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On a Subclass of Analytic Functions of Complex Order Defined by Al-Oboudi Generalized Operator

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Abstract

The main object of this paper is to study a new subclass of analytic functions in the open unit disk which is defined by Al-Oboudi fractional differential operator. Results of coefficients estimates, extreme points, radii of close-to convexity, starlikeness and convexity and integral means inequalities for functions belonging to this subclass are established.

Keywords and phrases: Analytic functions, Al-Oboudi fractional differential operator, coefficient estimates, extreme points, integral means inequalities.

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1. Introduction

Let A be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

that are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C}, \mid z \mid < 1\}$.

The fractional derivative is defined as follows (e.g.,[9,14])

Definition 1.1: *The Riemann-Liouville fractional derivative of order* $\alpha(0 \leq \alpha < 1)$ *is defined for the function* f *by*

$$D_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z (z-t)^{-\alpha} f(t) dt \tag{1.2}$$

where the function $f(z)$ *is analytic in a simply connected region of the* z*-plane containing the origin, and the multiplicity of* $(z-t)^{-\alpha}$ *is removed by requiring* $\log(z-t)$ *to be real when* $z-t > 0$.

Owa and Srivastava [10] introduced the operator $\Omega^\alpha : A \to A$ which is known as an extension of the fractional derivative and fractional integral , as follows:

$$\begin{aligned} \Omega^\alpha f(z) &= \Gamma(2-\alpha) z^\alpha D_z^\alpha f(z) \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} a_k z^k \end{aligned} \tag{1.3}$$

Note that $\Omega^0 f(z) = f(z)$.

Now we recall the linear fractional differential operator $D_\lambda^{n,\alpha} : A \to A$ introduced and studied by Al-Oboudi and Al-Amoudi [2,3]as follows

$$\begin{aligned} D_\lambda^{0,0} f(z) &= f(z) \\ D_\lambda^{1,\alpha} f(z) &= \lambda z (\Omega^\alpha f(z))' + (1-\lambda)\Omega^\alpha f(z) \equiv D_\lambda^\alpha f(z) \\ D_\lambda^{2,\alpha} f(z) &= D_\lambda^\alpha (D_\lambda^{1,\alpha} f(z)) \\ &\vdots \\ D_\lambda^{n,\alpha} f(z) &= D_\lambda^\alpha (D_\lambda^{n-1,\alpha} f(z)) \end{aligned} \tag{1.4}$$

for $n \in \mathbb{N}$, $\lambda \geq 0$ and $0 \leq \alpha < 1$.

If f is given by (1.1), then making use of (1.3) and (1.4) we conclude that

$$D_\lambda^{n,\alpha} f(z) = z + \sum_{k=2}^{\infty} [\psi_k(\alpha,\lambda)]^n a_k z^k, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \tag{1.5}$$

where

$$\psi_k(\alpha,\lambda) = \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} (1+\lambda(k-1)) \, , (k = 2, 3, ...) \tag{1.6}$$

when $\alpha = 0$ we get Al-Oboudi differential operator [1], when $\alpha = 0$ and $\lambda = 1$ the Sălăgean operator is obtained [12], and on setting $\lambda = 0$ and $n = 1$ we obtain the Owa-Srivastava fractional differential operator [10].

Definition 1.2: *A function $f \in A$ is said to be in the class $\mathcal{N}^{\alpha}_{m,n}(b,\delta,\beta,\lambda)$ if*

$$\operatorname{Re}\left\{1+\frac{1}{b}\left[\frac{D^{m,\alpha}_{\lambda}f(z)}{D^{n,\alpha}_{\lambda}f(z)}-1\right]\right\}>\beta\left|\frac{1}{b}\left[\frac{D^{m,\alpha}_{\lambda}f(z)}{D^{n,\alpha}_{\lambda}f(z)}-1\right]\right|+\delta \tag{1.7}$$

for some $\delta\ (0\le\delta<1)$, $\beta\ge 0$, $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $b\in\mathbb{C}\backslash\{0\}$, $\lambda\ge 0$ and $z\in\mathcal{U}$.

