Int. J. Open Problems Complex Analysis, Vol. 3, No. 1, March 2011
ISSN 2074-2827; Copyright ©ICSRS Publication, 2011
www.i-csrs.org

Asymptotics of Orthogonal Polynomials With a Generalized Szegő Condition

[1]R. Khaldi and A. Guezane-Lakoud[2]

[1]Laboratory of Food Safety and Environment
University Badji Mokhtar. Annaba. Algeria
e-mail:rkhadi@yahoo.fr
[2]Laboratory of Advanced Materials
University Badji Mokhtar B.P. 12, 23000, Annaba. Algeria
e-mail:a_guezane@yahoo.fr

Abstract

We focus on the pointwise asymptotics inside the unit disk for orthogonal polynomials with respect to a measure from polynomial Szegő class and perturbed by a sequence of point masses outside the unit circle. Moreover, we show that these asymptotics hold in L^2-sense on the unit circle.

Keywords: *Orthogonal Polynomials, Asymptotic Behavior, Blaschke product, Polynomial Szegő class.*
2000 Mathematics Subject Classification: 42C05; 30E15, 30E10, 30C40, 30D50.

1 Introduction

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle and $z_1, z_2, ..., z_m$ be m fixed points outside $\mathbb{T}$. Consider a measure on $\mathbb{T}\cup\{z_1, z_2, ..., z_m\}$ of the form

$$\alpha = \sigma + \sum_{j=1}^{m} A_j \delta_{z_j}$$

where $\sigma = \sigma_{ac} + \sigma_s$ is a Borel probability measure on the unit circle $\mathbb{T}$ and δ_{z_j} denotes the Dirac unit measure supported at the point z_j with mass $A_j > 0$, for $j = 1, ..., m$.. As usual, σ_{ac} denotes the absolutely continuous part of σ and σ_s the singular part.

Denote by $\mathcal{P}_n$ the set of polynomials of degree at most n and by $\psi_n(z) = \gamma_n z^n + ... \in \mathcal{P}_n(\gamma_n > 0)$ the polynomial of degree n orthonormal with respect to σ i.e.,

$$\frac{1}{2\pi}\int_0^{2\pi} \psi_n(z)\overline{z^k}d\sigma(\theta) + \sum_{j=1}^{m} A_j\psi_n(z_j)\overline{z_j^k} = \gamma_n^{-1}\delta_{kn}, \tag{1}$$

$k = 0, 1, ..., n$, $z = e^{i\theta}$, where δ_{kn} is the Kronecker's symbol.

Let $\varphi_n(z) = k_n z^n + ... \in \mathcal{P}_n(k_n > 0)$ be the polynomial of degree n orthonormal with respect to σ i.e.,

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_n(z)\overline{z^k}d\sigma(\theta) = k_n^{-1}\delta_{kn}, \tag{2}$$

$k = 0, 1, ..., n$, $z = e^{i\theta}$.

For a polynomial $p \in \mathcal{P}_n$, we put $p^*(z) = z^n\overline{p(1/\overline{z})}$. One can check that, for $z \in \mathbb{T}$, $|p^*(z)| = |p(z)|$.

It is well known from the recurrence formulae [8]

$$\begin{aligned} k_n\varphi_{n+1} &= k_{n+1}z\varphi_n + \varphi_{n+1}(0)\varphi_n^* \\ k_n\varphi_{n+1}^* &= k_{n+1}\varphi_n^* + \overline{\varphi_{n+1}(0)}z\varphi_n \end{aligned}$$

