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An Application of Differential Subordination for Starlikeness of Analytic Functions

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Abstract

In this paper, we find certain sufficient conditions for analytic functions to be starlike of order β, $0 \le \beta < 1$ in the open unit disk $\mathbb{E}$. Using differential subordination, we also extend some results of Hitoshi and Owa [1]. We use Mathematica 5.2 to plot the extended regions of the complex plane.

Keywords: *Analytic function, Univalent function, Starlike function, Differential subordination.*

2000 Mathematical Subject Classification: Primary 30C80, Secondary 30C45.

1 Introduction

Let $\mathcal{A}$ be the class of functions f, analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$.

A function $f \in \mathcal{A}$ is said to be starlike of order β, $0 \le \beta < 1$ if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta,\ z \in \mathbb{E}.$$

Let $\mathcal{S}^*(\beta)$ denote the class of all such functions. $\mathcal{S}^* = \mathcal{S}^*(0)$ is the class of univalent starlike functions with respect to the origin.

A function $f \in \mathcal{A}$ is said to be convex of order β, $0 \le \beta < 1$ if it satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta,\ z \in \mathbb{E}.$$

Let $\mathcal{K}(\beta)$ denote the class of all univalent convex functions of order β. $\mathcal{K} = \mathcal{K}(0)$ is the class of univalent convex functions.

Let f be analytic in $\mathbb{E}$, g analytic and univalent in $\mathbb{E}$ and $f(0) = g(0)$. Then, by the symbol $f(z) \prec g(z)$ (f subordinate to g) in $\mathbb{E}$, we shall mean $f(\mathbb{E}) \subset g(\mathbb{E})$.

Let $\psi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be an analytic function, p be an analytic function in $\mathbb{E}$, with $(p(z), zp'(z)) \in \mathbb{C} \times \mathbb{C}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$, then a function p is said to satisfy first order differential subordination if

$$\psi(p(z), zp'(z)) \prec h(z), \ \psi(p(0), 0) = h(0). \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1), is said to be the best dominant of (1).

Starlike functions play an important role in the theory of univalent functions, so it has always been of interest for the researchers to find a new criterion for starlikeness and improve the known criteria of starlikeness. In [1], Hitoshi and Owa proved the following results by making use of Jack's lemma.

Theorem 1.1 *If $f \in \mathcal{A}$ satisfies*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{\alpha + 1}{2(\alpha - 1)}, \ z \in \mathbb{E},$$

for some $\alpha (2 \leq \alpha < 3)$, or

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{5\alpha - 1}{2(\alpha + 1)}, \ z \in \mathbb{E},$$

for some $\alpha (1 < \alpha \leq 2)$, then

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha(1 - z)}{\alpha - z}, \ z \in \mathbb{E}$$

and

$$\left|\frac{zf'(z)}{f(z)} - \frac{\alpha}{\alpha + 1}\right| < \frac{\alpha}{\alpha + 1}, \ z \in \mathbb{E}.$$

This implies that $f \in \mathcal{S}^*$ and $\displaystyle\int_0^z \frac{f(t)}{t}\, dt \in \mathcal{K}$.

Theorem 1.2 *If $f \in \mathcal{A}$ satisfies*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{\alpha + 1}{2\alpha(\alpha - 1)}, \ z \in \mathbb{E},$$

for some $\alpha(\alpha \leq -1)$, *or*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{3\alpha+1}{2\alpha(\alpha+1)}, \; z \in \mathbb{E},$$

for some $\alpha(\alpha > 1)$, *then*

$$\frac{f(z)}{zf'(z)} \prec \frac{\alpha(1-z)}{\alpha - z}, \; z \in \mathbb{E}$$

and

$$f(z) \in \mathcal{S}^*\left(\frac{\alpha+1}{2\alpha}\right)$$

This implies that $\displaystyle\int_0^z \frac{f(t)}{t}\,dt \in \mathcal{K}\left(\frac{\alpha+1}{2\alpha}\right)$.

The main objective of this paper is to find certain sufficient conditions in terms of the convex operator $1 + \dfrac{zf''(z)}{f'(z)}$ for normalized analytic functions to be starlike of order β, $0 \leq \beta < 1$. We also extend the region of variability of the operator $1 + \dfrac{zf''(z)}{f'(z)}$ in above stated results of Hitoshi and Owa [1] for the conclusion of starlikeness. We use Mathematica 5.2 to show the extended regions of the complex plane, pictorially.

2 Preliminaries

We shall use the following lemma to prove our results.

