Int. J. Open Problems Complex Analysis, Vol. 3, No. 1, March 2011
ISSN 2074-2827; Copyright ©ICSRS Publication, 2011
www.i-csrs.org

Certain strong differential subordinations using a multiplier transformation and Ruscheweyh operator

Alina Alb Lupaş

Department of Mathematics and Computer Science
University of Oradea
str. Universitatii nr. 1, 410087 Oradea, Romania
e-mail: dalb@uoradea.ro

Abstract

In the present paper we establish several strong differential subordinations regarding the new operator $IR^m_{\lambda,l}$ ***defined by the Hadamard product of the multiplier transformation*** $I(m,\lambda,l)$ ***and the Ruscheweyh operator*** R^m***, given by*** $IR^m_{\lambda,l}: \mathcal{A}^*_\zeta \to \mathcal{A}^*_\zeta$, $IR^m_{\lambda,l} f(z,\zeta) = (I(m,\lambda,l) * R^m) f(z,\zeta)$, ***where*** $\mathcal{A}^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\}$ ***is the class of normalized analytic functions with*** $\mathcal{A}^*_{1\zeta} = \mathcal{A}^*_\zeta$.

Keywords: *strong differential subordination, univalent function, convex function, best dominant, differential operator, convolution product.*
2000 Mathematical Subject Classification: 30C45, 30A20, 34A40.

1 Introduction

Denote by U the unit disc of the complex plane $U = \{z \in C : |z| < 1\}$, $\overline{U} = \{z \in C : |z| \leq 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let

$$\mathcal{A}^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U, \zeta \in \overline{U}\},$$

with $\mathcal{A}^*_{1\zeta} = \mathcal{A}^*_\zeta$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq 2$, and

$$\begin{aligned} \mathcal{H}^*[a,n,\zeta] &= \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \ldots, \\ z &\in U, \zeta \in \overline{U}\}, \end{aligned}$$

for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq n$.

Denote by

$$\mathcal{H}_u(U) = \left\{ f \in \mathcal{H}^*[a, n, \zeta], f(z, \zeta) \text{ univalent in } U, \text{for all } \zeta \in \overline{U} \right\},$$

$$K^* = \left\{ f \in \mathcal{H}^*[a, n, \zeta],\ \operatorname{Re} \frac{zf''(z, \zeta)}{f'(z, \zeta)} + 1 > 0,\ z \in U, \text{for all } \zeta \in \overline{U} \right\},$$

the class of convex functions and

$$S^* = \left\{ f \in \mathcal{H}^*[a, n, \zeta],\ \operatorname{Re} \frac{zf'(z, \zeta)}{f(z, \zeta)} > 0,\ z \in U, \text{for all } \zeta \in \overline{U} \right\}$$

the class of starlike functions.

Definition 1.1 *[7] Let $f(z, \zeta)$, $H(z, \zeta)$ analytic in $U \times \overline{U}$. The function $f(z, \zeta)$ is said to be strongly subordinate to $H(z, \zeta)$ if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z, \zeta) = H(w(z), \zeta)$ for all $\zeta \in \overline{U}$. In such a case we write $f(z, \zeta) \prec\prec H(z, \zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

Remark 1.2 *[7] (i) Since $f(z, \zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and univalent in U, for all $\zeta \in \overline{U}$, definition 1.1 is equivalent to $f(0, \zeta) = H(0, \zeta)$, for all $\zeta \in \overline{U}$, and $f\left(U \times \overline{U}\right) \subset H\left(U \times \overline{U}\right)$.*

(ii) If $H(z, \zeta) \equiv H(z)$ and $f(z, \zeta) \equiv f(z)$, the strong subordination becomes the usual notion of subordination.

Lemma 1.3 *[9, p. 71] Let $h(z, \zeta)$ be a convex function with $h(0, \zeta) = a$ for every $\zeta \in \overline{U}$ and let $\gamma \in \mathbb{C}^*$ be a complex number with $Re\gamma \geq 0$. If $p \in \mathcal{H}^*[a, n, \zeta]$ and*

$$p(z, \zeta) + \frac{1}{\gamma} zp'(z, \zeta) \prec\prec h(z, \zeta),$$

then

$$p(z, \zeta) \prec\prec g(z, \zeta) \prec\prec h(z, \zeta),$$

where $g(z, \zeta) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t, \zeta) t^{\frac{\gamma}{n} - 1} dt$ is convex and it is the best dominant.

