\frac{I_\omega^{n+1}(\lambda,l)f(z)}{I_\omega^n(\lambda,l)f(z)-I_\omega^n(\lambda,l)f(-z)}-1}{\frac{I_\omega^{n+1}(\lambda,l)f(z)}{I_\omega^n(\lambda,l)f(z)-I_\omega^n(\lambda,l)f(-z)}+1}\right| =$$

$$= \left|\frac{(z-\omega)+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n\left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)-1+(-1)^k\right]a_k(z-\omega)^k}{(\alpha+2)(z-\omega)-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n\left[\alpha\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+1-(-1)^k\right]a_k(z-\omega)^k}\right|$$

$$< \beta$$

Since $|R_e z| < |(z-\omega)|$ for all z, we have

$$R_e\left\{\frac{(z-\omega)+\sum\limits_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n\left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)-1+(-1)^k\right]a_k(z-\omega)^k}{(\alpha+2)(z-\omega)-\sum\limits_{k=2}^{\infty}\left(\frac{1+(k-1)+l}{1+l}\right)^n\left[\alpha\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+1-(-1)^k\right]a_k(z-\omega)^k}\right\} < \beta \quad (11)$$

Choose values of z on the real axis so that $\frac{I_\omega^{n+1}(\lambda,l)f(z)}{I_\omega^n(\lambda,l)f(z)-I_\omega^n(\lambda,l)f(-z)}$ is real and $I_\omega^n(\lambda,l)f(z)-I_\omega^n(\lambda,l)f(-z)\neq 0$ for $z\neq\omega$. Upon clearing the denominator in (11) and letting $(z)\to(r+d)$ through real values, we obtain

$$1+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n\left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)-1+(-1)^k\right]$$

$$\leq \beta(\alpha+2)-\beta\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n\left[\alpha\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+1-(-1)^k a_k\right]$$

This gives the required condition.

Corollary 2.2. *Let the function $f(z)$ be defined by (6) be in the class $S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$. Then we have*

$$a_k \leq \frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n\left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)(1+\alpha\beta)+(\beta-1)[1-(-1)^k]\right](r+d)^{k-1}} \quad (12)$$

$$(k\geq 2 \quad, n\in N_0, l\geq 0, \lambda\geq 0).$$

The equality in (12) is attained for the function $f(z)$ given by

$$f(z)=(z-\omega)-\frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n\left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)(1+\alpha\beta)+(\beta-1)[1-(-1)^k]\right](r+d)^{k-1}}$$

$$\cdot(z-\omega)^k \quad (k\geq 2 \quad, n\in N_0, l\geq 0, \lambda\geq 0) \quad (13)$$

Theorem 2.3. *Let the function $f(z)$ be defined by (6). Then $f(z) \in S^*_{c,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$ if and only if*

$$\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n\left[\frac{1+\lambda(k-1)+l}{1+l}(1+\alpha\beta)+2(\beta-1)\right]a_k(r+d)^{k-1} \quad (14)$$
$$\leq \beta(\alpha+2)-1$$

Proof: Using Definition and (2), we have:

$$\left|\frac{I_\omega^{n+1}(\lambda,l)f(z)}{I_\omega^n(\lambda,l)f(z)+\overline{I_\omega^n(\lambda,l)f(\overline{z})}}-1\right|<\beta\left|\frac{\alpha I_\omega^{n+1}(\lambda,l)f^(z)}{I_\omega^n(\lambda,l)f(z)+\overline{I_\omega^n(\lambda,l)f(\overline{z})}}+1\right|$$

and we let

$$\overline{I_\omega^n(\lambda,l)f(\overline{z})}=(z-\omega)-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n a_k(z-\omega)^k$$

such that

$$\begin{aligned}
&|(z-\omega)-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^{n+1}a_k(z-\omega)^k-(z-\omega)\\
&+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n a_k(z-\omega)^k-(z-\omega)\\
&+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n a_k(z-\omega)^k|\\
&<\beta|\alpha(z-\omega)-\sum_{k=2}^{\infty}\alpha\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^{n+1}a_k(z-\omega)^k\\
&+(z-\omega)-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n a_k(z-\omega)^k+(z-\omega)\\
&-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n a_k(z-\omega)^k|
\end{aligned}$$

which gives

$$\begin{aligned}
&|-(z-\omega)-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n\left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)-2\right]a_k(z-\omega)^k|\\
&<\beta|(\alpha+2)(z-\omega)\\
&-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n\left[\alpha\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+2\right]a_k(z-\omega)^k|
\end{aligned}$$

that is,

$$|(z-\omega)|+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)-2\right]a_k|(z-\omega)^k|$$
$$\leq \beta(\alpha+2)|(z-\omega)|-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)\alpha\beta+2\beta\right]$$
$$\cdot a_k|(z-\omega)^k|$$

