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Integral Representation Formula For Maximal Surfaces In The Lorentzian Heisenberg Group $Heis_1^3$

Essin Turhan and Talat Körpinar

Firat University, Department of Mathematics, 23119, Elazig, TURKEY
e-mail: essin.turhan@gmail.com

Abstract

In this paper, we describe a method to derive a Weierstrass-type representation formula for simply connected immersed maximal surfaces in Lorentzian Heisenberg group $Heis_1^3$. We consider the Lorentzian left invariant metric and use some results of Levi-Civita connection. Furthermore, we show that any harmonic map of a simply connected coordinate region D into $Heis_1^3$ can be represented the form.

Keywords: *Weierstrass representation, Heisenberg group, maximal surface.*

1 Introduction

The Weierstrass representation formula for minimal surfaces in $\mathbb{R}^3$ has been a fundamental tool for producing examples and to prove general properties of such surfaces, since it allows to bring into the problem the theory of holomorphic function of one complex variable. In (see [7]) the authors describe a general Weierstrass representation formula for minimal surfaces in an arbitrary Riemannian manifold. The P.D.E. involved are, in general, too complicated to be solved explicitly.

Surface theory has been intensively studied in mathematics and physics. The application of the theory to solitary wave phenomena in physics yields socalled "soliton geometry". An important branch is the Weierstrass representation of the surface in constant curvature space. The representation makes us study surfaces and their properties by means of analysis methods. A classical example of such an approach is given by the Weierstrass representation for the minimal surface in $\mathbb{R}^3$.

Surfaces and their dynamics are key ingredients in a number of phenomena in physics too. They are, for instance, surface waves, propagation of flame fronts, growth of crystals, deformation of membranes, dynamics of vortex sheets, many problems of hydrodynamics connected with motion of boundaries between region of differing densities and viscosities. Number of papers has been devoted to a study and application of the integrals over surfaces in gauge field theories, string theory, quantum gravity and statistical physics (see [12]).

Analytic methods to study surfaces and their properties are of great interest both in mathematics and in physics. A classical example of such an approach is given by theWeierstrass representation for minimal surfaces (see [1–3]). This representation allows us to construct any minimal surface in the three-dimensional Euclidean space $\mathbb{R}^3$ via two holomorphic functions. It is the most powerful tool for the analysis of minimal surfaces.

It is well-known that the classical Weierstrass-Enneper representation formula describes minimal surfaces in Euclidean 3-space $\mathbb{R}^3$ in terms of their Gauss maps and auxiliary holomorphic functions (see [13]). More generally, a remarkable representation formula has been discovered by Kenmotsu (see [3]) for arbitrary surfaces in $\mathbb{R}^3$ with nonvanishing mean curvature, which describes these surfaces in terms of their Gauss maps and mean curvature functions. On the other hand, Kobayashi (see [6]) proved the Lorentzian version of the classical Weierstrass-Enneper representation formula for maximal surfaces in Minkowski 3-space $\mathbb{L}^3$ (see [8]) and applied it to the study of maximal surfaces with conelike singularities.

In this paper, we describe a method to derive a Weierstrass-type representation formula for simply connected immersed maximal surfaces in Lorentzian Heisenberg group $Heis_1^3$. We consider the Lorentzian left invariant metric and use some results of Levi-Civita connection. Furthermore, we show that any harmonic map of a simply connected coordinate region D into $Heis_1^3$ can be represented the form.

2 The Lorentzian Heisenberg Group $Heis_1^3$

Heisenberg group plays an important role in many branches of mathematics such as representation theory, harmonic analysis, PDEs or even quantum mechanics, where it was initially defined as a group of 3×3 matrices

$$\left\{ \begin{pmatrix} 1 & x^1 & x^3 \\ 0 & 1 & x^2 \\ 0 & 0 & 1 \end{pmatrix} : \; x^1, x^2, x^3 \in \mathbb{R} \right\}$$

with the usual multiplication rule.

