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On a Class of Harmonic Starlike Multivalent Meromorphic Functions

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Abstract.

We use the differential operator $HD^n_{\lambda,p} f$ to introduce a new class $\sum_{H_p}(n,\lambda,p)$ of multivalent meromorphic starlike harmonic functions in punctured open unit disc $U^* = \{z : 0 < |z| < 1\} = U \setminus \{0\}$. We give sufficient coefficient condition for this class. Also, this coefficient condition is shown to be necessary if the coefficients of the co-analytic part of the harmonic functions are negative. Furthermore, we determine the inclusion relation, extreme points, distortion and covering theorems, convolution and convex combination conditions for these functions.

Keywords: *Multivalent, Meromorphic, starlike, harmonic functions, differential operator.*

AMS Mathematics Subject Classification: *30C45.*

1 Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subset C$ if both u and v are real harmonic in D. In any simply connected domain D we write $f = h + \bar{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D. See Clunie and Sheil-Small [6]. There are numerous papers on harmonic functions defined in the exterior of the unit disc $\widetilde{U}=\{z:|z|>1\}$ (see [1], [7], [9]) or in punctured open unit disc $U^{*}=\{z:0<|z|<1\}$ (see [5], [10]).

Denote by Σ_{H_p} the class of p-valent harmonic functions $f=h+\overline{g}$ that are sense-preserving in U^{*} and h, g are of the form

$$h(z)=\frac{1}{z^{p}}+\sum_{k=p}^{\infty}a_{k}z^{k} \quad , \; g(z)=\sum_{k=2p}^{\infty}b_{k}z^{k} \quad . \tag{1.1}$$

Also, denote by $\Sigma_{H_p}^{-}$ the subclass of Σ_{H_p} consisting of all functions $f=h+\overline{g}$ where h, g are given by

$$h(z)=\frac{1}{z^{p}}+\sum_{k=p}^{\infty}a_{k}z^{k} \quad , \; g(z)=-\sum_{k=2p}^{\infty}b_{k}z^{k} \quad , a_{k}\, , b_{k}\geq 0 \tag{1.2}$$

We denote by $\Sigma_{H_p}^{*}(\alpha)$ the subclass of Σ_{H_p} consisting of starlike functions of order $\alpha(0\leq\alpha<p)$ in U^{*}. A necessary and sufficient condition for such f to be starlike of order α in U^{*} is

$$-\frac{\partial}{\partial\theta}\arg\left[f(re^{i\theta})\right]\geq\alpha\,, \quad z=re^{i\theta}, 0<r<1.$$

Similarly, we denote by $\Sigma_{H_p}^{c}(\alpha)$ the subclass of Σ_{H_p} consisting of convex functions of order $\alpha(0\leq\alpha<p)$ in U^{*}. A necessary and sufficient condition for such f to be convex of order α in U^{*} is

$$-\frac{\partial}{\partial\theta}\arg\left[\frac{\partial}{\partial\theta}f(re^{i\theta})\right]\geq\alpha\,, \quad z=re^{i\theta}, 0<r<1. \tag{1.3}$$

Remark 1.1. *For* $0\leq\alpha_{1}<\alpha_{2}<p,$ *we have*

$$\Sigma_{H_p}^{*}(\alpha_{2})\subset\Sigma_{H_p}^{*}(\alpha_{1})\subset\Sigma_{H_p}^{*}(0)=\Sigma_{H_p}^{*}\;,$$

$$\Sigma_{H_p}^{c}(\alpha_{2})\subset\Sigma_{H_p}^{c}(\alpha_{1})\subset\Sigma_{H_p}^{c}(0)=\Sigma_{H_p}^{c}\;.$$

The Convolution or Hadamard Product of $f(z)=\frac{1}{z^{p}}+\sum_{k=p}^{\infty}a_{k}z^{k}+\sum_{k=2p}^{\infty}\overline{b}_{k}\,\overline{z}^{k}$ and

$g(z)=\frac{1}{z^p}+\sum_{k=p}^{\infty}c_k z^k+\sum_{k=2p}^{\infty}\overline{d}_k\,\overline{z}^k$ defined as $(f*g)(z)=\frac{1}{z^p}+\sum_{k=p}^{\infty}a_k c_k z^k+\sum_{k=2p}^{\infty}\overline{b}_k\overline{d}_k\,\overline{z}^k$.