that the orthonormal polynomials $\{\varphi_n(z)\}_{n\geq 0}$ are uniquely determined by the so called Geronimus parameters $a_n = -\overline{\varphi_{n+1}(0)}/k_{n+1}, n = 0, 1, ...$ In therm of the orthogonal polynomials $\{\varphi_n(z)\}_{n\geq 0}$ recall the following classes of measures. We say that a probability measure σ belongs to the Nevai class (N) (denoted by $\sigma \in (N)$) if $\lim\limits_{n\to\infty} a_n = 0$. It follow by Rahmanov' theorem [4,5] that condition $\sigma' > 0$ a.e. on $\mathbb{T}$ implies $\sigma \in (N)$. The class of probability measures with $\sigma' > 0$ a.e. on $\mathbb{T}$ is called Erdős' class, denoted by (E). Lastly, σ is a Rahmanov measure (i.e. $\sigma \in (R)$) if

$$(*) - \lim_{n\to\infty} |\varphi_n|^2 d\sigma = dm$$

where m is the probability Lebesgue measure on $\mathbb{T}$ i.e.

$$dm(t) = dt/(2\pi i t) = (1/2\pi)\, d\theta, t = e^{i\theta} \in \mathbb{T}.$$

For these classes of measures we have the following inclusions:

$$(E) \subset (N) \subset (R)\,.$$

Many details concerning these classes of measures can be found in the monograph dedicated to orthogonal polynomials on the unite circle [7] or in [4].

We say that a measure σ belongs to the Szegő class (denoted by $\sigma \in (S)$) if the Radon–Nikodym derivative σ'_{ac} of σ with respect to the probability Lebesgue measure m satisfies the usual Szegő's condition:

$$\int_0^{2\pi} \log \sigma'_{ac}(e^{i\theta}) d\theta > -\infty.$$

If $\sigma \in (S)$ then one can construct the so-called Szegő function

$$D(z) = \exp\left\{ \frac{1}{4\pi} \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log \sigma'_{ac}(e^{i\theta}) d\theta \right\}$$

with the following properties

1) D is analytic in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $D(z) \neq 0$ in $\mathbb{D}$, and $D(0) > 0$.

2) $|D(t)| = \sigma'_{ac}(t)$ a.e. on $\mathbb{T}$.

It is well known (Szegő [8]; Geronimus [2]) that if $\sigma \in (S)$ then

$$\lim_{n\to\infty} D(z)\varphi_n^*(z) = 1$$

for every $z \in \mathbb{D}$. Moreover,

$$\lim_{n\to\infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |D\varphi_n^* - 1|^2 \, d\theta \right\} = 0.$$

It is interesting to establish the same asymptotics for different classes of measures. Recently in 2006 Denisov and Kupin in [1] have obtained similar results for large class of measures defined as follow:

Let p be a trigonometric polynomial such that $p(t) \geq 0$, $t \in \mathbb{T}$. Without loss of generality we can assume that

$$p(t) = \prod_{k=1}^{N} (t - \xi_k)^{2m_k},$$

where $\{\xi_k\}$ are points on $\mathbb{T}$ and $m_k > 0$ are their multiplicities

We say that a measure σ belongs to the polynomial Szegő class (denoted by $\sigma \in (pS)$) if the Radon–Nikodym derivative σ'_{ac} of the absolute part of σ with respect to the Lebesgue measure m satisfies the generalized Szegö's condition:

$$\int_0^{2\pi} p(e^{i\theta}) \log \sigma'_{ac}(e^{i\theta}) d\theta > -\infty.$$

It is easy to see that $(S) \subset (pS) \subset (E)$.

For a measure $\sigma \in (pS)$, Kupin and Denisov in [1], obtained pointwise asymptotics in the open unit disk $\mathbb{D}$ for the associated orthogonal polynomials

$\{\varphi_n(z)\}$ and proved these asymptotics in L^2-sense on the unit circle. In the case where the measure $\sigma \in (pS)$ is perturbed with points masses outside the unit circle, the problem is known as difficult and is an open problem in analysis. In this work, we study the pointwise asymptotics inside the unit disk for orthogonal polynomials with respect to a measure from polynomial Szegő class and perturbed by a finite Blaschke sequence of point masses outside the unit circle. The results of this paper are new and can be considered as a contribution to the evolution of this field.

2 Preliminaries

First we give some notations.