The following are the special cases of the class $\mathcal{N}^{\alpha}_{m,n}(b,\delta,\beta,\lambda)$:
i. $\mathcal{N}^{0}_{m,n}(b,\delta,\beta,\lambda)\equiv\mathcal{N}_{m,n}(b,\delta,\beta,\lambda)$, the class introduced by Mahzoon and Latha [8].
ii. $\mathcal{N}^{0}_{m,n}(1,\delta,\beta,1)\equiv\mathcal{N}_{m,n}(\delta,\beta)$, the class introduced by Eker and Owa [6].
iii. $\mathcal{N}^{0}_{1,0}(1,\delta,\beta,1)\equiv\mathcal{SD}(\delta,\beta)$ and $\mathcal{N}^{0}_{2,1}(1,\delta,\beta,1)\equiv\mathcal{KD}(\delta,\beta)$, the classes studied by Shams, Kulkarni and Jahangiri [13].
iv. $\mathcal{N}^{0}_{m,n}(1,\delta,0,1)\equiv\mathcal{K}_{m,n}(\delta)$, the class studied by Eker and Owa [5].
v. $\mathcal{N}^{0}_{1,0}(1,\delta,0,1)\equiv\mathcal{S}(\delta)$ and $\mathcal{N}^{0}_{2,1}(1,\delta,0,1)\equiv\mathcal{K}(\delta)$, the classes introduced by Robertson [11].

The object of the present paper is to investigate the coefficient bounds, extreme points, radii of close-to-convexity, starlikeness and convexity and integral means inequalities for functions belonging to a subclass of $\mathcal{N}^{\alpha}_{m,n}(b,\delta,\beta,\lambda)$.

2. Coefficient Inequalities

The following theorem gives a sufficient condition for functions $f\in A$ to belong to the class $\mathcal{N}^{\alpha}_{m,n}(b,\delta,\beta,\lambda)$.

Theorem 2.1: *Let $f(z)\in A$ satisfy*

$$\sum_{k=2}^{\infty}\Phi_k(m,n,\alpha,\lambda,\delta,\beta,b)\mid a_k\mid\le 2\,|b|\,(1-\delta) \tag{2.1}$$

where

$$\begin{aligned}\Phi_k(m,n,\alpha,\lambda,\delta,\beta,b) &= \mid(1+b\delta)(\psi_k(\alpha,\lambda))^n-(\psi_k(\alpha,\lambda))^m\mid\\ &\quad+[b(2-\delta)-1]\left((\psi_k(\alpha,\lambda))^n-(\psi_k(\alpha,\lambda))^m\right)\\ &\quad+2\beta\left|(\psi_k(\alpha,\lambda))^n-(\psi_k(\alpha,\lambda))^m\right|\end{aligned} \tag{2.2}$$

for some $\delta(0\le\delta<1)$, $\lambda\ge 0$, $\beta\ge 0$, $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $b\in\mathbb{C}\backslash\{0\}$ and $0\le\alpha<1$. *Then $f(z)\in\mathcal{N}^{\alpha}_{m,n}(b,\delta,\beta,\lambda)$.*

Proof: It is sufficient to show that

$$\begin{aligned}&\mid [b(2-\delta)-1]\, D_\lambda^{n,\alpha} f(z) + D_\lambda^{m,\alpha} f(z) - \beta e^{i\theta}\, |D_\lambda^{m,\alpha} f(z) - D_\lambda^{n,\alpha} f(z)| \mid \\ &\quad - \mid (1+b\delta) D_\lambda^{n,\alpha} f(z) - D_\lambda^{m,\alpha} f(z) + \beta e^{i\theta}\, |D_\lambda^{m,\alpha} f(z) - D_\lambda^{n,\alpha} f(z)| \mid > 0\end{aligned}$$