Lemma 1.4 *[8] Let $g(z, \zeta)$ be a convex function in U, for all $\zeta \in \overline{U}$, and let*

$$h(z, \zeta) = g(z, \zeta) + n\alpha zg'(z, \zeta), \quad z \in U, \zeta \in \overline{U},$$

where $\alpha > 0$ and n is a positive integer. If

$$p(z, \zeta) = g(0, \zeta) + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \dots, \quad z \in U, \zeta \in \overline{U},$$

is holomorphic in U, for all $\zeta \in \overline{U}$, and

$$p(z, \zeta) + \alpha zp'(z, \zeta) \prec\prec h(z, \zeta), \quad z \in U, \zeta \in \overline{U},$$

then

$$p(z,\zeta) \prec\prec g(z,\zeta)$$

and this result is sharp.

Definition 1.5 *For $f \in \mathcal{A}_\zeta^*$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, the operator $I(m,\lambda,l) : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$ is defined by the following infinite series*

$$I(m,\lambda,l) f(z,\zeta) := z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m a_j(\zeta) z^j.$$

Remark 1.6 *The operator $I(m,\lambda,l)$ verifies the property*

$$(l+1) I(m+1,\lambda,l) f(z,\zeta) =$$

$$[l+1-\lambda] I(m,\lambda,l) f(z,\zeta) + \lambda z (I(m,\lambda,l) f(z,\zeta))',$$

for $z \in U$, $\zeta \in \overline{U}$.

Remark 1.7 *For $l = 0$, $\lambda \geq 0$, the operator $D_\lambda^m = I(m,\lambda,0)$ was introduced and studied by Al-Oboudi [6], which reduced to the Sălăgean differential operator $S^m = I(m,1,0)$ [11] for $\lambda = 1$.*

Definition 1.8 *(Ruscheweyh [10]) For $f \in \mathcal{A}_\zeta^*$, $m \in \mathbb{N}$, the operator R^m is defined by $R^m : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$,*

$$\begin{aligned} R^0 f(z,\zeta) &= f(z,\zeta) \\ R^1 f(z,\zeta) &= z f'(z,\zeta) \\ &\ldots \\ (m+1) R^{m+1} f(z,\zeta) &= z (R^m f(z,\zeta))' + m R^m f(z,\zeta), z \in U, \zeta \in \overline{U}. \end{aligned}$$

Remark 1.9 *If $f \in \mathcal{A}_\zeta^*$, $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, then $R^m f(z,\zeta) = z + \sum_{j=2}^{\infty} C_{m+j-1}^m a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.*

2 Main Results

Definition 2.1 *[1] Let $\lambda, l \geq 0$ and $m \in \mathbb{N}$. Denote by $IR_{\lambda,l}^m$ the operator given by the Hadamard product (the convolution product) of the operator $I(m,\lambda,l)$ and the Ruscheweyh operator R^m, $IR_{\lambda,l}^m : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$,*

$$IR_{\lambda,l}^m f(z,\zeta) = (I(m,\lambda,l) * R^m) f(z,\zeta).$$

Remark 2.2 *If $f \in \mathcal{A}_\zeta^*$, $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, then $IR_{\lambda,l}^m f(z,\zeta) = z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m a_j^2(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.*

Remark 2.3 *For $l=0$, $\lambda\geq 0$, we obtain the Hadamard product DR_{λ}^{n} [3], [5] of the generalized Sălăgean operator D_{λ}^{n} and Ruscheweyh operator R^{n}.*

For $l=0$ and $\lambda=1$, we obtain the Hadamard product SR^{n} [2], [4] of the Sălăgean operator S^{n} and Ruscheweyh operator R^{n}.

