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Characterization Bertrand Curve in the Heisenberg Group Heis^3

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Abstract. In this paper, we study non-geodesic biharmonic curves in the Heisenberg group Heis^3. We characterize Bertrand mate of biharmonic curve in terms of curvature and torsion of biharmonic curve in the Heisenberg group Heis^3. Finally, we construct parametric equations of Bertrand mate of biharmonic curve.

Key words: Heisenberg group, biharmonic curve, Bertrand curve.
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1 Introduction

Bertrand curves are well-studied classical curves and may be defined by their property that any Bertrand curve shares its principal normals with another Bertrand curve, sometimes referred to as Bertrand mate [21]. Accordingly, Bertrand mates represent particular examples of offset curves [14] which are used in computer-aided design (CAD) and computer-aided manufacture (CAM). The distance between a Bertrand curve and its mate measured along the principal normal is known to be constant.

The aim of this paper is to study biharmonic curves in the Heisenberg group Heis^3.

Firstly, harmonic maps are given as follows:

Harmonic maps $f : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds are the critical points of the energy

$$E(f) = \frac{1}{2} \int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler–Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f) = \mathrm{trace}\nabla df. \tag{1.2}$$

Secondly, biharmonic maps are given as follows:

As suggested by Eells and Sampson in [8], we can define the bienergy of a map f by

$$E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [10], showing that the Euler–Lagrange equation associated to E_2 is

$$\begin{aligned} \tau_2(f) &= -\mathcal{J}^f(\tau(f)) = -\Delta\tau(f) - \mathrm{trace}R^N(df,\tau(f))\,df1.4 \\ &= 0, \end{aligned} \tag{1}$$

where $\mathcal{J}^f$ is the Jacobi operator of f . The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

Clearly, any harmonic map is biharmonic. However, the converse is not true. Nonharmonic biharmonic maps are said to be proper. It is well known that proper biharmonic maps, that is, biharmonic functions, play an important role in elasticity and hydrodynamics.

In this paper, we study non-geodesic biharmonic curves in the Heisenberg group Heis^3. We characterize Bertrand mate of biharmonic curve in terms of curvature and torsion of biharmonic curve in the Heisenberg group Heis^3. Finally, we construct parametric equations of Bertrand mate of biharmonic curve.

2 Heisenberg Group Heis^3

Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multipilcation:

$$(\overline{x},\overline{y},\overline{z})(x,y,z) = (\overline{x}+x,\overline{y}+y,\overline{z}+z-\frac{1}{2}\overline{x}y+\frac{1}{2}x\overline{y}) \tag{2.1}$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric g is given by

$$g = dx^2 + dy^2 + (dz + \frac{y}{2}dx - \frac{x}{2}dy)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$e_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}, \tag{2.2}$$

for which we have the Lie products

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = [e_3, e_1] = 0$$

with

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

We obtain

$$\begin{aligned}
\nabla_{e_1} e_1 &= \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = 0, \\
\nabla_{e_1} e_2 &= -\nabla_{e_2} e_1 = \frac{1}{2} e_3, \\
\nabla_{e_1} e_3 &= \nabla_{e_3} e_1 = -\frac{1}{2} e_2, \\
\nabla_{e_2} e_3 &= \nabla_{e_3} e_2 = \frac{1}{2} e_1.
\end{aligned}$$

We adopt the following notation and sign convention for Riemannian curvature operator on Heis^3 defined by

$$R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z,$$

while the Riemannian curvature tensor is given by

$$R(X, Y, Z, W) = g(R(X, Y)Z, W),$$

where X, Y, Z, W are smooth vector fields on Heis^3.

The components $\{R_{ijkl}\}$ of R relative to $\{e_1, e_2, e_3\}$ are defined by

$$g\left(R(e_i, e_j)e_k, e_l\right) = R_{ijkl.}$$

The non-vanishing components of the above tensor fields are

$$R_{121} = -\tfrac{3}{4} e_2, \quad R_{131} = \tfrac{1}{4} e_3, \quad R_{122} = \tfrac{3}{4} e_1,$$

$$R_{232} = \tfrac{1}{4} e_3, \quad R_{133} = -\tfrac{1}{4} e_1, \quad R_{233} = -\tfrac{1}{4} e_2,$$

and

$$R_{1212} = -\tfrac{3}{4}, \quad R_{1313} = R_{2323} = \tfrac{1}{4}. \tag{2.3}$$

3 Biharmonic Curves in the Heisenberg Group Heis^3

Let $I \subset \mathbb{R}$ be an open interval and $\gamma : I \longrightarrow (N, h)$ be a curve parametrized by arc length on a Riemannian manifold. Putting $\mathbf{T} = \gamma'$, we can write the tension field of γ as $\tau(\gamma) = \nabla_{\gamma'} \gamma'$ and the biharmonic map equation (1.1) reduces to

$$\nabla^3_{\mathbf{T}} \mathbf{T} + R(\mathbf{T}, \nabla_{\mathbf{T}} \mathbf{T})\mathbf{T} = 0. \tag{3.1}$$

A successful key to study the geometry of a curve is to use the Frenet frames along the curve, which is recalled in the following.