So, we have

$$\begin{aligned}&\mid [b(2-\delta)-1]\, D_\lambda^{n,\alpha} f(z) + D_\lambda^{m,\alpha} f(z) - \beta e^{i\theta}\, |D_\lambda^{m,\alpha} f(z) - D_\lambda^{n,\alpha} f(z)| \mid \\ &\quad - \mid (1+b\delta) D_\lambda^{n,\alpha} f(z) - D_\lambda^{m,\alpha} f(z) + \beta e^{i\theta}\, |D_\lambda^{m,\alpha} f(z) - D_\lambda^{n,\alpha} f(z)| \mid \\ &= \mid b(2-\delta) z + \sum_{k=2}^{\infty} \{[b(2-\delta)-1]\, (\psi_k(\alpha,\lambda))^n + (\psi_k(\alpha,\lambda))^m\} a_k z^k \\ &\quad -\beta e^{i\theta} \mid \sum_{k=2}^{\infty} \{(\psi_k(\alpha,\lambda))^m - (\psi_k(\alpha,\lambda))^n\} a_k z^k \mid \quad \mid \\ &\quad - \mid b\delta z + \sum_{k=2}^{\infty} \{[1+b\delta]\, (\psi_k(\alpha,\lambda))^n - (\psi_k(\alpha,\lambda))^m\} a_k z^k \\ &\quad +\beta e^{i\theta} \mid \sum_{k=2}^{\infty} \{(\psi_k(\alpha,\lambda))^m - (\psi_k(\alpha,\lambda))^n\} a_k z^k \mid \quad \mid \\ &\geq \mid b \mid (2-\delta) \mid z \mid - \sum_{k=2}^{\infty} \mid [b(2-\delta)-1]\, (\psi_k(\alpha,\lambda))^n + (\psi_k(\alpha,\lambda))^m \mid \quad \mid a_k \mid\mid z^k \mid \\ &\quad -\beta \mid e^{i\theta} \mid \sum_{k=2}^{\infty} \mid (\psi_k(\alpha,\lambda))^m - (\psi_k(\alpha,\lambda))^n \mid \quad \mid a_k \mid \quad \mid z^k \mid \\ &\quad - \mid b \mid \delta \mid z \mid - \sum_{k=2}^{\infty} \mid (1+b\delta)(\psi_k(\alpha,\lambda))^n - (\psi_k(\alpha,\lambda))^m \mid \quad \mid a_k \mid \quad \mid z^k \mid \\ &\quad -\beta \mid e^{i\theta} \mid \sum_{k=2}^{\infty} \mid (\psi_k(\alpha,\lambda))^m - (\psi_k(\alpha,\lambda))^n \mid \quad \mid a_k \mid \quad \mid z^k \mid \\ &\geq 2 \mid b \mid (1-\delta) - \sum_{k=2}^{\infty} \{\mid (1+b\delta)(\psi_k(\alpha,\lambda))^n - (\psi_k(\alpha,\lambda))^m \mid \\ &\quad +[(b(2-\delta)-1)(\psi_k(\alpha,\lambda))^n + (\psi_k(\alpha,\lambda))^m] \\ &\quad +2\beta \mid (\psi_k(\alpha,\lambda))^n - (\psi_k(\alpha,\lambda))^m \mid\} \quad \mid a_k \mid \geq 0\end{aligned}$$

which directly yields the inequality (2.1). □

Example: The function $f(z)$ given by

$$f(z) = z + \sum_{k=2}^{\infty} \frac{2(2+\sigma)\,|b|\,(1-\delta)\varepsilon_k}{(k+\sigma)(k+1+\sigma)\Phi_k(m,n,\alpha,\lambda,\delta,\beta,b)} z^k = z + \sum_{k=2}^{\infty} A_k z^k$$

with

$$A_k = \frac{2(2+\sigma)\,|b|\,(1-\delta)\varepsilon_j}{(k+\sigma)(k+1+\sigma)\Phi_k(m,n,\alpha,\lambda,\delta,\beta,b)}$$

belongs to the class $\mathcal{N}_{m,n}^{\alpha}(b,\delta,\beta,\lambda)$ for $\sigma > -2, \varepsilon_k \in \mathbb{C}$ with $|\varepsilon_k| = 1$ and the other parameters are as constrained in Theorem 2.1. Because we have

$$\begin{aligned}\sum_{k=2}^{\infty}\Phi_k(m,n,\alpha,\lambda,\delta,\beta,b)\,|A_k| &\leq \sum_{k=2}^{\infty}\frac{2(2+\sigma)\,|b|\,(1-\delta)}{(k+\sigma)(k+1+\sigma)}\\ &= 2(2+\sigma)\,|b|\,(1-\delta)\sum_{k=2}^{\infty}\left(\frac{1}{(k+\sigma)}-\frac{1}{(k+1+\sigma)}\right)\\ &= 2\,|b|\,(1-\delta).\end{aligned}$$

The coefficient inequality (2.1) has several known results as special cases. For example, setting $\alpha=0$ in Theorem 2.1, we get the result recently obtained by Mahzoon and Latha [8, Thm. 2.1, p.194]. Also on setting $\alpha=0, b=1$ and $\lambda=1$ in Theorem 2.1, the result of coefficient inequality obtained by Eker and Owa [6, Thm. 2.1, p.2] is established. Moreover, if we set $m=1$ and $n=0$, we get the result obtained by Shams et al. [13, Thm. 2.1, p.2959].