Theorem 2.4 *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. If $\lambda,l\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}_{\zeta}^{*}$ and the strong differential subordination*

$$\frac{1}{z}\left(\frac{m+1}{\lambda}IR_{\lambda,l}^{m+1}f(z,\zeta)-\frac{m-2}{\lambda}IR_{\lambda,l}^{m}f(z,\zeta)\right)+$$

$$\frac{\lambda(m-1)-(l+1)}{\lambda(l+1)}\left(IR_{\lambda,l}^{m}f(z,\zeta)\right)'+\left(1-\frac{m-1}{l+1}-\frac{2}{\lambda}\right)-$$

$$\frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)}\int_{0}^{z}\frac{IR_{\lambda,l}^{m}f(t,\zeta)-t}{t^{2}}dt\prec\prec h(z,\zeta),z\in U,\zeta\in\overline{U}\quad(1)$$

holds, then

$$\left(IR_{\lambda,l}^{m}f(z,\zeta)\right)'\prec\prec g(z,\zeta),z\in U,\zeta\in\overline{U}$$

and this result is sharp.

Proof With notation

$$p(z,\zeta)=\left(IR_{\lambda,l}^{m}f(z,\zeta)\right)'=1+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}ja_{j}^{2}(\zeta)z^{j-1}$$

and $p(0,\zeta)=1$, we obtain for $f(z,\zeta)=z+\sum_{j=2}^{\infty}a_{j}(\zeta)z^{j}$,

$p(z,\zeta)+zp'(z,\zeta)=1+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}ja_{j}^{2}(\zeta)z^{j-1}+$
$\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}j(j-1)a_{j}^{2}(\zeta)z^{j-1}=$
$1+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}j^{2}a_{j}^{2}(\zeta)z^{j-1}=$
$\frac{1}{z}\left(z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}C_{m+j}^{m+1}\frac{m+1}{\lambda}a_{j}^{2}(\zeta)z^{j}-\right.$
$\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}\frac{\lambda(m-1)-(l+1)}{\lambda(l+1)}ja_{j}^{2}(\zeta)z^{j}-$
$\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}\frac{m-2}{\lambda}a_{j}^{2}(\zeta)z^{j}-$
$\left.\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}\frac{1}{j-1}\frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)}a_{j}^{2}(\zeta)z^{j}\right)=$
$\frac{1}{z}\left[\frac{m+1}{\lambda}\left(z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}C_{m+j}^{m+1}a_{j}^{2}(\zeta)z^{j}\right)\right.$
$\left.-\frac{m-2}{\lambda}\left(z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}a_{j}^{2}(\zeta)z^{j}\right)\right]+$

$\left(1-\frac{m+1}{\lambda}-\frac{m-2}{\lambda}\right)+\left(1+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}a_j^2\left(\zeta\right)jz^{j-1}\right)\frac{\lambda(m-1)-(l+1)}{\lambda(l+1)}-$ $\frac{\lambda(m-1)-(l+1)}{\lambda(l+1)}-\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}\frac{1}{j-1}\frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)}a_j^2\left(\zeta\right)z^{j-1}=$ $\frac{1}{z}\left(\frac{m+1}{\lambda}IR_{\lambda,l}^{m+1}f\left(z,\zeta\right)-\frac{m-2}{\lambda}IR_{\lambda,l}^{m}f\left(z,\zeta\right)\right)+\frac{\lambda(m-1)-(l+1)}{\lambda(l+1)}\left(IR_{\lambda,l}^{m}f\left(z,\zeta\right)\right)'+$ $\frac{\lambda l-\lambda m+2\lambda-2l-2}{\lambda(l+1)}-\frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)}\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}\frac{1}{j-1}a_j^2\left(\zeta\right)z^{j-1}=$ $\frac{1}{z}\left(\frac{m+1}{\lambda}IR_{\lambda,l}^{m+1}f\left(z,\zeta\right)-\frac{m-2}{\lambda}IR_{\lambda,l}^{m}f\left(z,\zeta\right)\right)+\frac{\lambda(m-1)-(l+1)}{\lambda(l+1)}\left(IR_{\lambda,l}^{m}f\left(z,\zeta\right)\right)'+$ $\left(1-\frac{m-1}{l+1}-\frac{2}{\lambda}\right)-\frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)}\int_0^z\frac{IR_{\lambda,l}^{m}f(t,\zeta)-t}{t^2}dt.$

We have $p\left(z,\zeta\right)+zp'\left(z,\zeta\right)\prec\prec h\left(z,\zeta\right)=g\left(z,\zeta\right)+zg'\left(z,\zeta\right)$, $z\in U$, $\zeta\in\overline{U}$. By using Lemma 1.4 we obtain $p\left(z,\zeta\right)\prec\prec g\left(z,\zeta\right)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $\left(IR_{\lambda,l}^{m}f\left(z,\zeta\right)\right)'\prec\prec g\left(z,\zeta\right)$, $z\in U$, $\zeta\in\overline{U}$ and this result is sharp.