Hence,

$$\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)(1+\alpha\beta)+2(\beta-1)\right]a_k|(z-\omega)^k|$$
$$\leq \beta(\alpha+2)|(z-\omega)|-|(z-\omega)|$$

let $|z-\omega|=r+d$, then

$$\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)(1+\alpha\beta)+2(\beta-1)\right]a_k(r+d)^{k-1}$$
$$\leq \beta(\alpha+2)-1$$

Corollary 2.4. *Let function $f(z)$ defined by (6) be in the class $S^*_{c,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$. Then we have*

$$a_k \leq \frac{\beta(\alpha+2)-1}{\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+2(\beta-1)\right](r+d)^{k-1}} \tag{15}$$
$$(k\geq 2, n\in N_0, \lambda\geq 0, l\geq 0.)$$

The equality in (15) is attained for the function $f(z)$ given by

$$f(z)=(z-\omega)-$$
$$-\frac{\beta(\alpha+2)-1}{\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+2(\beta-1)\right](r+d)^{k-1}}(z-\omega)^k$$
$$(k\geq 2, n\in N_0, \lambda\geq 0, l\geq 0.) \tag{16}$$

Theorem 2.5. *Let the function $f(z)$ be defined by (6) be in the class $S^*_{sc,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$. Then we have*

$$\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^k]\right] \tag{17}$$
$$\cdot a_k(r+d)^{k-1}\leq[\beta(\alpha+2)-1]$$

Proof: Using definition 1.7, we have,

$$\left|\frac{I_\omega^{n+1}(\lambda,l)f(z)}{I_\omega^n(\lambda,l)f(z)-\overline{I_\omega^n(\lambda,l)f(\overline{-z})}}-1\right|<\beta\left|\frac{\alpha I_\omega^{n+1}(\lambda,l)f^(z)}{I_\omega^n(\lambda,l)f(z)-\overline{I_\omega^n(\lambda,l)f(\overline{-z})}}+1\right|,$$

here

$$\overline{I_\omega^n(\lambda,l)f(\overline{-z})}=-(z-\omega)-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n a_k(-1)^k(z-\omega)^k$$

and

$$\begin{aligned}&(z-\omega)-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n+1}a_k(z-\omega)^k-(z-\omega)\\&+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}a_k(z-\omega)^k-(z-\omega)\\&-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}a_k(-1)^k(z-\omega)^k|-\beta|\alpha(z-\omega)\\&-\sum_{k=2}^{\infty}\alpha\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n+1}a_k(z-\omega)^k\\&+(z-\omega)-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}a_k(z-\omega)^k+(z-\omega)\\&+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}a_k(-1)^k(z-\omega)^k<0\end{aligned}$$

which readily yields

$$\begin{aligned}&|-(z-\omega)-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)-1+(-1)^k\right]\\&\cdot a_k(z-\omega)^k|-\beta|(\alpha+2)(z-\omega)\\&-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[\alpha\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+1-(-1)^k\right]a_k(z-\omega)^k|<0\end{aligned}$$

that is,

$$\begin{aligned}&|(z-\omega)|+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[\frac{1+\lambda(k-1)+l}{1+l}-1+(-1)^k\right]\\&\cdot a_k|(z-\omega)^k|+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[\alpha\beta\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+\beta-\beta(-1)^k\right]\\&\cdot a_k|(z-\omega)^k|-\beta(\alpha+2)|(z-\omega)|<0\end{aligned}$$

and that

$$\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^{k}]\right]$$
$$\cdot a_k|(z-\omega)^k|-[\beta(\alpha+2)|(z-\omega)|-|(z-\omega)|]<0$$

Let $|(z-\omega)|=r+d$.
Hence

$$\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^{k}]\right]$$
$$\cdot a_k(r+d)^{k-1}-[\beta(\alpha+2)-1]<0$$