Lemma 2.1 *([3], p.132, Theorem 3.4 h) Let* q *be univalent in* $\mathbb{E}$ *and let* θ *and* ϕ *be analytic in a domain* $\mathbb{D}$ *containing* $q(\mathbb{E})$, *with* $\phi(w) \neq 0$, *when* $w \in q(\mathbb{E})$.
Set $Q(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ *and suppose that either*
(i) h *is convex, or*
(ii) Q *is starlike.*
In addition, assume that
(iii) $\Re\, \dfrac{zh'(z)}{Q(z)} > 0$, $z \in \mathbb{E}$.
If p *is analytic in* $\mathbb{E}$, *with* $p(0) = q(0), p(\mathbb{E}) \subset \mathbb{D}$ *and*

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p \prec q$ *and* q *is the best dominant.*

3 Main Results

Theorem 3.1 *Let* $f \in \mathcal{A}, \dfrac{zf'(z)}{f(z)} \neq 0, z \in \mathbb{E}$, *satisfy the differential subordination*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{(1-2\beta)z}{1+(1-2\beta)z} + \frac{1+2(1-\beta)z}{1-z} = h(z),$$

then $\dfrac{zf'(z)}{f(z)} \prec \dfrac{1+(1-2\beta)z}{1-z}, 0 \leq \beta < 1, z \in \mathbb{E}$ *i.e.* $f \in \mathcal{S}^*(\beta)$.

Proof. Let us define the functions θ and ϕ as follows:

$$\theta(w) = w,$$

and

$$\phi(w) = \frac{1}{w}.$$

Obviously, the functions θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0$ in $\mathbb{D}$.
Also define the functions Q and h as under:

$$Q(z) = zq'(z)\phi(q(z)) = \frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = q(z) + \frac{zq'(z)}{q(z)}.$$

Further, select the functions $p(z) = \dfrac{zf'(z)}{f(z)}$, $f \in \mathcal{A}$ and $q(z) = \dfrac{1+(1-2\beta)z}{1-z}$, $0 \leq \beta < 1$. Therefore,

$$Q(z) = \frac{2(1-\beta)z}{(1-z)[1+(1-2\beta)z]},$$

and

$$\frac{zQ'(z)}{Q(z)} = 1 + \frac{z}{1-z} - \frac{(1-2\beta)z}{1+(1-2\beta)z}.$$

It can easily be verified that $\Re\dfrac{zQ'(z)}{Q(z)} > 0$ in $\mathbb{E}$ for $0 \leq \beta < 1$ and hence Q is starlike in $\mathbb{E}$. We also have

$$h(z) = \frac{(1-2\beta)z}{1+(1-2\beta)z} + \frac{1+2(1-\beta)z}{1-z},$$

and

$$\frac{zh'(z)}{Q(z)} = 1 + \frac{z}{1-z} - \frac{(1-2\beta)z}{1+(1-2\beta)z} + \frac{1+(1-2\beta)z}{1-z}.$$

It is easy to verify that $\Re\dfrac{zh'(z)}{Q(z)} > 0$ in $\mathbb{E}$ for $0 \leq \beta < 1$. Hence, in view of Lemma 2.1, we obtain $\dfrac{zf'(z)}{f(z)} \prec \dfrac{1+(1-2\beta)z}{1-z}, 0 \leq \beta < 1, z \in \mathbb{E}$.

In view of Theorem 3.1, we have the following result.

Corollary 3.1 *If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0, z \in \mathbb{E}$, satisfies the condition*

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \begin{cases} \frac{\beta(1-2\beta)}{2(1-\beta)}, & 0 \leq \beta \leq \frac{1}{2} \\ \frac{(1+\beta)(2\beta-1)}{2\beta}, & \frac{1}{2} \leq \beta < 1 \end{cases},$$

then $f \in \mathcal{S}^*(\beta)$.

Remark 3.1 *By selecting $\beta = 1/2$ in Corollary 3.1, we obtain the following result of Marx [2] and Strohhäcker [4].*

If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0, z \in \mathbb{E}$, satisfies the condition

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > 0,$$

then $f \in \mathcal{S}^*(1/2)$.

By taking $\beta = 1/2$ in Theorem 3.1, we have the following result of Miller and Mocanu [3, Page 60] which is the subordination form of the above result.

Let $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0, z \in \mathbb{E}$, satisfy the differential subordination

$$1+\frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z},\ z \in \mathbb{E},$$

then $\dfrac{zf'(z)}{f(z)} \prec \dfrac{1}{1-z}$.