Theorem 2.5 *Let $h\left(z,\zeta\right)$ be a convex function such that $h\left(0,\zeta\right)=1$. If $\lambda,l\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}_{\zeta}^{*}$ and the strong differential subordination*

$$\frac{1}{z}\left(\frac{m+1}{\lambda}IR_{\lambda,l}^{m+1}f\left(z,\zeta\right)-\frac{m-2}{\lambda}IR_{\lambda,l}^{m}f\left(z,\zeta\right)\right)+$$
$$\frac{\lambda\left(m-1\right)-\left(l+1\right)}{\lambda\left(l+1\right)}\left(IR_{\lambda,l}^{m}f\left(z,\zeta\right)\right)'+\left(1-\frac{m-1}{l+1}-\frac{2}{\lambda}\right)-$$
$$\frac{2\left(l+1\right)\left(m-1\right)-2\lambda m}{\lambda\left(l+1\right)}\int_0^z\frac{IR_{\lambda,l}^{m}f\left(t,\zeta\right)-t}{t^2}dt\prec\prec h\left(z,\zeta\right),z\in U,\zeta\in\overline{U}\quad(2)$$

holds, then

$$\left(IR_{\lambda,l}^{m}f\left(z,\zeta\right)\right)'\prec\prec g\left(z,\zeta\right)\prec\prec h\left(z,\zeta\right),z\in U,\zeta\in\overline{U},$$

where $g\left(z,\zeta\right)=\frac{1}{z}\int_0^z h\left(t,\zeta\right)dt$ is convex and it is the best dominant.

Proof With notation

$$p\left(z,\zeta\right)=\left(IR_{\lambda,l}^{m}f\left(z,\zeta\right)\right)'=1+\sum_{j=2}^{\infty}\left(\frac{1+\lambda\left(j-1\right)+l}{l+1}\right)^{m}C_{m+j-1}^{m}ja_j^2\left(\zeta\right)z^{j-1}$$

and $p\left(0,\zeta\right)=1$, we obtain for $f(z,\zeta)=z+\sum_{j=2}^{\infty}a_j\left(\zeta\right)z^j$,

$p\left(z,\zeta\right)+zp'\left(z,\zeta\right)=\frac{1}{z}\left(\frac{m+1}{\lambda}IR_{\lambda,l}^{m+1}f\left(z,\zeta\right)-\frac{m-2}{\lambda}IR_{\lambda,l}^{m}f\left(z,\zeta\right)\right)+$ $\frac{\lambda(m-1)-(l+1)}{\lambda(l+1)}\left(IR_{\lambda,l}^{m}f\left(z,\zeta\right)\right)'+\left(1-\frac{m-1}{l+1}-\frac{2}{\lambda}\right)-$ $\frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)}\int_0^z\frac{IR_{\lambda,l}^{m}f(t,\zeta)-t}{t^2}dt.$

We have $p\left(z,\zeta\right)+zp'\left(z,\zeta\right)\prec\prec h\left(z,\zeta\right)$, $z\in U$, $\zeta\in\overline{U}$. Since $p\left(z,\zeta\right)\in\mathcal{H}^{*}\left[1,1,\zeta\right]$, using Lemma 1.3, for $n=1$ and $\gamma=1$, we obtain $p\left(z,\zeta\right)\prec\prec g\left(z,\zeta\right)\prec\prec h\left(z,\zeta\right)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $\left(IR_{\lambda,l}^{m}f\left(z,\zeta\right)\right)'\prec\prec g\left(z,\zeta\right)=\frac{1}{z}\int_0^z h\left(t,\zeta\right)dt\prec\prec h\left(z,\zeta\right)$, $z\in U$, $\zeta\in\overline{U}$, and $g\left(z,\zeta\right)$ is convex and it is the best dominant.