For $f\in\Sigma_{H_p}$ we define the differential operator $HD^n_{\lambda,p}f$ as

$$HD^n_{\lambda,p}f=D^n_{\lambda,p}h+\overline{D^n_{\lambda,p}g},\ n\in N_0=N\cup\{0\},\lambda\geq 0\ . \tag{1.4}$$

Where

$$\left.\begin{aligned} D^n_{\lambda,p}h(z)&=\frac{1}{z^p}+\sum_{k=p}^{\infty}[1+\lambda(k+p)]^n a_k z^k\\ D^n_{\lambda,p}g(z)&=\sum_{k=2p}^{\infty}[1+\lambda(k+p)]^n b_k z^k\end{aligned}\right\} \tag{1.5}$$

Both h and g satisfies the identity

$$D^{n+1}_{\lambda,p}h-(1+\lambda p)D^n_{\lambda,p}h=\lambda z\left(D^n_{\lambda,p}h\right)',\quad D^{n+1}_{\lambda,p}g-(1+\lambda p)D^n_{\lambda,p}g=\lambda z\left(D^n_{\lambda,p}g\right)',\quad \lambda\geq 0\ .$$

Remark 1.2. *I. If $p=1$ and the co-analytic part of f being identically zero, i.e., $g\equiv 0$, then $HD^n_{\lambda,p}f$ reduces to the differential operator which is introduced in [2].*

II. If the co-analytic part of f being identically zero, i.e., $g\equiv 0$, then $HD^n_{\lambda,p}f$ reduces to the differential operator which is introduced in [4].

Remark 1.3. *If $f\in\Sigma_{H_p}$ and the differential operator $HD^n_{\lambda,p}f$ is given by* (1.4) *and* (1.5), *then*

$$HD^{n+1}_{\lambda,p}f-(1+\lambda p)HD^n_{\lambda,p}f=\lambda z\left(HD^n_{\lambda,p}f\right)_z+\lambda\overline{z}\left(HD^n_{\lambda,p}f\right)_{\overline{z}}\ . \tag{1.6}$$

A function f in Σ_{H_p} is said to be in the class $\sum_{H_p}(n,\lambda,\alpha)$ if

$$-\operatorname{Re}\left\{\frac{z\left(HD^n_{\lambda,p}f(z)\right)_z-\overline{z}\left(HD^n_{\lambda,p}f\right)_{\overline{z}}}{HD^n_{\lambda,p}f(z)}\right\}\geq\alpha\ ,\ z\in U\ ,\ 0\leq\alpha<p, \tag{1.7}$$

Remark 1.4. *I. Putting $n=0$, we get $\sum_{H_p}(0,\lambda,\alpha)\equiv\sum^*_{H_p}(\alpha)$.*

II. If we consider $p=1$ *and the co-analytic part* $g\equiv 0$, *then* $\sum_{H_1}(n,\lambda,\alpha)\equiv MS_{\lambda}^{n}(\alpha)$ *the class of meromorphic starlike functions, studied by Al-Oboudi and Al-Zkeri [3].*

III. $f\in\Sigma_{H_p}(n,\lambda,\alpha) \quad \Leftrightarrow \quad HD_{\lambda,p}^{n}f\in\Sigma_{H_p}^{*}(\alpha)$.

Finally, we define the subclass $\sum_{H_p}^{-}(n,\lambda,\alpha)\equiv\sum_{H_p}(n,\lambda,\alpha)\cap\sum_{H_p}^{-}$.

We obtain sufficient bounds for functions in the class $\sum_{H_p}(n,\lambda,\alpha)$. This sufficient coefficient condition is shown to be also necessary for functions in the class $\sum_{H_p}^{-}(n,\lambda,\alpha)$. A representation theorem, inclusion properties, and distortion bounds for the class $\sum_{H_p}^{-}(n,\lambda,\alpha)$ are also obtained.

2 Main Results

We prove necessary and sufficient convolution condition for the class $\sum_{H_p}(n,\lambda,\alpha)$.

Theorem 2.1. *Let* $f\in\Sigma_{H_p}$. *Then* $f\in\Sigma_{H_p}(n,\lambda,\alpha)$ *if and only if*

$$h*\underbrace{u_{\lambda,p}*u_{\lambda,p}*\ldots*u_{\lambda,p}}_{n-times}*\left\{\frac{A}{z^p}+\frac{Bz^p}{1-z}+\frac{Cz^{p+1}}{(1-z)^2}\right\}$$

$$+\overline{g}*\underbrace{\overline{v}_{\lambda,p}*\overline{v}_{\lambda,p}*\ldots*\overline{v}_{\lambda,p}}_{n-times}*\left\{\frac{D\overline{z}^{2p+1}+E\overline{z}^{2p}}{(1-\overline{z})^2}\right\}\neq 0,$$