3. The Subclass $\widetilde{\mathcal{N}}_{m,n}^{\alpha}(b,\delta,\beta,\lambda)$

In view of Theorem 2.1, we now introduce the subclass

$$\widetilde{\mathcal{N}}_{m,n}^{\alpha}(b,\delta,\beta,\lambda)\subset\mathcal{N}_{m,n}^{\alpha}(b,\delta,\beta,\lambda)$$

which consists of functions

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \quad (a_k\geq 0\ ) \tag{3.1}$$

whose Taylor-Maclaurin coefficients satisfy the inequality (2.1). Thus an immediate corollary of Theorem 2.1 is written as

Corollary 3.1: *Let $f(z)\in\widetilde{\mathcal{N}}_{m,n}^{\alpha}(b,\delta,\beta,\lambda)$. Then*

$$a_k\leq\frac{2\,|b|\,(1-\delta)}{\Phi_k(m,n,\alpha,\lambda,\delta,\beta,b)},\quad (k=2,3,4,...) \tag{3.2}$$

for some $\delta(0\leq\delta<1), \lambda\geq 0, \beta\geq 0,\ \ m\in\mathbb{N},\ n\in\mathbb{N}_0\ , b\in\mathbb{C}\backslash\{0\}$ and $0\leq\alpha<1$.

Theorem 3.2: *If $f\in A$, then*

$$\widetilde{\mathcal{N}}_{m,n}^{\alpha}(b,\delta,\beta_2,\lambda)\subset\widetilde{\mathcal{N}}_{m,n}^{\alpha}(b,\delta,\beta_1,\lambda)$$

for some β_1 and β_2 such that $0 \leq \beta_1 \leq \beta_2$.

Proof: Easily verified since (2.2) implies

$$\sum_{k=2}^{\infty} \Phi_k(m,n,\alpha,\lambda,\delta,\beta_1,b) \mid a_k \mid \leq \sum_{k=2}^{\infty} \Phi_k(m,n,\alpha,\lambda,\delta,\beta_2,b) \mid a_k \mid$$

Therefore, if $f(z) \in \widetilde{\mathcal{N}}_{m,n}^{\alpha}(b,\delta,\beta_2,\lambda)$, then $f(z) \in \widetilde{\mathcal{N}}_{m,n}^{\alpha}(b,\delta,\beta_1,\lambda)$. Hence the proof is complete. □

Next we state the following theorem on extreme points for the class $\widetilde{\mathcal{N}}_{m,n}^{\alpha}(b,\delta,\beta_1,\lambda)$ without proof .

Theorem 3.3: *Let $f_1(z) = z$ and*

$$f_j(z) = z + \frac{2\,|b|\,(1-\delta)\varepsilon_j}{\Phi_j(m,n,\alpha,\lambda,\delta,\beta,b)} z^j, \quad (j = 2,3,4,...;|\varepsilon_j| = 1) \tag{3.3}$$

where $\Phi_j(m,n,\alpha,\lambda,\delta,\beta,b)$ is given by (2.2). Then $f \in \widetilde{\mathcal{N}}_{m,n}^{\alpha}(b,\delta,\beta,\lambda)$ if and only if it can be expressed in the form $f(z) = \sum_{j=1}^{\infty} \eta_j f_j(z)$ where $\eta_j \geq 0$ $(j = 1,2,3,...)$ and $\sum_{j=1}^{\infty} \eta_j = 1$.

Corollary 3.4: *The extreme points of $\widetilde{\mathcal{N}}_{m,n}^{\alpha}(b,\delta,\beta,\lambda)$ are the functions $f_1(z) = z$ and $f_j(z) = z + \frac{2\,|b|\,(1-\delta)\varepsilon_j}{\Phi_j(m,n,\alpha,\lambda,\delta,\beta,b)} z^j$, $(j = 2,3,4,...;|\varepsilon_j| = 1)$ where $\Phi_j(m,n,\alpha,\lambda,\delta,\beta,b)$ is given by (2.2).*

The results of extreme points of the subclass $\widetilde{\mathcal{N}}_{m,n}^{\alpha}(b,\delta,\beta,\lambda)$ obtained in Theorem 3.3 and Corollary 3.4 contain some known special cases. For example, when $\alpha = 0$, we get the results of extreme points recently obtained by Mahzoon and Latha [8, Thm. 4.1 and Cor. 4.2, pp. 196-197], and when $\alpha = 0, b = 1$ and $\lambda = 1$, we get the results obtained by Eker and Owa [6, Thm. 5.1 and Cor. 5.3, pp.7-8].

