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Certain Differential Inequalities of Meromorphic Functions Associated With Integral Operators

V. G. Gupta* and Bhavna Sharma**

*Department of Mathematics, University of Rajasthan

e-mail: Email: vgguptauor@rediffmail.com

**Department of Mathematics, Sri Balaji College of Engineering

e-mail: vallabhi_2007@yahoo.com

Abstract

The main object of the present paper is to derive certain differential inequalities for two integral operators P_β^α ***and*** Q_β^α ***which are introduced by Lashin [1].***

Keywords: *Meromorphic univalent function, Integral operators.*

1 Introduction

Let $\sum$ denote the class of functions of the form

(1.1) $$f(z) = \frac{1}{z} + \sum_{k==1}^{\infty} a_k z^k,$$

which are analytic in the punctured unit disk $U^* = \{z : 0 < |z| < 1\} = U \setminus \{0\}$, with a simple pole at the origin.

For the function $f(z) \in \sum$, given by (1.1) and $g(z) \in \sum$ defined by

(1.2) $$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k, \quad z \in U^*$$

the Hadamard product (or convolution) of f(z) and g(z) is given by

(1.3) $$(f*g)(z):=\frac{1}{z}+\sum_{k=1}^{\infty}a_kb_kz^k=:(g*f)(z).$$

Analogous to the operators defined by Jung, Kim and Srivastava [2] on the analytic functions Lashin [1] defines the following integral operators $P_\beta^\alpha, Q_\beta^\alpha:\Sigma\to\Sigma$:

(1.4) $$P_\beta^\alpha=P_\beta^\alpha f(z)=\frac{\beta^\alpha}{\Gamma(\alpha)}\frac{1}{z^{\beta+1}}\int_0^z t^\beta\left(\log\frac{z}{t}\right)^{\alpha-1}f(t)dt\ \ (\alpha>0,\beta>0;z\in U^*)$$

and

(1.5) $$Q_\beta^\alpha=Q_\beta^\alpha f(z)=\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)\Gamma(\alpha)}\frac{1}{z^{\beta+1}}\int_0^z t^\beta\left(1-\frac{t}{z}\right)^{\alpha-1}f(t)dt\ \ (\alpha>0,\beta>0;z\in U^*),$$

where $\Gamma(\alpha)$ is the familiar Gamma function. Using the integral representation of the Gamma and Beta functions given by (1.4) and (1.5), it can be shown that

$$P_\beta^\alpha f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{\beta}{k+\beta+1}\right)^\alpha a_kz^k,\ \ (\alpha>0,\beta>0)$$

and

(1.7) $$Q_\beta^\alpha f(z)=\frac{1}{z}+\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\beta+1)}{\Gamma(k+\beta+\alpha+1)}a_kz^k,\ \ (\alpha>0,\beta>0).$$

It is easily verified from (1.6) and (1.7) (see [1])

(1.8) $$z(P_\beta^\alpha f(z))'=\beta(P_\beta^{\alpha-1}f(z))-(\beta+1)(P_\beta^\alpha f(z))\ \ (\alpha>1,\beta>0)$$

and

(1.9) $$z(Q_\beta^\alpha f(z))'=(\beta+\alpha-1)(Q_\beta^{\alpha-1}f(z))-(\beta+\alpha)(Q_\beta^\alpha f(z))\ \ (\alpha>1,\beta>0).$$

Definition 1.1. Let H be the set of complex valued function $h(r,s,t):C^3\to C$ (C is the complex plane) such that:

(i) $h(r,s,t)$ is continuous in a domain $D\subset C^3$,

(ii) $(1,1,1)\in D$ and $|h(1,1,1)|<1$,

(iii) $$\left| h\left(e^{i\theta}, \frac{\zeta}{\beta} + e^{i\theta}, \frac{L+2\beta e^{i\theta}}{\beta(1+\frac{\beta}{\zeta}e^{i\theta})} + \frac{e^{i\theta}}{\beta} \right) \right| \geq 1,$$

whenever

$$\left| h\left(e^{i\theta}, \frac{\zeta}{\beta} + e^{i\theta}, \frac{L+2\beta e^{i\theta}}{\beta(1+\frac{\beta}{\zeta}e^{i\theta})} + \frac{e^{i\theta}}{\beta} \right) \right| \in D$$

with Re(L) > 0 for real θ and for $\zeta \geq 1$.