Theorem 2.6 *Let $g(z,\zeta)$ be a convex function, $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. If $\lambda,l\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and verifies the strong differential subordination*

$$\left(IR^m_{\lambda,l}f(z,\zeta)\right)'\prec\prec h(z,\zeta),z\in U,\zeta\in\overline{U}, \tag{3}$$

then

$$\frac{IR^m_{\lambda,l}f(z,\zeta)}{z}\prec\prec g(z,\zeta),z\in U,\zeta\in\overline{U},$$

and this result is sharp.

Proof For $f\in\mathcal{A}^*_\zeta$, $f(z,\zeta)=z+\sum_{j=2}^{\infty}a_j(\zeta)z^j$ we have $IR^m_{\lambda,l}f(z,\zeta)=z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1}a_j^2(\zeta)z^j$, $z\in U$, $\zeta\in\overline{U}$.

Consider $p(z,\zeta)=\frac{IR^m_{\lambda,l}f(z,\zeta)}{z}=\frac{z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1}a_j^2(\zeta)z^j}{z}=1+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1}a_j^2(\zeta)z^{j-1}$.

We have $p(z,\zeta)+zp'(z,\zeta)=\left(IR^m_{\lambda,l}f(z,\zeta)\right)'$, $z\in U$, $\zeta\in\overline{U}$.

Then $\left(IR^m_{\lambda,l}f(z,\zeta)\right)'\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$ becomes $p(z,\zeta)+zp'(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. By using Lemma 1.4 we obtain $p(z,\zeta)\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $\frac{IR^m_{\lambda,l}f(z,\zeta)}{z}\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Theorem 2.7 *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta)=1$. If $\lambda,l\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and verifies the strong differential subordination*

$$\left(IR^m_{\lambda,l}f(z,\zeta)\right)'\prec\prec h(z,\zeta),z\in U,\zeta\in\overline{U}, \tag{4}$$

then

$$\frac{IR^m_{\lambda,l}f(z,\zeta)}{z}\prec\prec g(z,\zeta)\prec\prec h(z,\zeta),z\in U,\zeta\in\overline{U},$$

where $g(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)\,dt$ is convex and it is the best dominant.

Proof The proof is the same with the proof of Theorem 2.6, using Lemma 1.3, for $n=1$ and $\gamma=1$.

Theorem 2.8 *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. If $\lambda,l\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and verifies the strong differential subordination*

$$\left(\frac{zIR^{m+1}_{\lambda,l}f(z,\zeta)}{IR^m_{\lambda,l}f(z,\zeta)}\right)'\prec\prec h(z,\zeta),z\in U,\zeta\in\overline{U}, \tag{5}$$

then

$$\frac{IR_{\lambda,l}^{m+1}f\left(z,\zeta\right)}{IR_{\lambda,l}^{m}f\left(z,\zeta\right)}\prec\prec g\left(z,\zeta\right),z\in U,\zeta\in\overline{U}$$

and this result is sharp.

Proof For $f\in\mathcal{A}_{\zeta}^{*}$, $f(z,\zeta)=z+\sum_{j=2}^{\infty}a_{j}\left(\zeta\right)z^{j}$ we have $IR_{\lambda,l}^{m}f\left(z,\zeta\right)=z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}a_{j}^{2}\left(\zeta\right)z^{j}$, $z\in U$, $\zeta\in\overline{U}$.

Consider $p\left(z,\zeta\right)=\frac{IR_{\lambda,l}^{m+1}f(z,\zeta)}{IR_{\lambda,l}^{m}f(z,\zeta)}=\frac{z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}C_{m+j}^{m+1}a_{j}^{2}(\zeta)z^{j}}{z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}a_{j}^{2}(\zeta)z^{j}}=\frac{1+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}C_{m+j}^{m+1}a_{j}^{2}(\zeta)z^{j-1}}{1+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}a_{j}^{2}(\zeta)z^{j-1}}$.