4. Close-to-Convexity, Starlikeness and Convexity

We determine the radii of close-to-convexity, starlikeness and convexity results for functions in the class $\widetilde{\mathcal{N}}_{m,n}^{\alpha}(b,\delta,\beta,\lambda)$ in the following theorems.

Theorem 4.1: *Let $f \in \widetilde{\mathcal{N}}_{m,n}^{\alpha}(b,\delta,\beta,\lambda)$. Then f is close-to-convex of order $\gamma(0 \leq \gamma < 1)$ in the disk $|z| = r_1$; where*

$$r_1 = \inf_{k\geq 2} \left\{ \frac{(1-\gamma)\Phi_k(m,n,\alpha,\lambda,\delta,\beta,b)}{2k\,|b|\,(1-\delta)} \right\}^{\frac{1}{k-1}} \tag{4.1}$$

Proof: Let f belongs to A. It is known that f is close-to-convex of order γ if it satisfies the condition

$$\mathrm{Re}\{f'(z)\} > \gamma$$

or equivalently

$$|f'(z) - 1| < 1 - \gamma \tag{4.2}$$

For the left hand side of (4.2), we have

$$|f'(z) - 1| = \left| \sum_{k=2}^{\infty} k a_k z^{k-1} \right| \leq \sum_{k=2}^{\infty} k\,|a_k|\,|z|^{k-1}$$

The last expression is bounded by $(1-\gamma)$ if

$$\sum_{k=2}^{\infty} \frac{k}{1-\gamma}\,|a_k|\,|z|^{k-1} < 1$$

By Theorem 2.1 and Corollary 3.1, $f \in \widetilde{\mathcal{N}}_{m,n}^{\alpha}(b,\delta,\beta,\lambda)$ if and only if

$$\sum_{k=2}^{\infty} \frac{\Phi_k(m,n,\alpha,\lambda,\delta,\beta,b)}{2\,|b|\,(1-\delta)} a_k \leq 1, \quad a_k \geq 0 \tag{4.3}$$

Hence, (4.2) holds true if

$$\frac{k}{1-\gamma}\,|z|^{k-1} < \frac{\Phi_k(m,n,\alpha,\lambda,\delta,\beta,b)}{2\,|b|\,(1-\delta)}$$

or equivalently

$$|z|^{k-1} < \frac{(1-\gamma)\Phi_k(m,n,\alpha,\lambda,\delta,\beta,b)}{2k\,|b|\,(1-\delta)}, \quad k \geq 2$$

which completes the proof. □

Theorem 4.2: *Let $f \in \widetilde{\mathcal{N}}_{m,n}^{\alpha}(b,\delta,\beta,\lambda)$. Then f is starlike of order $\gamma(0 \leq \gamma < 1)$ in the disk $|z| = r_2$; where*

$$r_2 = \inf_{k \geq 2} \left\{ \frac{(1-\gamma)\Phi_k(m,n,\alpha,\lambda,\delta,\beta,b)}{2\,(k-\gamma)\,|b|\,(1-\delta)} \right\}^{\frac{1}{k-1}} \tag{4.4}$$

Proof: Let f belongs to A. It is known that f is starlike of order γ if it satisfies the condition

$$\mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \gamma$$

or equivalently

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \gamma \tag{4.5}$$

Thus, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum\limits_{k=2}^{\infty} (k-1)\,|a_k|\,|z|^{k-1}}{1 - \sum\limits_{k=2}^{\infty} |a_k|\,|z|^{k-1}}$$

The last expression is bounded by $(1-\gamma)$ if

$$\sum_{k=2}^{\infty} \frac{k-\gamma}{1-\gamma}\,|a_k|\,|z|^{k-1} < 1$$

By Theorem 2.1 and Corollary 3.1, $f \in \widetilde{\mathcal{N}}^{\alpha}_{m,n}(b, \delta, \beta, \lambda)$ if and only if

$$\sum_{k=2}^{\infty} \frac{\Phi_k(m, n, \alpha, \lambda, \delta, \beta, b)}{2\,|b|\,(1-\delta)} a_k \leq 1, \quad a_k \geq 0$$