Definition 1.2. Let G be the set of complex valued function $g(r,s,t): C^3 \to C$ (C is the complex plane) such that:

(i) $g(r,s,t)$ is continuous in a domain $D \subset C^3$,

(ii) $(1,1,1) \in D$ and $|g(1,1,1)| < 1$,

(iii)

$$\left| g\left(e^{i\theta}, \frac{\zeta}{\beta+\alpha-2} + \frac{(\beta+\alpha-1)e^{i\theta}-1}{\beta+\alpha-2}, \frac{1}{\beta+\alpha-3}\left[\frac{(M-1)+2(\beta+\alpha-1)e^{i\theta}}{1+\left(\frac{e^{i\theta}(\beta+\alpha-1)}{\zeta} - \frac{1}{\zeta} \right)} - (\beta+\alpha-1) \right] e^{i\theta} \right) \right| \geq 1$$

whenever

$$\left| g\left(e^{i\theta}, \frac{\zeta}{\beta+\alpha-2} + \frac{(\beta+\alpha-1)e^{i\theta}-1}{\beta+\alpha-2}, \frac{1}{\beta+\alpha-3}\left[\frac{(M-1)+2(\beta+\alpha-1)e^{i\theta}}{1+\left(\frac{e^{i\theta}(\beta+\alpha-1)}{\zeta} - \frac{1}{\zeta} \right)} - (\beta+\alpha-1) \right] e^{i\theta} \right) \right| \in D \text{ for } \zeta \geq 1.$$

with Re(M) > 1 for real θ and for $\zeta \geq 1$.

2 Main Results

In proving our main result, we shall need the following lemma due to Miller and Mocanu [3].

Lemma 2.1 . Let $w(z) = a + w_n z^n + ...,$ be analytic in U with $w(z) \neq 0$ and $n \geq 1$. If $z_0 = r_0 e^{i\theta} (0 < r_0 < 1)$ and $|w(z_0)| = \max_{|z \leq z_0|} |w(z)|$. Then

(2.1) $$z_0 w'(z_0) = \zeta w(z_0),$$

and

(2.2) $$\mathrm{Re}\left\{1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right\} \geq \zeta,$$ where $\zeta \geq 1$ is a real number.

Theorem 2.1. Let $h(r,s,t) \in H$ and $f \in \Sigma$ satisfy

(2.3) $$\left(\left(\frac{(P_\beta^{\alpha-1} f(z))^{(j)}}{(P_\beta^{\alpha} f(z))^{(j)}}\right), \left(\frac{(P_\beta^{\alpha-2} f(z))^{(j)}}{(P_\beta^{\alpha-1} f(z))^{(j)}}\right), \left(\frac{(P_\beta^{\alpha-3} f(z))^{(j)}}{(P_\beta^{\alpha-2} f(z))^{(j)}}\right)\right) \in D \subset C^3$$

and

(2.4) $$\left| h\left(\frac{(P_\beta^{\alpha-1} f(z))^{(j)}}{(P_\beta^{\alpha} f(z))^{(j)}}, \frac{(P_\beta^{\alpha-2} f(z))^{(j)}}{(P_\beta^{\alpha-1} f(z))^{(j)}}, \frac{(P_\beta^{\alpha-3} f(z))^{(j)}}{(P_\beta^{\alpha-2} f(z))^{(j)}}\right)\right| < 1,$$

for all $z \in U; \alpha > 3, \beta > 0$ for some $\alpha \in R^+$.

Then we have

$$\left|\frac{(P_\beta^{\alpha-1} f(z))^{(j)}}{(P_\beta^{\alpha} f(z))^{(j)}}\right| < 1 \quad (z \in U).$$

Proof. Let

(2.5) $$\frac{(P_\beta^{\alpha-1} f(z))^{(j)}}{(P_\beta^{\alpha} f(z))^{(j)}} = w(z),$$

then it follows that $w(z)$ is analytic in U, $w(0) = 1$ and $w(z) \neq 1$. Differentiate (2.5) logarithmically and with the aid of the identity

(2.6) $$z(P_\beta^{\alpha} f(z))^{(j+1)} = \beta(P_\beta^{\alpha-1} f(z))^{(j)} - (\beta + j + 1)(P_\beta^{\alpha} f(z))^{(j)} \quad (\alpha > 1, \beta > 0),$$

and making some simple calculation, we obtain

(2.7) $$\frac{(P_\beta^{\alpha-2}f(z))^{(j)}}{(P_\beta^{\alpha-1}f(z))^{(j)}}=\frac{zw'(z)}{\beta w(z)}+w(z).$$

Again differentiate logarithmically and using (2.6), we easily get

(2.8) $$\frac{(P_\beta^{\alpha-3}f(z))^{(j)}}{(P_\beta^{\alpha-2}f(z))^{(j)}}=\frac{\left(\frac{zw''(z)}{w'(z)}+1\right)+2\beta w(z)}{\beta\left(1+\beta\frac{w(z)}{zw'(z)}.w(z)\right)}+\frac{w(z)}{\beta}.$$

