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Certain Differential Inequalities Involving Multivalent Functions

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Abstract

In the present paper, using Jack's lemma, we study certain differential inequalities involving multivalent functions in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$ and obtain sufficient conditions for uniformly p-valent close-to-convex and uniformly p-valent starlike functions.

Keywords: *Analytic function, Multivalent function, Uniformly close-to-convex function, Uniformly starlike function, Uniformly convex function.*

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1 Introduction

Let $\mathcal{A}$ denote the class of all analytic functions f, normalized by the conditions $f(0) = f'(0) - 1 = 0$. For $f \in \mathcal{A}$, the Taylor's series expansion is

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

By $\mathcal{A}_p$, we denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in \mathbb{N} \setminus \{1\}; z \in \mathbb{E}),$$

which are analytic and p-valent in the open unit disk $\mathbb{E}$. A function $f \in \mathcal{A}_p$ is said to be uniformly p-valent starlike in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - p\right|, \; z \in \mathbb{E}. \tag{1}$$

We denote by US_p^*, the class of uniformly p-valent starlike functions. A function $f \in \mathcal{A}_p$ is said to be uniformly p-valent close-to-convex in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \left|\frac{zf'(z)}{g(z)} - p\right|, \; z \in \mathbb{E}, \tag{2}$$

for some $g \in US_p^*$. Let UCC_p denote the class of all such functions. Note that US_1^* and UCC_1 are the usual classes of uniformly starlike functions and uniformly close-to-convex functions and will be denoted here by US^* and UCC, respectively. Also notice that the function $g(z) \equiv z^p \in \mathcal{A}_p$ and satisfies the condition (1). Therefore, when we select $g(z) \equiv z^p$, in condition (2), it becomes

$$\Re\left(\frac{f'(z)}{z^{p-1}}\right) > \left|\frac{f'(z)}{z^{p-1}} - p\right|, \; z \in \mathbb{E}. \tag{3}$$

Hence, a function $f \in \mathcal{A}_p$ is uniformly p-valent close-to-convex in $\mathbb{E}$ if it satisfies the condition (3).

In 1991, Goodman [1], introduced the concept of uniformly starlike and uniformly convex functions. He defined uniformly starlike and uniformly convex functions as functions $f \in \mathcal{A}$ with the geometric property that image of every circular arc contained in $\mathbb{E}$, with center $\zeta \in \mathbb{E}$, is starlike with respect to $f(\zeta)$ and convex respectively.

In 1993 Ronning [3], studied the class of uniformly convex functions and obtained an interesting criterion for $f \in \mathcal{A}$ to be uniformly convex in $\mathbb{E}$. He proved that a function $f \in \mathcal{A}$, is uniformly convex if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|, \; z \in \mathbb{E}.$$

Let us, now, define the parabolic domain Ω and the circular domain O as under:

$$\Omega = \left\{u + iv : u > \sqrt{(u-p)^2 + v^2}\;\right\}$$

and

$$O = \left\{u + iv : \sqrt{(u-p)^2 + v^2} < \frac{p}{2}\;\right\}.$$

Obviously $O \subset \Omega$. In particular when $p = 2$, the plot of the boundary curve of the parabolic domain Ω and that of the circular domain O is given below as in Figure 1.1.

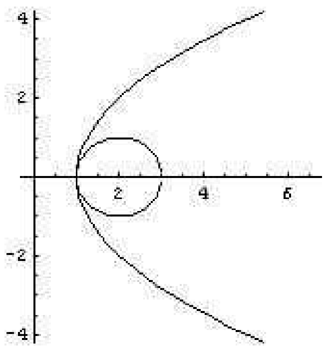

Figure 1.1

The main objective of this paper is to derive some sufficient conditions for uniformly p-valent starlikeness and uniformly p-valent close-to-convexity in terms of certain differential inequalities. In particular, we also obtain some sufficient conditions for uniformly starlikeness and uniformly close-to-convexity. We shall make use of Jack's lemma to prove our main results.

Lemma 1.1 *(Jack [2]). Suppose w be a nonconstant analytic function in $\mathbb{E}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value at a point $z_0 \in \mathbb{E}$ on the circle $|z| = r < 1$, then $z_0 w'(z_0) = mw(z_0)$, where $m \geq 1$, is some real number.*

2 Main Results

Theorem 2.1 *If $f \in \mathcal{A}_p$ satisfies the differential inequality*

$$\left|(1-\alpha)\frac{f'(z)}{z^{p-1}} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - p\right| < \frac{(1-\alpha)p}{2} + \frac{\alpha}{3}\ , \tag{4}$$

for some $0 \leq \alpha \leq 1$, then $f \in UCC_p$.

Proof. Let us write

$$\frac{f'(z)}{z^{p-1}} - p = \frac{p}{2}w(z),$$

where w be analytic in $\mathbb{E}$ with $w(0) = 0$. Now we will show that $|w(z)| < 1, z \in \mathbb{E}$. If $|w(z)| \not< 1$, by Lemma 1.1, there exists z_0, $|z_0| < 1$ such that $|w(z_0)| = 1$ and $z_0 w^{'}(z_0) = kw(z_0)$ where $k \geq 1$. When we put $w(z_0) = e^{i\theta}$, we have

$$\left|(1-\alpha)\frac{f'(z_0)}{z_0^{p-1}}+\alpha\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)-p\right|$$

$$\begin{aligned}&= \left|(1-\alpha)\left(p+\frac{p}{2}w(z_0)\right)+\alpha\left(p+\frac{z_0w'(z_0)}{2+w(z_0)}\right)-p\right|\\ &= \left|\frac{(1-\alpha)p}{2}e^{i\theta}+\frac{\alpha ke^{i\theta}}{2+e^{i\theta}}\right|\\ &= \left|\frac{(1-\alpha)p}{2}+\frac{\alpha k}{2+e^{i\theta}}\right|\geq\frac{(1-\alpha)p}{2}+\frac{\alpha}{3}\ ,\end{aligned}$$

which is a contradiction to (4). Therefore, we must have $|w(z)|<1, z\in\mathbb{E}$ and we have