where

$$|\zeta|=1,\ A=\alpha\zeta+\alpha-2p\ ,\ B=\zeta(2p+\alpha)+\alpha\ ,\ C=\zeta+1$$

$$D=\zeta(p-\alpha-1)+3p-\alpha-1,\ \ E=-\zeta(p-\alpha)-3p+\alpha$$

$$u_{\lambda,p}(z) = \frac{1}{z^p}+\frac{[1+2\lambda p]z^p-[1+\lambda(2p-1)]z^{p+1}}{(1-z)^2},$$

$$v_{\lambda,p}(z) = \frac{[1+3\lambda p]z^{2p}-[1+\lambda(3p-1)]z^{2p+1}}{(1-z)^2}.$$

Proof. The condition (1.7) is equivalent to the following

$$-\operatorname{Re}\left\{\frac{z\left(D^n_{\lambda,p}h(z)\right)' - \overline{z\left(D^n_{\lambda,p}g(z)\right)'} + \alpha D^n_{\lambda,p}h(z) + \alpha\overline{D^n_{\lambda,p}g(z)}}{D^n_{\lambda,p}h(z) + \overline{D^n_{\lambda,p}g(z)}}\right\} \geq 0 \ . \qquad (2.1)$$

Since

$$-\left\{\frac{z\left(D^n_{\lambda,p}h(z)\right)' - \overline{z\left(D^n_{\lambda,p}g(z)\right)'} + \alpha D^n_{\lambda,p}h(z) + \alpha\overline{D^n_{\lambda,p}g(z)}}{D^n_{\lambda,p}h(z) + \overline{D^n_{\lambda,p}g(z)}}\right\}\Bigg|_{z=0} = p - \alpha \ ,$$

The required condition (2.1) is equivalent to

$$-\frac{z\left(D^n_{\lambda,p}h(z)\right)' - \overline{z\left(D^n_{\lambda,p}g(z)\right)'} + \alpha D^n_{\lambda,p}h(z) + \alpha\overline{D^n_{\lambda,p}g(z)}}{D^n_{\lambda,p}h(z) + \overline{D^n_{\lambda,p}g(z)}} \neq p\,\frac{\zeta - 1}{\zeta + 1} \qquad (2.2)$$

where $|\zeta| = 1$ and $\zeta \neq -1$.

By a simple algebraic manipulation, inequality (2.2) yields

$$0 \neq (\zeta + 1)[z\left(D^n_{\lambda,p}h(z)\right)' - \overline{z\left(D^n_{\lambda,p}g(z)\right)'} + \alpha D^n_{\lambda,p}h(z) + \alpha\overline{D^n_{\lambda,p}g(z)}]$$

$$- p(1-\zeta)[D^n_{\lambda,p}h(z) + \overline{D^n_{\lambda,p}g(z)}]$$

$$= D^n_{\lambda,p}h(z) * \left\{(\zeta + 1)\left[p\,\frac{z^{2p} + z - 1}{z^p(1-z)} + \frac{z^{p+1}}{(1-z)^2}\right] + [\zeta(\alpha + p) + \alpha - p]\left[\frac{1 - z + z^{2p}}{z^p(1-z)}\right]\right\}$$

$$+ \overline{D^n_{\lambda,p}g(z)} * \left\{-(\zeta + 1)\overline{\left(\frac{2pz^{2p}}{1-z} + \frac{z^{2p+1}}{(1-z)^2}\right)} + [\zeta(\alpha + p) + \alpha - p]\overline{\left(\frac{z^{2p}}{1-z}\right)}\right\}$$

$$= D^n_{\lambda,p}h(z) * \left\{\frac{\alpha\zeta + \alpha - 2p}{z^p} + \frac{[\zeta(2p + \alpha) + \alpha]z^p}{1-z} + \frac{(\zeta + 1)z^{p+1}}{(1-z)^2}\right\}$$

$$+ \overline{D^n_{\lambda,p}g(z)} * \left\{\frac{[\zeta(p - \alpha - 1) + 3p - \alpha - 1]\bar{z}^{2p+1} - [\zeta(p - \alpha) + 3p - \alpha]\bar{z}^{2p}}{(1-\bar{z})^2}\right\} \ . \qquad (2.3)$$