We claim that $|w(z)|<1$ for $z\in U$ .Otherwise there exist a point $z_0\in U$ such that $\max_{|z|<|z_0|}|w(z)|=w(z_0)=1$. Letting $w(z_0)=e^{i\theta}$ and using the Lemma 2.1 with $a=n=1$, we can see that

$$\frac{(P_\beta^{\alpha-1}f(z_0))^{(j)}}{(P_\beta^{\alpha}f(z_0))^{(j)}}=e^{i\theta},$$

$$\frac{(P_\beta^{\alpha-2}f(z_0))^{(j)}}{(P_\beta^{\alpha-1}f(z_0))^{(j)}}=\frac{\zeta}{\beta}+e^{i\theta}$$

and

$$\frac{(P_\beta^{\alpha-3}f(z_0))^{(j)}}{(P_\beta^{\alpha-2}f(z_0))^{(j)}}=\frac{L+2\beta e^{i\theta}}{\beta(1+\frac{\beta}{\zeta}e^{i\theta})}+\frac{e^{i\theta}}{\beta}\qquad \text{where } L=\frac{z_0w''(z_0)}{w'(z_0)}$$

Since $h(r,s,t)\in H$, we have

$$\left|h\left(\frac{(P_\beta^{\alpha-1}f(z_0))^{(j)}}{(P_\beta^{\alpha}f(z_0))^{(j)}},\frac{(P_\beta^{\alpha-2}f(z_0))^{(j)}}{(P_\beta^{\alpha-1}f(z_0))^{(j)}},\frac{(P_\beta^{\alpha-3}f(z_0))^{(j)}}{(P_\beta^{\alpha-2}f(z_0))^{(j)}}\right)\right|$$

$$=\left|h\left(e^{i\theta},\frac{\zeta}{\beta}+e^{i\theta},\frac{L+2\beta e^{i\theta}}{\beta(1+\frac{\beta}{\zeta}e^{i\theta})}+\frac{e^{i\theta}}{\beta}\right)\right|\geq 1$$

where $\mathrm{Re}(L)\geq 0$ and $\zeta\geq 1$.

This contradicts the condition (2.4) of the Theorem 2.1. Therefore we conclude that

$$\left|\frac{(P_\beta^{\alpha-1} f(z_0))^{(j)}}{(P_\beta^{\alpha} f(z_0))^{(j)}}\right| < 1, \quad (z \in U).$$

This completes the proof of the theorem.

Corollary 2.1. Let $h(r,s,t) = s$ and $f \in \Sigma$ satisfy the condition in Theorem 2.1. Then

$$\left|\frac{(P_\beta^{\alpha+i-1} f(z))^{(j)}}{(P_\beta^{\alpha+i} f(z))^{(j)}}\right| < 1, \quad (i = 0,1,2,..., \alpha > 4, i, j \in N, z \in U). \tag{2.9}$$

Theorem 2.2. Let $g(r,s,t) \in G$ and $f \in \Sigma$ satisfy

$$\left(\frac{(Q_\beta^{\alpha-1} f(z))^{(j)}}{(Q_\beta^{\alpha} f(z))^{(j)}}, \frac{(Q_\beta^{\alpha-2} f(z))^{(j)}}{(Q_\beta^{\alpha-1} f(z))^{(j)}}, \frac{(Q_\beta^{\alpha-3} f(z))^{(j)}}{(Q_\beta^{\alpha-2} f(z))^{(j)}}\right) \in D \subset C^3 \tag{2.10}$$

and

$$\left| g\left(\frac{(Q_\beta^{\alpha-1} f(z))^{(j)}}{(Q_\beta^{\alpha} f(z))^{(j)}}, \frac{(Q_\beta^{\alpha-2} f(z))^{(j)}}{(Q_\beta^{\alpha-1} f(z))^{(j)}}, \frac{(Q_\beta^{\alpha-3} f(z))^{(j)}}{(Q_\beta^{\alpha-2} f(z))^{(j)}}\right)\right| < 1, \text{ for all } z \in U; \alpha, \beta > 0 \tag{2.11}$$

for some $\alpha \in N$ .Then we have

$$\left|\frac{(Q_\beta^{\alpha-1} f(z))^{(j)}}{(Q_\beta^{\alpha} f(z))^{(j)}}\right| < 1, \quad (j \in N, z \in U).$$

Proof. Let

$$\frac{(Q_\beta^{\alpha-1} f(z))^{(j)}}{(Q_\beta^{\alpha} f(z))^{(j)}} = w(z). \tag{2.12}$$