Corollary 2.6. *Let function $f(z)$ defined by (6) be in the class $S^*_{sc,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$. Then we have*

$$a_k\leq\frac{\beta(\alpha+2)-1}{\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^{k}]\right](r+d)^{k-1}} \tag{18}$$
$$(k\geq 2, n\in N_0, \lambda\geq 0, l\geq 0).$$

The equality in (18) is attained for the function $f(z)$ given by

$$f(z)=(z-\omega)-$$
$$-\frac{\beta(\alpha+2)-1}{\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^{k}](r+d)^{k-1}\right]}(z-\omega)^k$$
$$(k\geq 2, n\in N_0, \lambda\geq 0, l\geq 0.) \tag{19}$$

3 Distortion theorem

Theorem 3.1. *Let function $f(z)$ defined by (6) be in the class $S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$. Then we have*

$$|(z-\omega)|-\frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n+1-i}(1+\alpha\beta)(r+d)}\quad|(z-\omega)|^2\leq\left|I^i_\omega(\lambda,l)f(z)\right|$$
$$\leq|(z-\omega)|+\frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n+1-i}(1+\alpha\beta)(r+d)}\quad|(z-\omega)|^2 \tag{20}$$

for $z\in U$, where $0\leq i\leq n$, and ω is a fixed point in U. The result is sharp.

Proof: Note that $f(z) \in S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$, if and only if $I^i_\omega(\lambda,l)f(z) \in S^*_{s,n-i,\lambda,l}(\omega)T(\omega,\alpha,\beta)$ and

$$I^i_\omega(\lambda,l)f(z) = (z-\omega) - \sum_{k=2}^{\infty}\left(\frac{1+\lambda+l}{1+l}\right)^i a_k(z-\omega)^k \tag{21}$$

Using Theorem 2.1, we know that

$$\begin{aligned}&\left(\frac{1+\lambda+l}{1+l}\right)^{n+1-i}(1+\alpha\beta)(r+d)\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^i a_k\\ &\le \sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n\left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)(1+\alpha\beta)+(\beta-1)[1-(-1)^k]\right]\\ &\cdot a_k(r+d)^{k-1} \le \beta(2+\alpha)-1\end{aligned} \tag{22}$$

That is, that

$$\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^i a_k \le \frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n+1-i}(1+\alpha\beta)(r+d)} \tag{23}$$

It follows from (21) and (23) that

$$\begin{aligned}|I^i_\omega(\lambda,l)f(z)| &\ge |(z-\omega)| - |(z-\omega)|^2\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^i a_k\\ &\ge |(z-\omega)| - \frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n+1-i}(1+\alpha\beta)(r+d)}|(z-\omega)|^2\end{aligned} \tag{24}$$

and

$$\begin{aligned}|I^i_\omega(\lambda,l)f(z)| &\le |(z-\omega)| + |(z-\omega)|^2\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^i a_k\\ &\le |(z-\omega)| + \frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n+1-i}(1+\alpha\beta)(r+d)}|(z-\omega)|^2\end{aligned} \tag{25}$$

Finally, we note that the equality in (20) is attained by the function

$$I^i_\omega(\lambda,l)f(z) = (z-\omega) - \frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n+1-i}(1+\alpha\beta)(r+d)}\quad(z-\omega)^2 \tag{26}$$

or by

$$f(z) = (z-\omega) - \frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n+1}(1+\alpha\beta)(r+d)} \quad (z-\omega)^2 \tag{27}$$

Corollary 3.1: Let the function $f(z)$ defined by (6) be in the class $S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$. Then we have

$$|(z-\omega)| - \frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n+1}(1+\alpha\beta)(r+d)} \quad |(z-\omega)|^2 \leq |f(z)| \leq$$
$$|(z-\omega)| + \frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n+1}(1+\alpha\beta)(r+d)} \quad |(z-\omega)|^2 \tag{28}$$

for $z \in U$,where ω is a fixed point in U. The result is sharp for the function $f(z)$ given by (27).
Proof: Taking i = 0 in Theorem 3.1, we can easily show (28).