Now, if we consider

$$u_{\lambda,p}(z) = \frac{1}{z^p} + \sum_{k=p}^{\infty}[1+\lambda(k+p)]z^k = \frac{1}{z^p} + \frac{[1+2\lambda p]z^p - [1+\lambda(2p-1)]z^{p+1}}{(1-z)^2},$$

$$v_{\lambda,p}(z) = \sum_{k=2p}^{\infty}[1+\lambda(k+p)]z^k = \frac{[1+3\lambda p]z^{2p} - [1+\lambda(3p-1)]z^{2p+1}}{(1-z)^2},$$

we get

$$D^n_{\lambda,p}h = h * \underbrace{u_{\lambda,p} * u_{\lambda,p} * \ldots * u_{\lambda,p}}_{n-times}, \quad D^n_{\lambda,p}g = g * \underbrace{v_{\lambda,p} * v_{\lambda,p} * \ldots * v_{\lambda,p}}_{n-times}. \qquad (2.4)$$

Using (2.4) in (2.3), we obtain the condition required by this theorem. ■

The above theorem yields a sufficient coefficient condition for the class $\Sigma_{H_p}(n,\lambda,\alpha)$.

Corollary 2.1. Let $f \in \Sigma_{H_p}$. If

$$\sum_{k=p}^{\infty}[k+\alpha][1+\lambda(k+p)]^n|a_k| + \sum_{k=2p}^{\infty}[k-\alpha][1+\lambda(k+p)]^n|b_k| \le p-\alpha, \qquad (2.5)$$

then $f \in \Sigma_{H_p}(n,\lambda,\alpha)$. The result is sharp.

Proof. In view to the convolution condition given in Theorem 2.1, we note that for h and g as in (1.1), we have

$$\left| \frac{\alpha\zeta + \alpha - 2p}{z^p} + \sum_{k=p}^{\infty}[\zeta(k+\alpha+p) + k + \alpha - p][1+\lambda(k+p)]^n a_k z^k \right.$$

$$\left. + \sum_{k=2p}^{\infty}[\zeta(-k+p+\alpha) - k + \alpha - p][1+\lambda(k+p)]^n \overline{b_k}\,\overline{z}^k \right|$$

$$\ge \frac{|2p-\alpha-\alpha\zeta|}{|z|^p} - \sum_{k=p}^{\infty}|\zeta(k+\alpha+p) + k + \alpha - p|[1+\lambda(k+p)]^n|a_k||z|^k$$

$$- \sum_{k=2p}^{\infty}|\zeta(-k+\alpha+p) - k + \alpha - p|[1+\lambda(k+p)]^n|b_k||z|^k$$

$$\ge \frac{2(p-\alpha)}{|z|^p} - 2\sum_{k=p}^{\infty}[k+\alpha][1+\lambda(k+p)]^n|a_k||z|^k - 2\sum_{k=2p}^{\infty}[k-\alpha][1+\lambda(k+p)]^n|b_k||z|^k$$

$$> 2\left[p-\alpha-\sum_{k=p}^{\infty}[k+\alpha][1+\lambda(k+p)]^n|a_k| \; - \sum_{k=2p}^{\infty}[k-\alpha][1+\lambda(k+p)]^n|b_k| \right]$$

This last expression is nonnegative by the hypothesis of the corollary and so $f\in\Sigma_{H_p}(n,\lambda,\alpha)$. ■

The condition (2.5) given in Corollary 2.1 is sharp for the function

$$f(z)=\frac{1}{z^p}+\sum_{k=p}^{\infty}\frac{x_k}{[k+\alpha][1+\lambda(k+p)]^n}z^k+\sum_{k=2p}^{\infty}\frac{\bar{y}_k}{[k-\alpha][1+\lambda(k+p)]^n}\bar{z}^k,$$

where $\sum_{k=p}^{\infty}|x_k|+\sum_{k=2p}^{\infty}|y_k|=p-\alpha\cdot$

In the next theorem we use Theorem 2.1 to obtain inclusion relation with respect to n for the class $\Sigma_{H_p}(n,\lambda,\alpha)$.

Theorem 2.2. *For all* $n\in N_0,\ \Sigma_{H_p}(n+1,\lambda,\alpha)\subset\Sigma_{H_p}(n,\lambda,\alpha)$.