Hence, (4.2) holds true if

$$\frac{k-\gamma}{1-\gamma}\,|z|^{k-1} < \frac{\Phi_k(m, n, \alpha, \lambda, \delta, \beta, b)}{2\,|b|\,(1-\delta)}$$

or equivalently

$$|z|^{k-1} < \frac{(1-\gamma)\Phi_k(m, n, \alpha, \lambda, \delta, \beta, b)}{2\,(k-\gamma)\,|b|\,(1-\delta)}, \quad k \geq 2$$

which yields the assertion (4.4). □

Now, $f \in A$ is convex of order $\gamma (0 \leq \gamma < 1)$ if and only if $zf'(z)$ is starlike of order γ, that f satisfies the condition

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma$$

or equivalently

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1 - \gamma$$

So, following similar steps to that in the proof of Theorem 4.2, we get the theorem:

Theorem 4.3: *Let $f \in \widetilde{\mathcal{N}}^{\alpha}_{m,n}(b,\delta,\beta,\lambda)$. Then f is convex of order $\gamma (0 \leq \gamma < 1)$ in the disk $|z| = r_3$; where*

$$r_3 = \inf_{k\geq 2}\left\{\frac{(1-\gamma)\Phi_k(m,n,\alpha,\lambda,\delta,\beta,b)}{2k\,(k-\gamma)\,|b|\,(1-\delta)}\right\}^{\frac{1}{k-1}} . \tag{4.6}$$

By setting specified values of the parameters involved in the subclass $\widetilde{\mathcal{N}}^{\alpha}_{m,n}(b,\delta,\beta,\lambda)$, we can get results for radii of starlikeness, convexity and close-to-convexity for a wide range of more specified subclasses of analytic functions found in the literature.

5. Integral Means Inequalities

Definition 5.1: *For two functions f and g analytic in $\mathcal{U}$, we say that $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$ written as $f(z) \prec g(z)$ if there exists a Schwarz function $\omega(z)$, analytic in $\mathcal{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$, $z \in \mathcal{U}$.*
In particular, if the function g is univalent in $\mathcal{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Now we write down the following lemma of Littlewood [7] (see,e.g. Duren [4])

Lemma 5.2: *If the functions $f(z)$ and $g(z)$ are analytic in $\mathcal{U}$ with $g(z) \prec f(z)$, then for $\gamma > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$*

$$\int_0^{2\pi} |\, g(re^{i\theta})\,|^{\gamma}\, d\theta \leq \int_0^{2\pi} |\, f(re^{i\theta})\,|^{\gamma}\, d\theta. \tag{5.1}$$

Theorem 5.3: *Let $f(z) \in \widetilde{\mathcal{N}}^{\alpha}_{m,n}(b,\delta,\beta,\lambda)$ and suppose that*

$$f_j(z) = z + \frac{2\,|b|\,(1-\delta)\varepsilon_j}{\Phi_j(m,n,\alpha,\lambda,\delta,\beta,b)} z^j, \quad (j = 2,3,4,...;|\varepsilon_j| = 1) \tag{5.2}$$

If there exists an analytic function $\omega(z)$ given by

$$\{\,\omega(z)\}^{j-1} = \frac{\Phi_j(m,n,\alpha,\lambda,\delta,\beta,b)}{2\,|b|\,(1-\delta)\varepsilon_j}\sum_{j=2}^{\infty} a_j z^{j-1} \tag{5.3}$$

then for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |\, f(re^{i\theta})\,|^{\gamma}\, d\theta \leq \int_0^{2\pi} |\, f_j(re^{i\theta})\,|^{\gamma}\, d\theta \tag{5.4}$$

Proof: By Lemma 5.2, it would suffice to show that

$$1+\sum_{j=2}^{\infty} a_j z^{j-1} \prec z + \frac{2\,|b|\,(1-\delta)\varepsilon_j}{\Phi_j(m,n,\alpha,\lambda,\delta,\beta,b)} z^{j-1}, \quad (j=2,3,4,...;|\varepsilon_j|=1)$$