Corollary 3.2. *Let the function $f(z)$ defined by (6) be in the class $S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$. Then we have*

$$1 - \frac{2\beta(2+\alpha)-2}{\left(\frac{1+\lambda+l}{1+l}\right)^{n+1}(1+\alpha\beta)(r+d)} \quad |(z-\omega)| \leq |f'(z)|$$
$$\leq 1 + \frac{2\beta(2+\alpha)-2}{\left(\frac{1+\lambda+l}{1+l}\right)^{n+1}(1+\alpha\beta)(r+d)} \quad |(z-\omega)| \tag{29}$$

for $z \in U$, where ω is a fixed point in U. The result is sharp for the function $f(z)$ giving by (27)

Theorem 3.3. *Let the function be defined by (6) be in the class $S^*_{c,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$. Then we have*

$$|(z-\omega)| - \frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n-i}\left[\left(\frac{1+\lambda+l}{1+l}\right)\alpha\beta + \frac{\lambda+(2\beta-1)(1+l)}{1+l}\right](r+d)} \quad |(z-\omega)|^2 \leq |I^i_\omega(\lambda,l)f(z)|$$
$$\leq |(z-\omega)| + \frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n-i}\left[\left(\frac{1+\lambda+l}{1+l}\right)\alpha\beta + \frac{\lambda+(2\beta-1)(1+l)}{1+l}\right](r+d)} \quad |(z-\omega)|^2 \tag{30}$$

for $z \in U$, where $0 \leq i \leq n$, and ω is a fixed point in U. The result is sharp, *for the function $f(z)$ giving by*

$$I_\omega^i(\lambda,l)f(z) = (z-\omega) - \frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n-i}\left[\left(\frac{1+\lambda+l}{1+l}\right)\alpha\beta + \frac{\lambda+(2\beta-1)(1+l)}{1+l}\right](r+d)} \quad |(z-\omega)|^2 \tag{31}$$

or by

$$f(z) = (z-\omega) - \frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n}\left[\left(\frac{1+\lambda+l}{1+l}\right)\alpha\beta + \frac{\lambda+(2\beta-1)(1+l)}{1+l}\right](r+d)} \quad (z-\omega)^2 \tag{32}$$

Corollary 3.4. *Let the function $f(z)$ defined by (6) be in the class $S^*_{c,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$. Then we have*

$$|(z-\omega)| - \frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n}\left[\left(\frac{1+\lambda+l}{1+l}\right)\alpha\beta + \frac{\lambda+(2\beta-1)(1+l)}{1+l}\right](r+d)} \quad |(z-\omega)|^2 \leq |f(z)|$$

$$\leq |(z-\omega)| + \frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n}\left[\left(\frac{1+\lambda+l}{1+l}\right)\alpha\beta + \frac{\lambda+(2\beta-1)(1+l)}{1+l}\right](r+d)} \quad |(z-\omega)|^2 \tag{33}$$

for $z \in U$, where ω is a fixed point in U. The result is sharp for the function $f(z)$ given by (32).

Corollary 3.5. *Let the function $f(z)$ defined by (6) be in the class $S^*_{c,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$. Then we have*

$$1 - \frac{2\beta(2+\alpha)-2}{\left(\frac{1+\lambda+l}{1+l}\right)^{n}\left[\left(\frac{1+\lambda+l}{1+l}\right)\alpha\beta + \frac{\lambda+(2\beta-1)(1+l)}{1+l}\right](r+d)} \quad |(z-\omega)| \leq f'(z)$$

$$\leq 1 + \frac{2\beta(2+\alpha)-2}{\left(\frac{1+\lambda+l}{1+l}\right)^{n}\left[\left(\frac{1+\lambda+l}{1+l}\right)\alpha\beta + \frac{\lambda+(2\beta-1)(1+l)}{1+l}\right](r+d)} \quad |(z-\omega)| \tag{34}$$

for $z \in U$, and ω is a fixed point in U. The result is sharp for the function given by (32).

Theorem 3.6. *Let function $f(z)$ be defined by (6) be in the class $S^*_{sc,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$. Then we have*

$$|(z-\omega)| - \frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n+1-i}(1+\alpha\beta)(r+d)} \quad |(z-\omega)|^2 \leq |I_\omega^i(\lambda,l)f(z)|$$

$$\leq |(z-\omega)| + \frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n+1-i}(1+\alpha\beta)(r+d)} \quad |(z-\omega)|^2 \tag{35}$$

for $z \in U$. where $0 \leq i \leq n$, and ω is a fixed point in U. The result is sharp.