Proof. Let $f\in\Sigma_{H_p}(n,\lambda,\alpha)$, then for the values of A, B, C, D and E given in Theorem 2.1 we have

$$h*\underbrace{u_{\lambda,p}*u_{\lambda,p}*\ldots*u_{\lambda,p}}_{n+1-times}*\underbrace{\left\{\frac{A}{z^p}+\frac{Bz^p}{1-z}+\frac{Cz^{p+1}}{(1-z)^2}\right\}}_{\mathrm{T}}$$

$$+\overline{g}*\underbrace{\bar{v}_{\lambda,p}*\bar{v}_{\lambda,p}*\ldots*\bar{v}_{\lambda,p}}_{n+1-times}*\underbrace{\left\{\frac{D\bar{z}^{2p+1}+E\bar{z}^{2p}}{(1-\bar{z})^2}\right\}}_{\overline{\Psi}}\neq 0,$$

by commutative property of the convolution, we get

$$u_{\lambda,p}*h*\underbrace{u_{\lambda,p}*u_{\lambda,p}*\ldots*u_{\lambda,p}}_{n-times}*\mathrm{T}+\bar{v}_{\lambda,p}*\overline{g}*\underbrace{\bar{v}_{\lambda,p}*\bar{v}_{\lambda,p}*\ldots*\bar{v}_{\lambda,p}}_{n-times}*\overline{\Psi}\neq 0,$$

hence

$$\left(u_{\lambda,p}+\bar{v}_{\lambda,p}\right)*\left(h*\underbrace{u_{\lambda,p}*u_{\lambda,p}*\ldots*u_{\lambda,p}}_{n-times}*\mathrm{T}+\overline{g}*\underbrace{\bar{v}_{\lambda,p}*\bar{v}_{\lambda,p}*\ldots*\bar{v}_{\lambda,p}}_{n-times}*\overline{\Psi}\right)\neq 0,$$

then we have

$$h * \underbrace{u_{\lambda,p} * u_{\lambda,p} * \ldots * u_{\lambda,p}}_{n-times} * \mathrm{T} + \overline{g} * \underbrace{\overline{v}_{\lambda,p} * \overline{v}_{\lambda,p} * \ldots * \overline{v}_{\lambda,p}}_{n-times} * \overline{\Psi} \neq 0,$$

this yields that $f \in \Sigma_{H_p}(n,\lambda,\alpha)$. ■

Remark 2.1. *From Theorem 2.2 and Remark 1.4 (I), we deduce that functions in the class* $\Sigma_{H_p}(n,\lambda,\alpha)$ *are meromorphic starlike of order* α.

Now we show that the sufficient condition (2.5) is also necessary for functions in the class $\Sigma^{-}_{H_p}(n,\lambda,\alpha)$.

Theorem 2.3. *A necessary and sufficient condition for f to be in the class* $\Sigma^{-}_{H_p}(n,\lambda,\alpha)$ *is that*

$$\sum_{k=p}^{\infty}[k+\alpha][1+\lambda(k+p)]^n a_k + \sum_{k=2p}^{\infty}[k-\alpha][1+\lambda(k+p)]^n b_k \leq p-\alpha . \qquad (2.6)$$

Proof. In view of Corollary 2.1 and since $\Sigma^{-}_{H_p}(n,\lambda,\alpha) \subset \Sigma_{H_p}(n,\lambda,\alpha)$, we need only to show that $f \notin \Sigma^{-}_{H_p}(n,\lambda,\alpha)$ if the condition (2.6) does not hold.

Since

$$-\operatorname{Re}\left\{\frac{z\left(HD^n_{\lambda,p} f(z)\right)_z - \overline{z}\left(HD^n_{\lambda,p} f\right)_{\overline{z}}}{HD^n_{\lambda,p} f(z)}\right\} \geq \alpha \ , \ z \in U \ , \ 0 \leq \alpha < p,$$

we have

$$-\operatorname{Re}\left\{\frac{z\left(HD^n_{\lambda,p} f(z)\right)_z - \overline{z}\left(HD^n_{\lambda,p} f\right)_{\overline{z}}}{HD^n_{\lambda,p} f(z)} + \alpha\right\}$$

$$= -\operatorname{Re}\left\{\frac{z\left(D^n_{\lambda,p}h(z)\right)' - \overline{z\left(D^n_{\lambda,p}g(z)\right)'} + \alpha D^n_{\lambda,p}h(z) + \alpha\overline{D^n_{\lambda,p}g(z)}}{D^n_{\lambda,p}h(z) + \overline{D^n_{\lambda,p}g(z)}}\right\}$$

$$= \operatorname{Re}\left\{\frac{p-\alpha-\sum_{k=p}^{\infty}[k+\alpha][1+\lambda(k+p)]^n a_k z^{k+p} - \sum_{k=2p}^{\infty}[k-\alpha][1+\lambda(k+p)]^n b_k \overline{z}^k z^p}{1+\sum_{k=p}^{\infty}[1+\lambda(k+p)]^n a_k z^{k+p} - \sum_{k=2p}^{\infty}[1+\lambda(k+p)]^n b_k \overline{z}^k z^p}\right\}$$