Remark 3.2 *In this remark, we discuss the fact that the result in Theorem 3.1 is better than Corollary 3.1 for $\dfrac{1}{2} \leq \beta < 1$. We notice that the region of variability of the differential operator $1+\dfrac{zf''(z)}{f'(z)}$ can be extended for the validity of result in Corollary 3.1 for $\dfrac{1}{2} \leq \beta < 1$, using Theorem 3.1.*

For $\beta = \frac{3}{4}$, the constant on the right hand side in Corollary 3.1 reduces to $\frac{7}{12}$. In Figure 3.1, we plot the dashing line $\Re\, z = \frac{7}{12}$ and the curve $h(z)$ (given in Theorem 3.1). The result in Corollary 3.1 holds when the operator $1+\frac{zf''(z)}{f'(z)}$ lies in the portion of the plane right to the line $\Re\, z = \frac{7}{12}$ whereas in view of Theorem 3.1, the same result holds when $1+\frac{zf''(z)}{f'(z)}$ lies in the portion of the plane right to the curve $h(z)$. Therefore, the region between the dashing line and the curve is the claimed extension over the result by Corollary 3.1. We show the extended region below, pictorially.

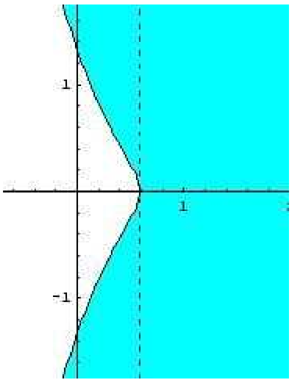

Figure 3.1 (when $\beta = 3/4$)

Theorem 3.2 *Let $\alpha > 1$ be a real number. If $f \in \mathcal{A}$, $\frac{zf'(z)}{f(z)} \neq 0, z \in \mathbb{E}$, satisfies the differential subordination*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{\alpha + (1-\alpha)z}{\alpha - z} - \frac{z}{1-z} = h_1(z),$$

then $\frac{zf'(z)}{f(z)} \prec \frac{\alpha(1-z)}{\alpha - z}$.

Proof. Define the functions $\theta,\ \phi,\ Q,\ h_1$ similar as in Theorem 3.1.

Further, select the functions $p(z) = \frac{zf'(z)}{f(z)}$, $f \in \mathcal{A}$ and $q(z) = \frac{\alpha(1-z)}{\alpha - z}$, $\alpha > 1$. Therefore,

$$Q(z) = \frac{(1-\alpha)z}{(1-z)(\alpha - z)},$$

and

$$\frac{zQ'(z)}{Q(z)} = 1 + \frac{z}{1-z} + \frac{z}{\alpha - z}.$$

Obviously $\Re \dfrac{zQ'(z)}{Q(z)} > 0$ in $\mathbb{E}$ for $\alpha > 1$ and hence Q is starlike in $\mathbb{E}$. We also have

$$h_1(z) = \frac{\alpha(1-z)}{\alpha - z} + \frac{(1-\alpha)z}{(1-z)(\alpha - z)},$$

and

$$\frac{zh_1'(z)}{Q(z)} = 1 + \frac{z}{1-z} + \frac{z}{\alpha - z} + \frac{\alpha(1-z)}{\alpha - z}.$$

It can be easily verified that $\Re \dfrac{zh_1'(z)}{Q(z)} > 0$ in $\mathbb{E}$ for $\alpha > 1$. Therefore, in view of Lemma 2.1, we obtain $\dfrac{zf'(z)}{f(z)} \prec \dfrac{\alpha(1-z)}{\alpha - z}, \alpha > 1, z \in \mathbb{E}$.

Remark 3.3 *The result in Theorem 3.2 extends the region of variability of the operator* $1 + \dfrac{zf''(z)}{f'(z)}$ *over the result in Theorem 1.1 due to Hitoshi and Owa [1] for the required implication. We also show our claim, pictorially in Figure 3.2.*

For $\alpha = \dfrac{3}{2}$, *the constant on right hand side of Theorem 1.1 reduces to* $\dfrac{13}{10}$. *In Figure 3.2, we plot* $h_1(z)$ *(given in Theorem 3.2) and the dashing line* $\Re\ z = \dfrac{13}{10}$. *The result in Theorem 1.1 holds only if the operator* $1 + \dfrac{zf''(z)}{f'(z)}$ *lies in the portion of the plane left to the dashing line* $\Re\ z = \dfrac{13}{10}$, *but our result shows that the result also holds when this operator lies in the portion of the plane left to the curve* $h_1(z)$. *Thus the region bounded by the dashing line* $\Re\ z = \dfrac{13}{10}$ *and the curve* $h_1(z)$ *is our claimed extension.*