4 Extreme points

Theorem 4.1. *The class* $S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$ *is closed under convex linear combination.*

Proof: Let the functions

$$f_j(z) = (z-\omega) - \sum_{k=2}^{\infty} a_{k,j}(z-\omega)^k \quad (a_{k,j} \geq 0; j = 1, 2)$$

be in the class $S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$ and ω is a fixed point in U. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \lambda f_1(z) + (1-\lambda)f_2(z) \quad (0 \leq \lambda \leq 1) \tag{36}$$

is in the class $S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$. Since for $0 \leq \lambda \leq 1$,

$$h(z) = (z-\omega) - \sum_{k=2}^{\infty}[\lambda a_{k,1} + (1-\lambda)a_{k,2}](z-\omega)^k$$

With the aid of Theorem 2.1, we have

$$\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n\left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)(1+\alpha\beta) + (\beta-1)[1-(-1)^k](r+d)^{k-1}\right]$$
$$\cdot\,[\lambda a_{k,1} + (1-\lambda)a_{k,2}] \leq \beta(\alpha+2) - 1$$

which implies that $h(z) \in S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$. As a consequence of Theorem 2.1, there exist extreme points of the class $S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$.

Theorem 4.2. *Let* $f_1(z) = (z-\omega)$ *and*

$$f_k(z) = (z-\omega) -$$
$$-\frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right) + (\beta-1)[1-(-1)^k]\right](r+d)^{k-1}}(z-\omega)^k \tag{37}$$

for $0 \leq \alpha \leq 1, l \geq 0$, $\lambda \geq 0$, $0 < \beta \leq 1$, $n \in N_o$, *and* ω *is a fixed point in* U. *Then* $f(z)$ *is in the class* $S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha\beta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=2}^{\infty}\lambda_k f_k(z) \tag{38}$$

where $\lambda_k > 0 \quad (k > 1)$ *and*

$$\sum_{k=2}^{\infty}\lambda_k = 1$$

Proof: Suppose that

$$f(z)=\sum_{k=2}^{\infty}\lambda_k f_k(z)=(z-\omega)-$$
$$-\frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^k]\right](r+d)^{k-1}}$$
$$\cdot\,\lambda_k(z-\omega)^k, \tag{39}$$

then we get that

$$\sum_{k=2}^{\infty}\frac{\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^k]\right](r+d)^{k-1}}{\beta(2+\alpha)-1}\times$$
$$\times\frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^k]\right](r+d)^{k-1}}\lambda_k$$
$$=\sum_{k=2}^{\infty}\lambda_k=1-\lambda_1\le 1. \tag{40}$$

By virtue of Theorem 2.1, this shows that $f(z)\in S^{*}_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$.
On the other hand, suppose that the function $f(z)$ defined by (6) is in the class $S^{*}_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$. Again, by using Theorem 2.1, we can show that

$$a_k\le\frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^k]\right](r+d)^{k-1}}$$
$$(k\ge 2, n\in N_0) \tag{41}$$

setting

$$\lambda_k=\frac{\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^k]\right](r+d)^{k-1}}{\beta(2+\alpha)-1}$$
$$(k\ge 2, n\in N_0) \tag{42}$$

and

$$\lambda_1=1-\sum_{k=2}^{\infty}\lambda_k \tag{43}$$

We can see that $f(z)$ can be expressed in the form (38). This completes the proof of Theorem 4.2.

Corollary 4.3. *The extreme points of the class $S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$ are the function $f_k(z)$ $(k \geq 1)$ given by Theorem 4.2.*

Theorem 4.4. *Let $f_1(z) = (z-\omega)$ and*

$$f_k(z) = (z-\omega) - \frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n \left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right) + 2(\beta-1)\right](r+d)^{k-1}}(z-\omega)^k \quad (k \geq 2)$$

for $0 \leq \alpha \leq 1$, $\lambda \geq 0$, $l \geq 0$ and $0 < \beta \leq 1$ and $n \in N_0$, where ω is a fixed point in U. Then $f(z)$ is in the class $S^*_{c,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=2}^{\infty} \lambda_k f_k(z)$$

where $\lambda_k > 0$ $(k > 1)$ and

$$\sum_{k=2}^{\infty} \lambda_k = 1$$

Corollary 4.5. *The extreme points of the class $S^*_{c,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$ are the function $f_k(z)$ $(k > 1)$ given by Theorem 4.4.*