We will use the following complex definition of the Heisenberg group.

$$Heis_1^3 = \mathbb{C} \times \mathbb{R} = \{(w, z) : w \in \mathbb{C}, z \in \mathbb{R}\}$$

with

$$(w, z) * (\overline{w}, \overline{z}) = (w + \overline{w}, z + \overline{z} + \operatorname{Im}(\langle w, \overline{w} \rangle)), \tag{2.1}$$

where $\langle , \rangle$ is the usual Hermitian product in $\mathbb{C}$.

The identity of the group is $(0, 0, 0)$ and the inverse of (x^1, x^2, x^3) is given by $(-x^1, -x^2, -x^3)$.

Let $a = (w_1, z_1)$, $b = (w_2, z_2)$ and $c = (w_3, z_3)$. The commutator of the elements $a, b \in Heis_1^3$ is equal to

$$\begin{aligned} [a, b] &= a * b * a^{-1} * b^{-1} \\ &= (w_1, z_1) * (w_2, z_2) * (-w_1, -z_1) * (-w_2, -z_2) \\ &= (w_1 + w_2 - w_1 - w_2, z_1 + z_2 - z_1 - z_2) \\ &= (0, \alpha), \end{aligned}$$

where $\alpha \neq 0$ in general. For example

$$[(1, 0), (i, 0)] = (0, 2) \neq (0, 0).$$

Which shows that $Heis^3$ is not abelian.

On the other hand , for any $a, b, c \in Heis_1^3$, their double commutator is

$$\begin{aligned} [[a, b], c] &= [(0, \alpha), (w_3, z_3)] \\ &= (0, 0). \end{aligned}$$

This implies that $Heis_1^3$ is a nilpotent Lie group with nilpotency 2.

The left-invariant Lorentz metric on $Heis_1^3$ is

$$g = -\left(dx^1\right)^2 + \left(dx^2\right)^2 + (x^1 dx^2 + dx^3)^2. \tag{2.2}$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{ \mathbf{e}_1 = \frac{\partial}{\partial x^3}, \; \mathbf{e}_2 = \frac{\partial}{\partial x^2} - x\frac{\partial}{\partial x^3}, \; \mathbf{e}_3 = \frac{\partial}{\partial x^1} \right\}. \tag{2.3}$$

The characterising properties of this algebra are the following commutation relations:

$$[\mathbf{e}_2, \mathbf{e}_3] = 2\mathbf{e}_1, \quad [\mathbf{e}_3, \mathbf{e}_1] = 0, \quad [\mathbf{e}_2, \mathbf{e}_1] = 0, \tag{2.4}$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1, \quad g(\mathbf{e}_3, \mathbf{e}_3) = -1. \tag{2.5}$$

Lemma 2.1 *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2.6}$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

3 Integral Representation Formula in the Lorentzian Heisenberg Group $Heis_1^3$

In this section, we obtain an integral representation formula for spacelike maximal surfaces in the Lorentzian Heisenberg group $Heis_1^3$.

We will denote with $\Omega \subseteq \mathbb{C} \cong \mathbb{R}^2$ a simply connected domain with a complex coordinate $z = u + iv$, $u, v \in \mathbb{R}$. Also we will use the standard notations for complex derivatives:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right). \tag{3.1}$$

For $X \in \chi\left(Heis_1^3\right)$, denote by $\mathsf{ad}(X)^*$ the adjoint operator of $\mathsf{ad}(X)$, i.e., it satisfies the equation

$$g\left([X,Y], Z\right) = g\left(Y, \mathsf{ad}(X)^*(Z)\right), \tag{3.2}$$

for any $Y, Z \in \chi\left(Heis_1^3\right)$. Let U be the symmetric bilinear operator on $\chi\left(Heis_1^3\right)$ defined by

$$U(X,Y) := \frac{1}{2}\left\{\mathsf{ad}(X)^*(Y) + \mathsf{ad}(Y)^*(X)\right\}. \tag{3.3}$$