For $\sigma \in (pS)$ we introduce the functions

$$\widetilde{D}(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} K(e^{i\theta}, z)\log\sigma'_{ac}(e^{i\theta})d\theta\right\}, \tag{3}$$

$$\widetilde{\varphi}_n^*(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} K(e^{i\theta}, z)\log\left|\varphi_n^*(e^{i\theta})\right| d\theta\right\}, \tag{4}$$

$$\widetilde{\psi}_n^*(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} K(e^{i\theta}, z)\log\left|\psi_n^*(e^{i\theta})\right| d\theta\right\} \tag{5}$$

where $K(e^{i\theta}, z)$ is the modified Schwarz kernel defined by

$$K(t, z) = \frac{t+z}{t-z}\frac{q(t)}{q(z)},$$

and $q(t) = \prod_{k=1}^{N}(t-\varsigma_k)^{2m_k}/t^{N'}$, $N' = \sum_k m_k$, $t = e^{i\theta} \in \mathbb{T}$.

The functions $\{\widetilde{\varphi}_n^*\}$ and $\left\{\widetilde{\psi}_n^*\right\}$ are called the modified reversed orthogonal polynomials with respect to σ and α respectively and satisfy the following Lemma

Lemma 2.1 *([1]) Let $\sigma \in (pS)$, the functions $\widetilde{D}(z), \widetilde{\varphi}_n^*(z)$ and $\widetilde{\psi}_n^*(z)$ be defined by (3),(4) and (5). Then*

(i) $\left|\widetilde{D}(t)\right|^2 = \sigma'_{ac}$ *a.e.on* $\mathbb{T}$,

(ii) $|\widetilde{\varphi}_n^*(t)| = |\varphi_n^*(t)|$ *a.e.on* $\mathbb{T}$,

(iii) $\left|\widetilde{\psi}_n^*(t)\right| = |\psi_n^*(t)|$ *a.e.on* $\mathbb{T}$.

Next, we cite two useful Theorems from [1].

Theorem 2.2 *([1]) Let $\sigma \in (pS)$. Then*

$$\lim_{n\to\infty} \widetilde{D}(z)\widetilde{\varphi}_n^*(z) = 1$$

for every $z \in \mathbb{D}$.

Theorem 2.3 *([1]) Let $\alpha \in (pS)$. Then*

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \left|\widetilde{D}(e^{i\theta})\widetilde{\varphi}_n^*(e^{i\theta}) - 1\right|^2 d\theta = 0.$$

Remark 2.4 *([1]) Notice that from the following inequality*

$$\left\|\widetilde{D}\widetilde{\varphi}_n^*\right\|_{L^2(\mathbb{T})} - \|1\|_{L^2(\mathbb{T})} \le \left\|\widetilde{D}\widetilde{\varphi}_n^* - 1\right\|_{L^2(\mathbb{T})}$$

and the fact that the right hand side tends to 0 as $n \to \infty$ (see Theorem 2.3), then

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \left|D(e^{i\theta})\varphi_n(e^{i\theta})\right|^2 d\theta = 1,$$

holds and it implies

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \left|\varphi_n(e^{i\theta})\right|^2 d\sigma_s = 0,$$

since $\|\varphi_n\|_{L^2(\sigma)}^2 = 1$.

3 Main results

We recall the asymptotics of the ratio for the two orthonormal polynomials $\{\psi_n(z)\}$ and $\{\varphi_n(z)\}$.

Theorem 3.1 *([3]) If $\alpha \in (N)$, then*

$$\lim_{n\to\infty} \frac{\psi_n(z)}{\varphi_n(z)} = B(z),$$

uniformly for $|z| \ge 1$. Where

$$B(z) = \prod_{k=1}^{m} \frac{(z - z_k)}{(\overline{z_k}z - 1)} \frac{|z_k|}{z_k}$$

is the finite Blaschke product.