$= p +$

$$-\operatorname{Re}\left\{\frac{\alpha+\sum_{k=p}^{\infty}[k+\alpha+p][1+\lambda(k+p)]^n a_k z^{k+p}+\sum_{k=2p}^{\infty}[k-\alpha-p][1+\lambda(k+p)]^n b_k \bar{z}^k z^p}{1+\sum_{k=p}^{\infty}[1+\lambda(k+p)]^n a_k z^{k+p}-\sum_{k=2p}^{\infty}[1+\lambda(k+p)]^n b_k \bar{z}^k z^p}\right\}$$

$\geq p +$

$$-\left\{\frac{\alpha+\sum_{k=p}^{\infty}[k+\alpha+p][1+\lambda(k+p)]^n a_k |z|^{k+p}+\sum_{k=2p}^{\infty}[k-\alpha-p][1+\lambda(k+p)]^n b_k |z|^{k+p}}{1-\sum_{k=p}^{\infty}[1+\lambda(k+p)]^n a_k |z|^{k+p}-\sum_{k=2p}^{\infty}[1+\lambda(k+p)]^n b_k |z|^{k+p}}\right\}$$

$$=\frac{p-\alpha-\sum_{k=p}^{\infty}[k+\alpha+2p][1+\lambda(k+p)]^n a_k |z|^{k+p}-\sum_{k=2p}^{\infty}[k-\alpha][1+\lambda(k+p)]^n b_k |z|^{k+p}}{1-\sum_{k=p}^{\infty}[1+\lambda(k+p)]^n a_k |z|^{k+p}-\sum_{k=p}^{\infty}[1+\lambda(k+p)]^n b_k |z|^{k+p}} \geq 0, \tag{2.7}$$

if (2.6) does not hold, then the numerator of (2.7) is negative. So, we can find $0 < z_o < 1$ such (2.7) is negative and this contradict the fact $f \notin \sum_{H_p}^{-}(n,\lambda,\alpha)$. Hence the condition(2.6) holds.■

3 Representation & Distortion theorems

Theorem 3.1. *Let* $f = h + \bar{g}$ *where* h *and* g *are of the form (1.2). Then* $f \in \Sigma_{H_p}^{-}(n,\lambda,\alpha)$ *if and only if* f *can be expressed as*

$$f(z) = \sum_{k=p-1}^{\infty} x_k h_k(z) + \sum_{k=2p}^{\infty} y_k g_k(z).$$

Where $h_{p-1}(z) = \frac{1}{z^p}$, $h_k(z) = \frac{1}{z^p} + \frac{p-\alpha}{[k+\alpha][1+\lambda(k+p)]^n} z^k \ (k \geq p)$,

$$g_k(z) = \frac{1}{z^p} - \frac{p-\alpha}{[k-\alpha][1+\lambda(k+p)]^n}\bar{z}^k \ \ (k \geq 2p)\ ,\ x_k,\ y_k \geq 0 \ \textit{and} \ \sum_{k=p-1}^{\infty} x_k + \sum_{k=2p}^{\infty} y_k = 1.$$

Proof. Suppose

$$f(z) = \sum_{k=p-1}^{\infty} x_k h_k(z) + \sum_{k=2p}^{\infty} y_k g_k(z)$$

$$= \frac{1}{z^p} + \sum_{k=p}^{\infty} \frac{(p-\alpha)x_k}{[k+\alpha][1+\lambda(k+p)]^n} z^k + \sum_{k=2p}^{\infty} \frac{(p-\alpha)y_k}{[k-\alpha][1+\lambda(k+p)]^n} \bar{z}^k.$$

Hence

$$\sum_{k=p}^{\infty}[k+\alpha][1+\lambda(k+p)]^n a_k + \sum_{k=2p}^{\infty}[k-\alpha][1+\lambda(k+p)]^n b_k$$

$$= \sum_{k=p}^{\infty} \frac{[k+\alpha][1+\lambda(k+p)]^n (p-\alpha)x_k}{[k+\alpha][1+\lambda(k+p)]^n} + \sum_{k=2p}^{\infty} \frac{[k-\alpha][1+\lambda(k+p)]^n (p-\alpha)y_k}{[k-\alpha][1+\lambda(k+p]^n}$$

$$= (p-\alpha)\left[\sum_{k=p}^{\infty} x_k + \sum_{k=2p}^{\infty} y_k\right] = (p-\alpha)[1-x_{p-1}] \leq p-\alpha\ ,$$

and so $f \in \Sigma_{H_p}^{-}(n,\lambda,\alpha)$.