Lemma 3.1 *Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the orthonormal basis for an orthonormal basis for the corresponding Lie algebra defined in (2.3). Then*

$$\begin{aligned} U(\mathbf{e}_1,\mathbf{e}_1) &= 0, \quad U(\mathbf{e}_1,\mathbf{e}_2) = -\mathbf{e}_3, \; U(\mathbf{e}_1,\mathbf{e}_3) = -\mathbf{e}_2, \\ U(\mathbf{e}_2,\mathbf{e}_1) &= -\mathbf{e}_3, \; U(\mathbf{e}_2,\mathbf{e}_2) = 0, \quad U(\mathbf{e}_2,\mathbf{e}_3) = 0, \\ U(\mathbf{e}_3,\mathbf{e}_1) &= -\mathbf{e}_2, \; U(\mathbf{e}_3,\mathbf{e}_2) = 0, \quad U(\mathbf{e}_3,\mathbf{e}_3) = 0. \end{aligned}$$

Proof. Using (3.2) and (3.3), we have

$$2g(U(X,Y),Z) = g([X,Z],Y) + g([Y,Z],X).$$

Thus, direct computations lead to the table of U above. Lemma 3.1 is proved.

Lemma 3.2 *(see [17]) Let D be a simply connected domain. A smooth map $\varphi : D \longrightarrow Heis_1^3$ is harmonic if and only if*

$$\left(\varphi^{-1}\varphi_u\right)_u + \left(\varphi^{-1}\varphi_v\right)_v - \mathsf{ad}\left(\varphi^{-1}\varphi_u\right)^*\left(\varphi^{-1}\varphi_u\right) - \mathsf{ad}\left(\varphi^{-1}\varphi_v\right)^*\left(\varphi^{-1}\varphi_v\right) = 0 \quad (3.4)$$

holds.

Let $z = u + iv$. Then in terms of complex coordinates z, $\bar{z}$, the harmonic map equation (3.4) can be written as

$$\frac{\partial}{\partial \bar{z}}\left(\varphi^{-1}\frac{\partial \varphi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\varphi^{-1}\frac{\partial \varphi}{\partial \bar{z}}\right) - 2U\left(\varphi^{-1}\frac{\partial \varphi}{\partial z}, \varphi^{-1}\frac{\partial \varphi}{\partial \bar{z}}\right) = 0. \quad (3.5)$$

Let $\varphi^{-1}d\varphi = Adz + \bar{A}d\bar{z}$. Then, (3.5) is equivalent to

$$A_{\bar{z}} + \bar{A}_z = 2U\left(A, \bar{A}\right). \quad (3.6)$$

The Maurer–Cartan equation is given by

$$A_{\bar{z}} - \bar{A}_z = \left[A, \bar{A}\right]. \quad (3.7)$$

(3.6) and (3.7) can be combined to a single equation

$$A_{\bar{z}} = U\left(A, \bar{A}\right) + \frac{1}{2}\left[A, \bar{A}\right]. \quad (3.8)$$

(3.8) is both the integrability condition for the differential equation $\varphi^{-1}d\varphi = Adz + \bar{A}d\bar{z}$ and the condition for φ to be a harmonic map.

Let $D(z,\bar{z})$ be a simply connected domain and $\varphi : D \longrightarrow Heis_1^3$ a smooth map. If we write $\varphi(z) = \left(x^1(z), x^2(z), x^3(z)\right)$, then by direct calculation

$$A = x_z^1\mathbf{e}_1 + x_z^2\mathbf{e}_2 + \left(x^1x_z^2 + x_z^3\right)\mathbf{e}_3. \quad (3.9)$$

It follows from the harmonic map equation (3.6) that

Theorem 3.3 *$\varphi : D \longrightarrow Heis_1^3$ is harmonic if and only if the following equations hold:*

$$x^1_{z\bar{z}} = 0, \tag{3.10}$$

$$x^2_{z\bar{z}} + x^1_z\left(x^1x^2_{\bar{z}} + x^3_{\bar{z}}\right) + x^1_{\bar{z}}\left(x^1x^2_z + x^3_z\right) = 0, \tag{3.11}$$

$$x^1_{\bar{z}}x^2_z + x^1_z x^2_{\bar{z}} + x^1 x^2_{z\bar{z}} + x^3_{z\bar{z}} = 0. \tag{3.12}$$