Now we prove our main results, first we begin by establish the pointwise asymptotics in the open unit disc for the orthogonal polynomials $\{\psi_n(z)\}$.

Theorem 3.2 *Let a measure* $\alpha = \sigma + \sum_{k=1}^{m} A_k \delta_{z_k}$*, such that* $\sigma \in (pS)$*. Associate with the measure* α *the functions* $\widetilde{D}$ *and* $\widetilde{\psi}_n^*$ *given by (3), (5) then we have*

$$\lim_{n\to\infty} \widetilde{D}(z)\widetilde{\psi}_n^*(z) = 1,$$

for every $z \in \mathbb{D}$*.*

Proof. Consider the following sequence of functions

$$h_n(z) = \frac{\widetilde{\psi}_n^*(z)}{\widetilde{\varphi}_n^*(z)} = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} K(e^{i\theta}, z)\log\left|\frac{\psi_n^*(e^{i\theta})}{\varphi_n^*(e^{i\theta})}\right| d\theta\right\},$$

using Lemma 2.1, it yields

$$h_n(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} K(e^{i\theta}, z)\log\left|\frac{\psi_n(e^{i\theta})}{\varphi_n(e^{i\theta})}\right| d\theta\right\}.$$

By passing to the limit when $n \to \infty$ and using Theorem 3.1 and the fact that $\left|B(e^{i\theta})\right| = 1$, we obtain

$$\lim_{n\to\infty} h_n(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} K(e^{i\theta}, z)\log\left|B(e^{i\theta})\right| d\theta\right\} = 1. \tag{6}$$

Finally, (6) and Theorem 2.2 imply

$$\lim_{n\to\infty} \widetilde{D}(z)\widetilde{\psi}_n^*(z) = \lim_{n\to\infty}\left[\widetilde{D}(z)\widetilde{\varphi}_n^*(z)\right] h_n(z) = 1.$$

This achieves the proof of the Theorem.

To prove the second main result we introduce the following Lemma

Lemma 3.3 *Under the assumptions of Theorem 3.2, we have*

$$\lim_{n\to\infty} 2\mathcal{R}e\int_0^{2\pi} \widetilde{D}(e^{i\theta})\widetilde{\psi}_n^*(e^{i\theta})d\theta = 4\pi.$$

Proof. It is proved in [1,p.22] for $\widetilde{\varphi}_n^*$ that

$$2\mathcal{R}e\int_0^{2\pi} \widetilde{D}(e^{i\theta})\widetilde{\varphi}_n^*(e^{i\theta})d\theta = 4\pi\mathcal{R}e\left[\widetilde{D}(\xi_0)\widetilde{\varphi}_n^*(\xi_0)\right],$$

for a certain $\xi_0 \in \mathbb{D}$, the proof is based on the fact that (see Theorem 2.2),

$$\lim_{n\to\infty}\left[\widetilde{D}(z)\widetilde{\varphi}_n^*(z)\right] = 1, for, z \in \mathbb{D},$$

then by Theorem 3.2 it is also true for $\widetilde{\psi}_n^*$.

Next we give the asymptotics of the modified reversed orthogonal polynomials with respect to α in $L^2(\mathbb{T})$.

Theorem 3.4 *Under the assumptions of Theorem 3.2, we have*

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|\widetilde{D}(e^{i\theta})\widetilde{\psi}_n^*(e^{i\theta})-1\right|^2 d\theta = 0.$$

Proof. First, we transform the integral in the Theorem in the following sum:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left|\widetilde{D}(e^{i\theta})\widetilde{\psi}_n^*(e^{i\theta})-1\right|^2 d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\left|\widetilde{D}(e^{i\theta})\widetilde{\psi}_n^*(e^{i\theta})\right|^2 d\theta \qquad (7)\\ &\quad -2\mathcal{R}e\frac{1}{2\pi}\int_0^{2\pi}\widetilde{D}(e^{i\theta})\widetilde{\psi}_n^*(e^{i\theta})d\theta + 1. \qquad (8)\end{aligned}$$