Conversely, suppose that $f \in \Sigma_{H_p}^{-}(n,\lambda,\alpha)$, then we have

$$a_k \leq \frac{p-\alpha}{[k+\alpha][1+\lambda(k+p)]^n}\ ,\ k \geq p\ , \qquad b_k \leq \frac{p-\alpha}{[k-\alpha][1+\lambda(k+p)]^n}\ ,\ k \geq 2p$$

Set

$$x_k = \frac{[k+\alpha][1+\lambda(k+p)]^n}{p-\alpha} a_k\ ,\ k \geq p\ , \qquad y_k = \frac{[k-\alpha][1+\lambda(k+p)]^n}{p-\alpha} b_k\ ,\ k \geq 2p$$

Then note that by Theorem 2.3, $x_k \geq 0$ for $k \geq p$ and $y_k \geq 0$ for $k \geq 2p$.

We define

$$x_{p-1} = 1 - \sum_{k=p}^{\infty} x_k - \sum_{k=2p}^{\infty} y_k\ ,$$

and note that, by Theorem 2.3, $x_{p-1} \geq 0$. Consequently, we obtain $f(z) = \sum_{k=p-1}^{\infty} x_k h_k(z) + \sum_{k=2p}^{\infty} y_k g_k(z)$. ■

We shall obtain distortion bounds for functions in the class $\Sigma^-_{H_p}(n,\lambda,\alpha)$.

Theorem 3.2. *If* $f\in\Sigma^-_{H_p}(n,\lambda,\alpha)$ *for* $0<|z|=r<1$, *then*

$$|f(z)|\le \frac{1}{r^p}+r^p\sum_{k=p}^{2p-1}a_k+r^{2p}\left[\frac{p-\alpha-\sum_{k=p}^{2p-1}[k+\alpha][1+\lambda(k+p)]^n a_k}{[2p-\alpha][1+3\lambda p]^n}\right],$$

and

$$|f(z)|\ge \frac{1}{r^p}-r^p\sum_{k=p}^{2p-1}a_k-r^{2p}\left[\frac{p-\alpha-\sum_{k=p}^{2p-1}[k+\alpha][1+\lambda(k+p)]^n a_k}{[2p-\alpha][1+3\lambda p]^n}\right].$$

Proof. We prove the left hand side inequality for $|f|$. The proof for the right hand side inequality can be done using similar arguments.

Let $f\in\Sigma^-_{H_p}(n,\lambda,\alpha)$, then by Theorem 2.3, we obtain

$$|f(z)|\ge \frac{1}{r^p}-r^p\sum_{k=p}^{2p-1}a_k-r^{2p}\sum_{k=2p}^{\infty}(a_k+b_k)$$

$$\ge \frac{1}{r^p}-r^p\sum_{k=p}^{2p-1}a_k-\frac{r^{2p}}{[2p-\alpha][1+3\lambda p]^n}\sum_{k=2p}^{\infty}\Big\{[k+\alpha][1+\lambda(k+p)]^n a_k$$

$$+[k-\alpha][1+\lambda(k+p)]^n b_k\Big\}$$

$$\ge \frac{1}{r^p}-r^p\sum_{k=p}^{2p-1}a_k-r^{2p}\left[\frac{p-\alpha-\sum_{k=p}^{2p-1}[k+\alpha][1+\lambda(k+p)]^n a_k}{[2p-\alpha][1+3\lambda p]^n}\right]. \blacksquare$$

The following covering result follows from the left hand side inequality in Theorem 3.2.

Corollary 3.1. *If* $f\in\Sigma^-_{H_p}(n,\lambda,\alpha)$, *then the set*

$$\left\{ w : |w| < 1 - \frac{p-\alpha}{[2p-\alpha][1+3\lambda p]^n} + \sum_{k=p}^{2p-1} \left[\frac{[k+\alpha][1+\lambda(k+p)]^n}{[2p-\alpha][1+3\lambda p]^n} - 1 \right] a_k \right\} \subset f(U^*)$$

4 Convolution & Convex Linear Combination

First, we show that the class $\Sigma_{H_p}^{-}(n,\lambda,\alpha)$ is closed under the convolution of its members.