Let $\gamma : I \longrightarrow Heis^3$ be a curve on $Heis^3$ parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to Heis^3 along γ defined as follows: T is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and B is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} - \tau\mathbf{B}, 3.2 \\ \nabla_{\mathbf{T}}\mathbf{B} &= \tau\mathbf{N}, \end{aligned} \tag{2}$$

where $\kappa = |\nabla_{\mathbf{T}}\mathbf{T}|$ is the curvature of γ and τ is its torsion. With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T} \times \mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned}$$

Theorem 3.1.*(see [11]) Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesic curve on $Heis^3$ parametrized by arc length. Then γ is a non-geodesic biharmonic curve if and only if*

$$\begin{aligned} \kappa &= \text{constant} \neq 0, \\ \kappa^2 + \tau^2 &= \frac{1}{4} - B_3^2, 3.3 \\ \tau' &= N_3 B_3. \end{aligned} \tag{3}$$

Theorem 3.2*.(see [11]) Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesic curve on the Heisenberg group $Heis^3$ parametrized by arc length. If κ is constant and $N_1 B_1 \neq 0$, then γ is not biharmonic.*

4 Bertrand Mate of Biharmonic Curve in Heisenberg Group Heis^3

Definition 4.1. *A curve $\gamma : I \longrightarrow Heis^3$ with $\kappa \neq 0$ is called a Bertrand curve if there exist a curve $\beta : I \longrightarrow Heis^3$ such that the principal normal lines of γ and β at $s \in I$ are equal. In this case β is called a Bertrand mate of γ [14].*

Theorem 4.2. *Let $\gamma : I \longrightarrow Heis^3$ be a Bertrand curve parametrized by arc length . A Bertrand mate of γ is as follows:*

$$\beta(s) = \gamma(s) + \lambda\mathbf{N}(s), \quad \forall s \in I,$$

where λ is constant [14].

Theorem 4.3. *Let $\gamma : I \longrightarrow Heis^3$ be a biharmonic curve parametrized by arc length. If β is a Bertrand mate of γ, then the parametric equations of β are*

$$\begin{aligned} x_\beta(s) &= \frac{\lambda}{\kappa}\sin\varphi(\cos\varphi - \Re)\sin[\Re s+\rho] \\ &\quad +\frac{1}{\Re}\sin\varphi\sin[\Re s+\rho], \\ y_\beta(s) &= -\frac{\lambda}{\kappa}\sin\varphi(\cos\varphi - \Re)\cos[\Re s+\rho]4.1 \\ &\quad -\frac{1}{\Re}\sin\varphi\cos[\Re s+\rho], \\ z_\beta(s) &= (\cos\varphi + \frac{1}{4\Re}\sin^2\varphi)s, \end{aligned} \tag{4}$$

where ρ is constant of integration and $\Re = \frac{\cos\varphi \pm\sqrt{5(\cos\varphi)^2-4}}{2}$.

Proof. The covariant derivative of the vector field $\mathbf{T}$ is:

$$\nabla_{\mathbf{T}}\mathbf{T} = (T_1' + T_2T_3)\mathbf{e}_1 + (T_2' - T_1T_3)\mathbf{e}_2 + T_3'\mathbf{e}_3. \tag{4.2}$$

Thus using Theorem 3.2, we have

$$\mathbf{T} = \sin\varphi\cos[\Re s+\rho]\mathbf{e}_1 + \sin\varphi\sin[\Re s+\rho]\mathbf{e}_2 + \cos\varphi\mathbf{e}_3, \tag{4.3}$$

where $\Re = \frac{\cos\varphi \pm\sqrt{5(\cos\varphi)^2-4}}{2}$.

Using (2.2) in (4.3), we obtain

$$\begin{aligned} \mathbf{T} &= (\sin\varphi\cos[\Re s+\rho], \sin\varphi\sin[\Re s+\rho], \\ &\quad \cos\varphi - \frac{1}{2}y(s)\sin\varphi\cos[\Re s+\rho] + \frac{1}{2}x(s)\sin\varphi\sin[\Re s+\rho]). \end{aligned}$$

From (2.2), we get

$$\begin{aligned} \mathbf{T} &= (\sin\varphi\cos[\Re s+\rho], \sin\varphi\sin[\Re s+\rho], \\ &\quad \cos\varphi + \frac{1}{2\Re}\sin^2\varphi\cos^2[\Re s+\rho] + \frac{1}{2\Re}\sin^2\varphi\sin^2[\Re s+\rho]). \end{aligned}$$

On the other hand, suppose that $\beta(s)$ is a Bertrand mate of γ. Then by the definition we can assume that

$$\beta(s) = \gamma(s) + \lambda\mathbf{N}(s). \tag{4.4}$$

From (4.2) and (4.3), we get

$$\nabla_{\mathbf{T}}\mathbf{T} = \sin\varphi(\cos\varphi - \Re)(\sin[\Re s+\rho]\mathbf{e}_1 - \cos[\Re s+\rho]\mathbf{e}_2).$$

By the use of Frenet formulas, we get

$$\begin{aligned}\mathbf{N} &= \frac{1}{\kappa}\nabla_{\mathbf{T}}\mathbf{T}\\ &= \frac{1}{\kappa}[\sin\varphi\,(\cos\varphi-\Re)\,(\sin[\Re s+\rho]\mathbf{e}_1-\cos[\Re s+\rho]\mathbf{e}_2)].\end{aligned}$$

Substituting (2.2) in (4.5), we have

$$\mathbf{N}=\frac{1}{\kappa}\sin\varphi\,(\cos\varphi-\Re)\,(\sin[\Re s+\rho],-\cos[\Re s+\rho],0). \tag{4.5}$$

Finally, we substitute (4.3) and (4.5) into (4.4), we get (4.1). The proof is completed.

Corollary 4.4. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed non-geodesic biharmonic curve . Then, the parametric equations of γ are*

$$\begin{aligned}x\,(s) &= \frac{1}{\Re}\sin\varphi\sin[\Re s+\rho],\\ y\,(s) &= -\frac{1}{\Re}\sin\varphi\cos[\Re s+\rho],\\ z\,(s) &= (\cos\varphi+\frac{1}{4\Re}\sin^2\varphi)s,\end{aligned}$$

where $\Re=\frac{\cos\varphi\pm\sqrt{5(\cos\varphi)^2-4}}{2}$.

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