We have $p'\left(z,\zeta\right)=\frac{\left(IR_{\lambda,l}^{m+1}f(z,\zeta)\right)'}{IR_{\lambda,l}^{m}f(z,\zeta)}-p\left(z,\zeta\right)\cdot\frac{\left(IR_{\lambda,l}^{m}f(z,\zeta)\right)'}{IR_{\lambda,l}^{m}f(z,\zeta)}$.

Then $p\left(z,\zeta\right)+zp'\left(z,\zeta\right)=\left(\frac{zIR_{\lambda,l}^{m+1}f(z,\zeta)}{IR_{\lambda,l}^{m}f(z,\zeta)}\right)'$.

Relation (5) becomes $p\left(z,\zeta\right)+zp'\left(z,\zeta\right)\prec\prec h\left(z,\zeta\right)=g\left(z,\zeta\right)+zg'\left(z,\zeta\right)$, $z\in U$, $\zeta\in\overline{U}$ and by using Lemma 1.4 we obtain $p\left(z,\zeta\right)\prec\prec g\left(z,\zeta\right)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $\frac{IR_{\lambda,l}^{m+1}f(z,\zeta)}{IR_{\lambda,l}^{m}f(z,\zeta)}\prec\prec g\left(z,\zeta\right)$, $z\in U$, $\zeta\in\overline{U}$.

Theorem 2.9 *Let $h\left(z,\zeta\right)$ be a convex function, $h\left(0,\zeta\right)=1$. If $\lambda,l\geq0$, $m\in\mathbb{N}$, $f\in\mathcal{A}_{\zeta}^{*}$ and verifies the strong differential subordination*

$$\left(\frac{zIR_{\lambda,l}^{m+1}f\left(z,\zeta\right)}{IR_{\lambda,l}^{m}f\left(z,\zeta\right)}\right)'\prec\prec h\left(z,\zeta\right),z\in U,\zeta\in\overline{U},\tag{6}$$

then

$$\frac{IR_{\lambda,l}^{m+1}f\left(z,\zeta\right)}{IR_{\lambda,l}^{m}f\left(z,\zeta\right)}\prec\prec g\left(z,\zeta\right)\prec\prec h\left(z,\zeta\right),z\in U,\zeta\in\overline{U},$$

where $g\left(z,\zeta\right)=\frac{1}{z}\int_{0}^{z}h\left(t,\zeta\right)dt$ is convex and it is the best dominant.

Proof The proof is the same with the proof of Theorem 2.7, using Lemma 1.3, for $n=1$ and $\gamma=1$.

3 Open Problem

The definitions, theorems and corollaries we established in this paper can be extended by using the notion of strong subordination. One can use the operator from definition 2.1 and the same technique to prove earlier results and to obtain a new set of results.

References

[1] A. Alb Lupaş, *A note on a certain subclass of analytic functions defined by multiplier transformation*, Journal of Computational Analysis and Applications, Vol. 12, No. 1-B, 2010, 369-373.

[2] A. Alb Lupaş, A. Cătaş, *Certain differential subordinations using Sălăgean and Ruscheweyh operators*, Acta Universitatis Apulensis, submitted, 2010.

[3] A. Alb Lupaş, *Certain differential subordinations using a generalized Sălăgean operator and Ruscheweyh operator*, Journal of Mathematics and Applications, submitted, 2009.

[4] A. Alb Lupaş, *Certain strong differential subordinations using Sălăgean and Ruscheweyh operators*, Advances in Applied Mathematical Analysis, to appear.

[5] A. Alb Lupaş, *Certain strong differential subordinations using a generalized Sălăgean operator and Ruscheweyh operator*, International Journal of Applied Mathematics, submitted, 2010.

[6] F.M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Ind. J. Math. Math. Sci., 2004, no.25-28, 1429-1436.

[7] G.I. Oros, Gh. Oros, *Strong differential subordination*, Turkish Journal of Mathematics, 33 (2009), 249-257.

[8] S.S. Miller, P.T. Mocanu, *On some classes of first-order differential subordinations*, Michigan Math. J., 32 (1985), no. 2, 185-195.

[9] S.S. Miller, P.T. Mocanu, *Differential Subordinations. Theory and Applications*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 225, Marcel Dekker Inc., New York, Basel, 2000.

[10] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amet. Math. Soc., **49** (1975), 109-115.

[11] G.St. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, **1013** (1983), 362-372.