Theorem 4.6. *Let $f_k(z) = (z-\omega)$ and*

$$f_k(z) = (z-\omega) - \frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n \left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right) + (\beta-1)[1-(-1)^k]\right](r+d)^{k-1}}(z-\omega)^k \quad (k \geq 2)$$

for $0 \leq \alpha \leq 1$, $\lambda \geq 0$, $l \geq 0$ and $0 < \beta \leq 1$ and $n \in N_0$, where ω is a fixed point in U. Then $f(z)$ is in the class $S^*_{sc,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=2}^{\infty} \lambda_k f_k(z)$$

where $\lambda_k > 0$ $(k > 1)$ and

$$\sum_{k=2}^{\infty} \lambda_k = 1$$

Corollary 4.7. *The extreme points of the class* $S^*_{sc,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$ *are the functions* $f_k(z)$ $(k > 1)$ *given by Theorem 4.6.*

5 Radii of close-to-convexity,starlikeness and convexity

Theorem 5.1. *Let the function* $f(z)$ *defined by (6) be in the class* $S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$. *Then* $f(z)$ *is* ω-n-λ-l- *close-to-convex of order* δ $(0 < \delta < 1)$ *in* $|(z-\omega)| < r_1$: *where*

$r_1 =$

$$= \inf_k \left[\frac{(1-\delta)\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^{n-1}\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^k]\right](r+d)^{k-1}}{\beta(2+\alpha)-1} \right]^{\frac{1}{k-1}}$$

$(k \geq 2) \quad (43*)$

The result is sharp with the extremal function given by (13).

Proof: for ω-n-λ-l-close-to-convexity it is sufficient to show that

$$|f'(z) - 1| \leq 1 - \delta$$

for $|(z-\omega)| < r_1$. we have

$$|f'(z) - 1| \leq \sum_{k=2}^{\infty} \left(\frac{1+\lambda(k-1)+l}{1+l} \right) a_k |(z-\omega)|^{k-1}.$$

Thus

$$|f'(z) - 1| \leq 1 - \delta$$

if

$$\sum_{k=2}^{\infty} \left(\frac{1+\lambda(k-1)+l)}{(1+l)(1-\delta)} \right) a_k |(z-\omega)|^{k-1} \leq 1 \tag{44}$$

According to Theorem 2.1, we have

$$\sum_{k=2}^{\infty} \frac{\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^n \left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^k]\right]}{\beta(2+\alpha)-1} (r+d)^{k-1} a_k \leq 1(45)$$

Hence (41) will be true if

$$\left(\frac{1+\lambda(k-1)+l)}{(1+l)(1-\delta)}\right)|(z-\omega)|^{k-1} \leq$$
$$\leq \frac{\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^{n}\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^{k}]\right]}{\beta(2+\alpha)-1}(r+d)^{k-1}$$

Or if

$$|(z-\omega)| <$$
$$< \left[\frac{(1-\delta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n-1}\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^{k}]\right](r+d)^{k-1}}{\beta(2+\alpha)-1}\right]^{\frac{1}{k-1}}$$
$$(k>2) \tag{46}$$

The theorem follows from (46)

Theorem 5.2. *Let the function $f(z)$ defined by (6) be in the class $S^{*}_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$. Then $f(z)$ is starlike of order δ $(0 \leq \delta \leq 1)$ in $|(z-\omega)| < r_2$, where*

$$r_2 = \inf_{k}\left[\frac{(1-\delta)\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^{n}\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^{k}]\right](r+d)^{k-1}}{\left(\frac{\lambda(k-1)+(1+l)(1-\delta)}{1+l}\right)[\beta(2+\alpha)-1]}\right]^{\frac{1}{k-1}}$$
$$(k \geq 2) \quad (46*)$$

The result is sharp with the extremal function given by (13) and r_2 attains its infimum for $k=2$

Proof: it is sufficient to show that

$$\left|\frac{(z-\omega)f'(z)}{f(z)}-1\right| \leq 1-\delta \quad for \quad |(z-\omega)| < r_2.$$

We have

$$\left|\frac{(z-\omega)f'(z)}{f(z)}-1\right| \leq \frac{\sum_{k=2}^{\infty}\left(\frac{\lambda(k-1)}{1+l}\right)a_k|z-\omega|^{k-1}}{1-\sum_{k=2}^{\infty}a_k|(z-\omega)|^{k-1}}$$