By setting

$$z+\sum_{j=2}^{\infty} a_j z^{j-1} = z + \frac{2\,|b|\,(1-\delta)\varepsilon_j}{\Phi_j(m,n,\alpha,\lambda,\delta,\beta,b)}\{\omega(z)\}^{j-1},$$

we find that

$$\{\omega(z)\}^{j-1} = \frac{\Phi_j(m,n,\alpha,\lambda,\delta,\beta,b)}{2\,|b|\,(1-\delta)\varepsilon_j}\sum_{j=2}^{\infty} a_j z^{j-1}$$

which implies $\omega(0)=0$. Furthermore

$$\begin{aligned} |\omega(z)|^{j-1} &= \left| \frac{\Phi_j(m,n,\alpha,\lambda,\delta,\beta,b)}{2\,|b|\,(1-\delta)\mid \varepsilon_j \mid} \sum_{j=2}^{\infty} a_j z^{j-1} \right| \\ &\leq \frac{\Phi_j(m,n,\alpha,\lambda,\delta,\beta,b)}{2\,|b|\,(1-\delta)} \sum_{k=2}^{\infty} \mid a_j \mid\mid z^{j-1} \mid \\ &\leq |z| \leq 1 \end{aligned}$$

by using (3.2). Hence $f(z) \prec f_j(z)$ which readily yields the integral means inequality (4.4). □

On setting $\alpha = 0$ in Theorem 5.3, we get the result of integral means inequality established by Mahzoon and Latha [8, Thm. 5.2, p.198]. More special results of integral means inequalities can be obtained by specifying the parameters of the subclass $\widetilde{\mathcal{N}}^{\alpha}_{m,n}(b,\delta,\beta,\lambda)$.

Open Problem:

The class $\mathcal{N}^{\alpha}_{m,n}(b,\delta,\beta,\lambda)$ and the subclass $\widetilde{\mathcal{N}}^{\alpha}_{m,n}(b,\delta,\beta,\lambda)$ can be redefined by using a different multiplier operator or by using the concept of convolution to get new classes. So, new results similar or parallel to what obtained in this paper can be derived for the new classes.

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References

[1] F.M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator.* Int. J. Math. Math. Sc., **2004(27)**(2004),1429-1436.

[2] F.M. Al-Oboudi and K.A. Al-Amoudi, *On classes of analytic functions related to conic domains.* J. Math. Anal. Appl., **339(1)**(2008), 655-667.

[3] F.M. Al-Oboudi and K.A. Al-Amoudi, *Subordination results for classes of analytic functions related to conic domains defined by a fractional operator.* J. Math. Anal. Appl., **354**(2009), 412-420.

[4] P.L. Duren, Univalent Functions. Springer-Verlag, New York, 1983.

[5] S. S. Eker and S. Owa, *New applications of classes of analytic functions involving Sălăgean operator*, International Symposium on Complex Function Theory and Applications, Brasov, Romania, 1 - 5 (2006).

[6] S. S. Eker and S. Owa, *Certain classes of analytic functions involving Sălăgean operator*, J. Ineq. Pure Appl. Math. **10(1)** (2009), Article 22, 12 pp.

[7] J.E. Littlewood , *On inequalities in the theory of functions.* Proc. London Math. Soc., **23** (1925), 481-519.

[8] H. Mahzoon and S. Latha, *On classes of analytic functions involving Al-Oboudi operator.* Int. J. Math. Anal. **4(4)**(2010), 193-199.

[9] S. Owa, *On the distortion theorems-I.* Kyungpook Math. J., **18(1)**(1978), 53-59.

[10] S. Owa and H.M. Srivastava, *Univalent and starlike generalized hypergeometric functions.* Canad. J. Math, **39(5)**(1987),1057-1077.

[11] M. S. Robertson, *On the theory of univalent functions.* Annals Math., **37** (1936), 374 - 406.

[12] S.G. Sălăgean, *Subclasses of univalent functions.* in: Complex Analysis, Proc. 5th Rom.-Finn. Semin. Part 1, Bucharest, 1981, in: Lecture Notes in Math., Vol. 1013, 1983, 362-372.

[13] S. Shams, S. R. Kulkarni and J. M. Jahangiri, *Classes of uniformly starlike and convex functions.* Int. J. Math. Math. Sci., **55**(2004), 2959 - 2961.

[14] H.M. Srivastava and S. Owa, Eds., Univalent Functions, Fractional Calculus and Their Applications. Ellis Horwood Series: Mathematics and Its Applications. Ellis Horwood, Chichester, UK, 1989 .