Proof. From (3.9), we have

$$\bar{A} = x^1_{\bar{z}}\mathbf{e}_1 + x^2_{\bar{z}}\mathbf{e}_2 + \left(x^1x^2_{\bar{z}} + x^3_{\bar{z}}\right)\mathbf{e}_3. \tag{3.13}$$

Using (3.9) and (3.13), we obtain

$$\begin{aligned} U\left(A, \bar{A}\right) &= -x^1_z x^2_{\bar{z}}\mathbf{e}_3 - x^1_z\left(x^1x^2_{\bar{z}} + x^3_{\bar{z}}\right)\mathbf{e}_2 \\ &\quad -x^1_{\bar{z}}x^2_z\mathbf{e}_3 - x^1_{\bar{z}}\left(x^1x^2_z + x^3_z\right)\mathbf{e}_2. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} A_{\bar{z}} &= x^1_{z\bar{z}}\mathbf{e}_1 + x^2_{z\bar{z}}\mathbf{e}_2 + \left(x^1_{\bar{z}}x^2_z + x^1x^2_{z\bar{z}} + x^3_{z\bar{z}}\right)\mathbf{e}_3, \\ \bar{A}_z &= x^1_{\bar{z}z}e_1 + x^2_{\bar{z}z}e_2 + \left(x^1_z x^2_{\bar{z}} + x^1x^2_{\bar{z}z} + x^3_{\bar{z}z}\right)\mathbf{e}_3. \end{aligned}$$

Hence, using (3.6) we obtain (3.10)-(3.12). This completes the proof of the Theorem.

The exterior derivative d is decomposed as

$$d = \partial + \bar{\partial}, \quad \partial = \frac{\partial}{\partial z}dz, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}}d\bar{z}, \tag{3.14}$$

with respect to the conformal structure of D.

Let

$$\wp^1 = x^1_z dz, \quad \wp^2 = x^2_z dz, \quad \wp^3 = \left(x^1x^2_z + x^3_z\right)dz. \tag{3.15}$$

Theorem 3.4 *The triplet $\{\wp^1, \wp^2, \wp^3\}$ of $(1,0)$-forms satisfies the following differential system:*

$$\bar{\partial}\wp^1 = 0, \tag{3.16}$$

$$\bar{\partial}\wp^2 = -\left(\wp^1 \wedge \overline{\wp^3} + \overline{\wp^1} \wedge \wp^3\right), \tag{3.17}$$

$$\bar{\partial}\wp^3 = -\left(\wp^1 \wedge \overline{\wp^2} + \overline{\wp^1} \wedge \wp^2\right). \tag{3.18}$$

Proof. From (3.10), we have (3.16) and Equation (3.17) is obtained by (3.12). Finally, using (3.13) and (3.15) we obtain (3.18).

Theorem 3.5 *Let $\{\wp^1, \wp^2, \wp^3\}$ be a solution to (3.16)-(3.18) on a simply connected coordinate region D. Then*

$$\varphi(z, \bar{z}) = 2\,\mathrm{Re}\int_{z_0}^{z} \left(\wp^1, \wp^2, \wp^3 - x^1\wp^2\right) \tag{3.19}$$

is a harmonic map into $Heis_1^3$.

Conversely, any harmonic map of D into $Heis_1^3$ can be represented in this form.

Proof. By theorem (3.3) we see that $\varphi(z, \bar{z})$ is a harmonic curve if and only if $\varphi(z, \bar{z})$ satisfy (3.10)-(3.12).

From (3.15), we have

$$x^1 = 2\,\mathrm{Re}\int_{z_0}^{z} \wp^1,\ x^2 = 2\,\mathrm{Re}\int_{z_0}^{z} \wp^2,\ x^3 = 2\,\mathrm{Re}\int_{z_0}^{z} \left(\wp^3 - x^1\wp^2\right),$$

which proves the theorem.

4 Open Problem

In this work, we obtain relationship between harmonic maps and representation formula. Additionally, problems such as investigation of relationship between biharmonic maps and representation formula. In addition to, resarcher can show umbilic points on a minimal surface are flat.

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