Thus

$$\left|\frac{f'(z)}{f(z)}-1\right| \leq 1-\delta$$

if

$$\sum_{k=2}^{\infty}\left(\frac{\lambda(k-1)+(1+l)(1-\delta)}{(1+l)(1-\delta)}\right)a_k\left|(z-\omega)\right|^{k-1}\leq 1 \tag{47}$$

Hence by using (45)and (47) will be true if

$$\left(\frac{\lambda(k-1)+(1+l)(1-\delta)}{(1+l)(1-\delta)}\right)|z-\omega|^{k-1}$$
$$\leq\frac{\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^{n}\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^k]\right](r+d)^{k-1}}{\beta(2+\alpha)-1}$$

$(k\geq 2)$ or if

$$|z-\omega|<\left[\frac{(1-\delta)\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^{n}\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^k]\right](r+d)^{k-1}}{\left(\frac{\lambda(k-1)+(1+l)(1-\delta)}{1+l}\right)[\beta(2+\alpha)-1]}\right]^{\frac{1}{k-1}}$$
$$(k\geq 2) \tag{48}$$

The theorem follows easily from (48)
Remarks: It is clear that r_2 attains its infimum at k=2 for the function $f(z)$ given by

$$f(z)=(z-\omega)-\frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n+1}(1+\alpha\beta)(r+d)^{k-1}}(z-\omega)^2$$

Also , we have

$$\left|\frac{(z-\omega)f'(z)}{f(z)}-1\right|=|(z-\omega)|\left|\frac{\beta(2+\alpha)-1}{\left(\frac{1+\lambda+l}{1+l}\right)^{n+1}(1+\alpha\beta)(r+d)-[\beta(2+\alpha)-1]\,z-\omega}\right|$$

Then

$$\frac{[\beta(2+\alpha)-1]\,|(z-\omega)|}{\left(\frac{1+\lambda+l}{1+l}\right)^{n+1}(1+\alpha\beta)(r+d)-[\beta(2+\alpha)-1]\,|(z-\omega)|}\leq 1-\delta$$

that is, we have

$$(2-\delta)[\beta(2+\alpha)-1]\,|(z-\omega)|\leq(1-\delta)\left[\left(\frac{1+\lambda+l}{1+l}\right)^{n+1}(1+\alpha\beta)(r+d)\right]$$

Then we have

$$|(z-\omega)|\leq\frac{(1-\delta)\left[\left(\frac{1+\lambda+l}{1+l}\right)^{n+1}(1+\alpha\beta)(r+d)\right]}{(2-\delta)[\beta(2+\alpha)-1]}$$

Corollary 5.3. *Let the function $f(z)$ defined by (6) be in the class $S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$. Then $f(z)$ is convex of order δ $(0 \leq \delta < 1)$ in $|(z-\omega)| < r_3$ where*

$$r_3 = \inf_k \left[\frac{(1-\delta)\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^{n-1} \left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right) + (\beta-1)[1-(-1)^k(r+d)^{k-1}]\right]}{\left(\frac{\lambda(k-1)+(1+l)(1-\delta)}{(1+l)(1-\delta)}\right)[\beta(2+\alpha)-1]} \right]^{\frac{1}{k-1}}$$

$(k \geq 2)$ $(48*)$

The result is sharp with the extreme function given by (13).

6 Integral operators

Theorem 6.1. *Let the function $f(z)$ defined by (6) be in the class $S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha\beta)$ and c be a real number such that $c > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{(z-\omega)^c} \int_\omega^z (t-\omega)^{c-1} f(t)dt \tag{49}$$

also belongs to the class $S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$.

Proof: From the representation of $F(z)$, it follows that

$$F(z) = (z-\omega) - \sum_{k=2}^{\infty} b_k (z-\omega)^k \tag{50}$$

where

$$b_k = \left(\frac{(c+1)(1+l)}{\lambda(k-1)+(c+1)(1+l)}\right) a_k \tag{51}$$

Therefore

$$\sum_{k=2}^{\infty} \left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n \left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)(1+\alpha\beta) + (\beta-1)[1-(-1)^k]\right]$$

$$\cdot (r+d)^{k-1} b_k = \sum_{k=2}^{\infty} \left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n$$

$$\cdot \left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)(1+\alpha\beta) + (\beta-1)[1-(-1)^k]\right]$$

$$\cdot (r+d)^{k-1} \left(\frac{(c+1)(1+l)}{\lambda(k-1)+(c+1)(1+l)}\right) a_k$$

$$\leq \sum_{k=2}^{\infty} \left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n \left[\left(\frac{1+\lambda(k-1)+l}{1+l}\right)(1+\alpha\beta) + (\beta-1)[1-(-1)^k]\right]$$

$$\cdot (r+d)^{k-1} a_k \leq \beta(2+\alpha) - 1$$

Since $f(z) \in S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$. Hence, by Theorem 2.1, $F(z) \in S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$.