We start with the second term on the right-hand side of (7). From Lemma 3.3, it holds

$$\lim_{n\to\infty} 2\mathcal{R}e\frac{1}{2\pi}\int_0^{2\pi}\widetilde{D}(e^{i\theta})\widetilde{\psi}_n^*(e^{i\theta})d\theta = 2. \tag{9}$$

Now for the first term on the right-hand side of (7), we have the estimate

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left|\widetilde{D}(e^{i\theta})\widetilde{\psi}_n^*(e^{i\theta})\right|^2 d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\left|D(e^{i\theta})\psi_n^*(e^{i\theta})\right|^2 d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\psi_n(e^{i\theta})}{\varphi_n(e^{i\theta})}\right|^2\left|D(e^{i\theta})\varphi_n(e^{i\theta})\right|^2 d\theta\\ &\le \sup_{t\in\mathbb{T}}\left|\frac{\psi_n(t)}{\varphi_n(t)}\right|^2\left[\frac{1}{2\pi}\int_0^{2\pi}\left|\varphi_n(e^{i\theta})\right|^2\alpha'_{a.c.}d\theta\right] \qquad (10)\end{aligned}$$

Since $\sup_{t\in\mathbb{T}}\left|\frac{\psi_n(t)}{\varphi_n(t)}\right|^2 \to |B(t)|$ as $n\to\infty$ (Theorem 3.1) and $\frac{1}{2\pi}\int_0^{2\pi}\left|\varphi_n(e^{i\theta})\right|^2\alpha'_{ac}d\theta \to 1$ as $n\to\infty$ (see Remark 2.4), then letting $n\to\infty$ in the inequality (9), we get

$$\limsup_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|\widetilde{D}(e^{i\theta})\widetilde{\psi}_n^*(e^{i\theta})\right|^2 d\theta \le |B(t)| = 1. \tag{11}$$

Finally by passing to the limit when $n\to\infty$ in (7) and using (8) and (10), we obtain

$$\begin{aligned}0 &\le \liminf_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|\widetilde{D}(e^{i\theta})\widetilde{\psi}_n^*(e^{i\theta})-1\right|^2 d\theta\\ &\le \limsup_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|\widetilde{D}(e^{i\theta})\widetilde{\psi}_n^*(e^{i\theta})-1\right|^2 d\theta \le 0.\end{aligned}$$

This achieves the proof of the Theorem.

Remark 3.5 *As a consequence of Theorem 3.4 we have*
(i) $\lim\limits_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|\psi_n(e^{i\theta})\right|^2 d\sigma_s = 0$
(ii) $\lim\limits_{n\to\infty}\sum_{j=1}^{m} A_j\left|\psi_n(z_j)\right|^2 = 0$
This is obvious, since from Theorem 3.1, it yields

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|\psi_n(e^{i\theta})\right|^2 d\sigma_{ac} = 1$$

and

$$\|\psi_n\|^2_{L^2(\alpha)} = \frac{1}{2\pi}\int_0^{2\pi}\left|\psi_n(e^{i\theta})\right|^2 d\sigma + \sum_{j=1}^{m} A_j\left|\psi_n(z_j)\right|^2 = 1.$$

Note that the relation (ii) implies that the points z_j*,* $j = 1,..,m$ *attract the zeros of the orthonormal polynomials* $\psi_n(z)$*.*

4 Open problems

1- In this work, we have studied the pointwise asymptotics inside the unit disk for orthogonal polynomials with respect to a measure perturbed by a finite Blaschke sequence of point masses outside the unit circle. The case of a measure supported on the unit circle with an infinite discrete part still remain open problem.

2- The second open problem is the sudy of the pointwise asymptotics of *Lp* extremal polynomials ($p > 0$) for measures concentrated on curve, arc and segment and perturbed by an infinite masses points. The sudy of extremal polynomials contributed in the resolution of other important and open problems in mathematics.

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