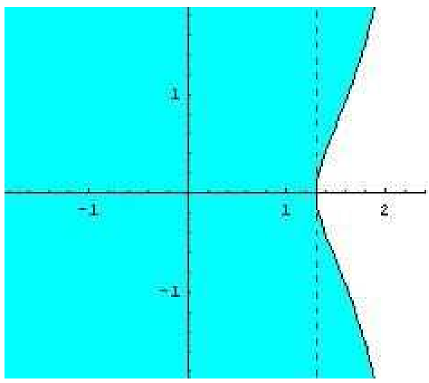

Figure 3.2 (when $\alpha = 3/2$)

Theorem 3.3 *Let α be a real number such that $|\alpha| > 1$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0, z \in \mathbb{E}$, satisfies the differential subordination*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{\alpha - z}{\alpha(1-z)} + \frac{z}{1-z} - \frac{z}{\alpha - z} = h_2(z),$$

then $\dfrac{zf'(z)}{f(z)} \prec \dfrac{\alpha - z}{\alpha(1-z)}$.

Proof. Define the functions $\theta,\ \phi,\ Q,\ h_2$ similar as in Theorem 3.1.

Further, select the functions $p(z) = \dfrac{zf'(z)}{f(z)}$, $f \in \mathcal{A}$ and $q(z) = \dfrac{\alpha - z}{\alpha(1-z)}$, $|\alpha| > 1$. Therefore,

$$Q(z) = \frac{(\alpha - 1)z}{(1-z)(\alpha - z)},$$

and

$$\frac{zQ'(z)}{Q(z)} = 1 + \frac{z}{1-z} + \frac{z}{\alpha - z}.$$

Obviously $\Re\dfrac{zQ'(z)}{Q(z)} > 0$ in $\mathbb{E}$ for $|\alpha| > 1$ and hence Q is starlike in $\mathbb{E}$. We also have

$$h_2(z) = \frac{\alpha - z}{\alpha(1-z)} + \frac{(\alpha - 1)z}{(1-z)(\alpha - z)},$$

and

$$\frac{zh_2'(z)}{Q(z)} = 1 + \frac{z}{1-z} + \frac{z}{\alpha - z} + \frac{\alpha - z}{\alpha(1-z)}.$$

It is easy to verify that $\Re\dfrac{zh_2'(z)}{Q(z)} > 0$ in $\mathbb{E}$ for $|\alpha| > 1$. Thus, in view of Lemma 2.1, we obtain $\dfrac{zf'(z)}{f(z)} \prec \dfrac{\alpha(1-z)}{\alpha - z}, |\alpha| > 1, z \in \mathbb{E}$.

Remark 3.4 *Similar to Remark 3.3, we use Theorem 3.3 to show the extension of the result in Theorem 1.2 of Hitoshi and Owa [1] for the required conclusion. For $\alpha = 2$, the constant on right hand side of Theorem 1.2 reduces to $\dfrac{7}{12}$. In Figure 3.3, we plot $h_2(z)$ (given in Theorem 3.3) and the dashing line $\Re\ z = \dfrac{7}{12}$. According to Theorem 1.2, the result holds only if the operator $1 + \dfrac{zf''(z)}{f'(z)}$ lies in the portion of the plane right to the dashing line $\Re\ z = \dfrac{7}{12}$, but our result shows that the result holds when this operator lies in the portion of the plane right to the curve $h_2(z)$. Thus the region bounded by the dashing line $\Re\ z = \dfrac{7}{10}$ and the curve $h_2(z)$ is the extension. This justifies our claim.*

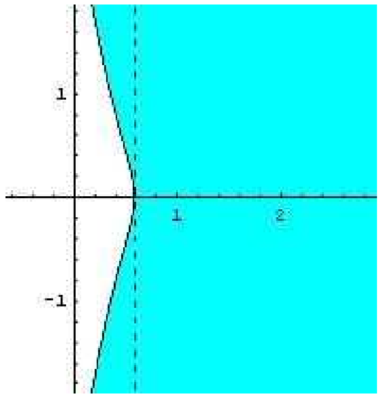

Figure 3.3 (when $\alpha = 2$)

4 Open Problem

One can look at the case for $\alpha \leq 1$ in Theorem 3.2 and for $|\alpha| \leq 1$ in Theorem 3.3.

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