Theorem 6.2. *Let c be a real number such that* $c > -1$. *If* $F(z) \in S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$, *then the function F(z) defined by (49) is univalent in* $|(z-\omega)| < r^*$, *where*

$$r^* = \inf_k \left[\frac{\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^{n-1} \left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)(\beta-1)[1-(-1)^k]\right](r+d)^{k-1}}{\left(\frac{\lambda(k-1)+(c+1)(1+l)}{(c+1)(1+l)}\right)[\beta(2+\alpha)-1]} \right]^{\frac{1}{k-1}}$$

$(k \geq 2)$ (52∗)

The result is sharp.

Proof:
Let $F(z) = (z-\omega) - \sum_{k=2}^{\infty} a_k (z-\omega)^k \quad (a_k \geq 0)$ It follows from (49) that

$$f(z) = \frac{(z-\omega)^{1-c}[(z-\omega)^c F(z)]'}{c+1} = (z-\omega) - \sum_{k=2}^{\infty} \left(\frac{\lambda(k-1)+(c+1)(1+l)}{(c+1)(1+l)}\right) a_k (z-\omega)^k$$

$$(c > -1) \tag{52}$$

In order to obtain the required result it suffices to show that $|f'(z)-1|$ in $|(z-\omega)| < r^*$. Now

$$|f'(z)-1| < \sum_{k=2}^{\infty} \left(\frac{1+\lambda(k-1)+l}{1+l}\right)\left(\frac{\lambda(k-1)+(c+1)(1+l)}{(c+1)(1+l)}\right) a_k \left|(z-\omega)\right|^{k-1}$$

Thus

$$|f'(z)-1| < 1$$

if

$$\sum_{k=2}^{\infty} \left(\frac{1+\lambda(k-1)+l}{1+l}\right)\left(\frac{\lambda(k-1)+(c+1)(1+l)}{(c+1)(1+l)}\right) a_k \left|(z-\omega)\right|^{k-1} < 1 \tag{53}$$

Hence by using (49), (54) will be satisfied if

$$\left(\frac{1+\lambda(k-1)+l}{1+l}\right)\left(\frac{\lambda(k-1)+(c+1)(1+l)}{(c+1)(1+l)}\right)\left|(z-\omega)\right|^{k-1}$$

$$< \frac{\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^{n}\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^k]\right](r+d)^{k-1}}{[\beta(2+\alpha)-1]}$$

that is, if

$$|(z-\omega)| < \left[\frac{\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^{n-1} \left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)(\beta-1)[1-(-1)^k]\right](r+d)^{k-1}}{\left(\frac{\lambda(k-1)+(c+1)(1+l)}{(c+1)(1+l)}\right)[\beta(2+\alpha)-1]} \right]^{\frac{1}{k-1}} \quad (k \geq 2) \tag{54}$$

Therefore $F(z)$ is univalent in $|(z-\omega)| < r^*$. Sharpness follows if we take

$$f(z) = (z-\omega) - \frac{\left(\frac{\lambda(k-1)+(c+1)(1+l)}{(c+1)(1+l)}\right)[\beta(2+\alpha)-1]}{\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^{n-1}\left[(1+\alpha\beta)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)+(\beta-1)[1-(-1)^k]\right](r+d)^{k-1}}(z-\omega)^k$$

$$(k \geq 2 : n \in N_0 : c > -1)$$

Conclusively, with various choices of all the parameters involved the results in [6,7,8,9,10,11,12,13,14,15,16,17] could be obtained and some other new ones could be derived.

7 Open Problem

The authors suggest to study the properties of Hadamard product for two functions from the classes $S^*_{s,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$, $S^*_{c,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$ and $S^*_{sc,n,\lambda,l}(\omega)T(\omega,\alpha,\beta)$.

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