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|<\frac{p}{2},$$

which shows that $\frac{f'(z)}{z^{p-1}}$ lies in the circular domain O. Therefore, for all z in $\mathbb{E}$, $\frac{f'(z)}{z^{p-1}}$ takes values in the domain Ω and hence

$$\Re\left(\frac{f'(z)}{z^{p-1}}\right)>\left|\frac{f'(z)}{z^{p-1}}-p\right|,$$

which completes the proof.

Theorem 2.2 *If $f\in\mathcal{A}_p$ satisfies the differential inequality*

$$\left|(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right|<\frac{p}{2}+\frac{\alpha}{3}\ , \tag{5}$$

for some $\alpha\geq 0$, then $f\in US_p^*$.

Proof. Let us write

$$\frac{zf'(z)}{f(z)}-p=\frac{p}{2}w(z),$$

where w be analytic in $\mathbb{E}$ with $w(0)=0$. Now we will show that $|w(z)|<1, z\in\mathbb{E}$. If $|w(z)|\not<1$, by Lemma 1.1, there exists z_0, $|z_0|<1$ such that $|w(z_0)|=1$ and $z_0w^{'}(z_0)=kw(z_0)$ where $k\geq 1$. When we put $w(z_0)=e^{i\theta}$, we have

$$\left|(1-\alpha)\frac{z_0f'(z_0)}{f(z_0)}+\alpha\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)-p\right|$$

$$= \left|(1-\alpha)\left(p+\frac{p}{2}w(z_0)\right)+\alpha\left(p+\frac{p}{2}w(z_0)+\frac{z_0w'(z_0)}{2+w(z_0)}\right)-p\right|$$
$$= \left|\frac{p}{2}e^{i\theta}+\frac{\alpha k e^{i\theta}}{2+e^{i\theta}}\right|$$
$$= \left|\frac{p}{2}+\frac{\alpha k}{2+e^{i\theta}}\right| \geq \frac{p}{2}+\frac{\alpha}{3}\ ,$$

which is a contradiction to (5). Therefore, we must have $|w(z)|<1, z\in\mathbb{E}$ and we have

$$\left|\frac{zf'(z)}{f(z)}-p\right|<\frac{p}{2},$$

which shows that $\frac{zf'(z)}{f(z)}$ lies in a circular domain O for all $z\in\mathbb{E}$. Therefore, $\frac{zf'(z)}{f(z)}$ takes values in the domain Ω for all z in $\mathbb{E}$ and hence

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>\left|\frac{zf'(z)}{f(z)}-p\right|,\ z\in\mathbb{E},$$

which completes the proof.

3 Deductions

By taking $\alpha=1$ in Theorem 2.1, we have the following result.

Corollary 3.1 *If $f\in\mathcal{A}_p$ satisfies the condition*

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right|<\frac{1}{3},\ z\in\mathbb{E},$$

then $f\in UCC_p$.

On writing $p=1$ in Theorem 2.1, we obtain:

Corollary 3.2 *Let $f\in\mathcal{A}$ satisfy the differential inequality*

$$\left|(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right|<\frac{3-\alpha}{6},\ z\in\mathbb{E},$$

for some $0\leq\alpha\leq 1$, then $f\in UCC$.

Setting $p=1$ and $\alpha=\frac{1}{2}$ in Theorem 2.1, we get:

Corollary 3.3 *Suppose $f \in \mathcal{A}$ satisfies the condition*

$$\left|f'(z) + \frac{zf''(z)}{f'(z)} - 1\right| < \frac{5}{6},\ z \in \mathbb{E},$$

then $f \in UCC$.

Letting $p = 1$ and $\alpha = 1$ in Theorem 2.1, we have:

Corollary 3.4 *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{1}{3},\ z \in \mathbb{E},$$

then $f \in UCC$.

For $\alpha = 1$ in Theorem 2.2, we obtain.

Corollary 3.5 *If $f \in \mathcal{A}_p$ satisfies*

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| < \frac{p}{2} + \frac{1}{3},\ z \in \mathbb{E},$$

then $f \in US_p^*$.

Writing $p = 1$ in Theorem 2.2, we get the following result.

Corollary 3.6 *If $f \in \mathcal{A}$ satisfies the differential inequality*

$$\left|(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right| < \frac{\alpha}{3} + \frac{1}{2},\ z \in \mathbb{E},$$

then $f \in US^*$.

Writing $p = 1$ and $\alpha = \frac{1}{2}$ in Theorem 2.2, we get:

Corollary 3.7 *If $f \in \mathcal{A}$ satisfies the condition*

$$\left|\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} - 1\right| < \frac{4}{3},\ z \in \mathbb{E},$$

then $f \in US^*$.

Letting $p = 1$ and $\alpha = 1$ in Theorem 2.2, we obtain:

Corollary 3.8 *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{5}{6},\ z \in \mathbb{E},$$

then $f \in US^*$.

4 Open Problem

The case of Theorem 2.1 when $\alpha < 0$ & $\alpha > 1$ is still open to settle and it would also be of interest to investigate the case of Theorem 2.2 for $\alpha < 0$.

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