Theorem 4.1. *Let* $f \in \Sigma_{H_p}^{-}(n,\lambda,\alpha)$ *and* $F \in \Sigma_{H_p}^{-}(n,\lambda,\beta)$ *for* $0 \le \beta \le \alpha < p$ *and given by* $f(z) = \frac{1}{z^p} + \sum_{k=p}^{\infty} a_k z^k - \sum_{k=2p}^{\infty} b_k \bar{z}^k$, $F(z) = \frac{1}{z^p} + \sum_{k=p}^{\infty} A_k z^k - \sum_{k=2p}^{\infty} B_k \bar{z}^k$, *then*

$$f * F \in \Sigma_{H_p}^{-}(n,\lambda,\alpha) \subset \Sigma_{H_p}^{-}(n,\lambda,\beta)$$

Proof. It's enough to show that the condition (2.6) holds for the function $f * F$. That is

$$\sum_{k=p}^{\infty}[k+\alpha][1+\lambda(k+p)]^n a_k A_k + \sum_{k=2p}^{\infty}[k-\alpha][1+\lambda(k+p)]^n b_k B_k$$

$$\le \sum_{k=p}^{\infty}[k+\alpha][1+\lambda(k+p)]^n a_k + \sum_{k=2p}^{\infty}[k-\alpha][1+\lambda(k+p)]^n b_k \qquad (A_k, B_k < 1)$$

$$\le p-\alpha ,$$

since $f \in \Sigma_{H_p}^{-}(n,\lambda,\alpha)$. Hence, we get that $f * F \in \Sigma_{H_p}^{-}(n,\lambda,\alpha) \subset \Sigma_{H_p}^{-}(n,\lambda,\beta)$.■

Our next result is on the convex combinations of the members of the family $\Sigma_{H_p}^{-}(n,\lambda,\alpha)$.

Theorem 4.2. *The class* $\Sigma_{H_p}^{-}(n,\lambda,\alpha)$ *is closed under convex combination.*

Proof. For $i = 1,2,3,\ldots$ suppose that $f_i \in \Sigma_{H_p}^{-}(n,\lambda,\alpha)$ where f_i is given by

$$f_i(z) = \frac{1}{z^p} + \sum_{k=p}^{\infty} a_{k_i} z^k - \sum_{k=2p}^{\infty} b_{k_i} \bar{z}^k.$$

Then, by (2.6),

$$\sum_{k=p}^{\infty}[k+\alpha][1+\lambda(k+p)]^n a_{k_i} + \sum_{k=2p}^{\infty}[k-\alpha][1+\lambda(k+p)]^n b_{k_i} \le p-\alpha. \qquad (4.1)$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \le t_i \le 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = \frac{1}{z^p} + \sum_{k=p}^{\infty}\left[\sum_{i=1}^{\infty} t_i a_{k_i}\right] z^k - \sum_{k=2p}^{\infty}\left[\sum_{i=1}^{\infty} t_i b_{k_i}\right] \bar{z}^k .$$

Then by (4.1),

$$\sum_{k=p}^{\infty}[k+\alpha][1+\lambda(k+p)]^n \sum_{i=1}^{\infty} t_i a_{k_i} + \sum_{k=2p}^{\infty}[k-\alpha][1+\lambda(k+p)]^n \sum_{i=1}^{\infty} t_i b_{k_i}$$

$$= \sum_{i=1}^{\infty} t_i \left[\sum_{k=p}^{\infty}[k+\alpha][1+\lambda(k+p)]^n a_{k_i} + \sum_{k=2p}^{\infty}[k-\alpha][1+\lambda(k+p)]^n b_{k_i}\right]$$

$$\le \left\{\sum_{i=1}^{\infty} t_i\right\}(p-\alpha) = p-\alpha ,$$

and so $\sum_{i=1}^{\infty} t_i f_i \in \Sigma_{H_p}^{-}(n,\lambda,\alpha)$. ■

5 Open problems

1. Consider the function $-\frac{z}{p}h' + \overline{\frac{z}{p}g'} \in \Sigma_{H_p}(n,\lambda,\alpha)$, where h and g are defined in (1.1). What can you say about the function $-\frac{z}{p}\left(D_{\lambda,p}^{n}h\right)' + \overline{\frac{z}{p}\left(D_{\lambda,p}^{n}g\right)'}$?

2. **(a)** Define an integral operator $HI_{\lambda,p}^{n} f$ on the harmonic functions $f \in \Sigma_{H_p}(n,\lambda,\alpha)$, which generalized the integral operator $I_{\lambda,p}^{n} f$ defined on multivalent meromorphic functions [4].

(b) Find the relation between the integral and differential operators, by using convolution properties.

3. Jakubowski et al [8] studied the class of harmonic functions with positive real part. Use these functions to obtain integral representation for functions of the class $\Sigma_{H_p}(n,\lambda,\alpha)$.

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