Obviously $w(z)$ is analytic in U, $w(0) = 1$ and $w(z) \neq 1$. With the aid of the identity

$$z(Q_\beta^{\alpha} f(z))^{(j+1)} = (\beta + \alpha - 1)(Q_\beta^{\alpha-1} f(z))^{(j)} - (\beta - j - 1)(Q_\beta^{\alpha} f(z))^{(j)} \quad (\alpha > 1, \beta > 0),$$

and proceed exactly same method describe in Theorem 2.1, we easily get

$$\frac{(Q_\beta^{\alpha-2} f(z))^{(j)}}{(Q_\beta^{\alpha-1} f(z))^{(j)}} = \frac{zw'(z)}{(\beta + \alpha - 2)w(z)} + \frac{(\beta + \alpha - 1)w(z) - 1}{(\beta + \alpha - 2)},$$

and

$$\frac{(Q_\beta^{\alpha-3} f(z))^{(j)}}{(Q_\beta^{\alpha-2} f(z))^{(j)}} = \frac{1}{\beta+\alpha-3}\left\{\frac{\frac{zw''(z)}{w'(z)} + 2(\beta+\alpha-1)w(z)}{1+(\beta+\alpha-1)\frac{(w(z))^2}{zw'(z)} - \frac{w(z)}{zw'(z)}}\right\} - (\beta+\alpha-1)w(z).$$

We can see that

$$\frac{(Q_\beta^{\alpha-1} f(z_0))^{(j)}}{(Q_\beta^{\alpha} f(z_0))^{(j)}} = e^{i\theta},$$

$$\frac{(Q_\beta^{\alpha-2} f(z_0))^{(j)}}{(Q_\beta^{\alpha-1} f(z_0))^{(j)}} = \frac{\zeta}{\beta+\alpha-2} + \frac{(\beta+\alpha-1)e^{i\theta}-1}{(\beta+\alpha-2)},$$

and

$$\frac{(Q_\beta^{\alpha-3} f(z_0))^{(j)}}{(Q_\beta^{\alpha-2} f(z_0))^{(j)}} = \frac{1}{\beta+\alpha-3}\left\{\frac{(M-1)+2(\beta+\alpha-1)e^{i\theta}}{1+\frac{(\beta+\alpha-1)}{\zeta}e^{i\theta} - \frac{1}{\zeta}}\right\} - (\beta+\alpha-1)e^{i\theta}$$

where $M = \frac{z_0 w''(z_0)}{w'(z_0)}$ and $\zeta \geq 1$.

Since $g(r,s,t) \in G$, we have

$$\left| g\left(\frac{(Q_\beta^{\alpha-1} f(z_0))^{(j)}}{(Q_\beta^{\alpha} f(z_0))^{(j)}}, \frac{(Q_\beta^{\alpha-2} f(z_0))^{(j)}}{(Q_\beta^{\alpha-1} f(z_0))^{(j)}}, \frac{(Q_\beta^{\alpha-3} f(z_0))^{(j)}}{(Q_\beta^{\alpha-2} f(z_0))^{(j)}} \right) \right|$$

$$= \left| g\left(e^{i\theta}, \frac{\zeta}{\beta+\alpha-2} + \frac{(\beta+\alpha-1)e^{i\theta}-1}{(\beta+\alpha-2)}, \frac{1}{\beta+\alpha-3}\left\{\frac{(M-1)+2(\beta+\alpha-1)e^{i\theta}}{1+\frac{(\beta+\alpha-1)}{\zeta}e^{i\theta} - \frac{1}{\zeta}}\right\} - (\beta+\alpha-1)e^{i\theta} \right) \right| \geq 1,$$

where $\mathrm{Re}(M) \geq 1$.

This contradicts the condition (2.10) of the Theorem 2.2. Therefore we conclude that

$$\left| \frac{(Q_\beta^{\alpha-1} f(z_0))^{(j)}}{(Q_\beta^{\alpha} f(z_0))^{(j)}} \right| < 1, \quad (j \in N, z \in U).$$

This completes the proof of the theorem.

Corollary 2.2. Let $g(r,s,t)=s$ and $f\in\Sigma$ satisfy the condition in Theorem 2.2. Then

(2.13) $$\left|\frac{(Q_\beta^{\alpha+i-1}f(z))^{(j)}}{(Q_\beta^{\alpha+i}f(z))^{(j)}}\right|<1,\quad (i=0,1,2,\ldots,\alpha>4,i,j\in N,z\in U).$$

3 An Open Problem

In this paper, we obtain some differential inequalities for the two integral operators $P_\beta^\alpha f(z)$ and $Q_\beta^\alpha f(z)$. Is it possible to generalize these results for meromorphic multivalent functions?

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