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Angular Estimates For Certain Analytic Univalent Functions

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Abstract

In this present paper, we shall derive the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Some results on angular estimates of functions belonging to the class are obtained. In addition, we derive some interesting conditions for the class of strongly starlike and strongly convex of order α in the open unit disk.

Keywords: Analytic function; Univalent function; Angular estimate; Convex function; Starlike function; Strongly starlike; Strongly convex.

AMS Mathematics Subject Classification (2000): 30C45.

1 Introduction and Definition

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad a_k \quad \text{is complex number} \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane $\mathbb{C}$. Next, we state basic ideas on the familiar subclasses of $\mathcal{A}$ consisting of functions that are starlike of order α in U, convex of order α in U, strongly starlike of order α in U and strongly convex of order α in U. Thus by definitions, we have a function f belonging to $\mathcal{A}$ is said to be starlike of order α if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in U), \tag{1.2}$$

for some α $(0 \leq \alpha < 1)$. We denote by $S^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of order α in U. Also, a function f belonging to $\mathcal{A}$ is said to be convex of order α if it satisfies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in U), \tag{1.3}$$

for some α $(0 \leq \alpha < 1)$. We denote by $C(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are convex of order α in U. In particular, the classes $S^*(0) = S^*$ and $C(0) = C$ are the familiar classes of starlike and convex, respectively.

If f belonging to $\mathcal{A}$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\alpha \qquad (z \in U), \tag{1.4}$$

for some α $(0 < \alpha \leq 1)$, then the function f is said to be strongly starlike of order α in U, and this class denoted by $\overline{S}^*(\alpha)$ the class of all such functions.

If f belonging to $\mathcal{A}$ satisfies

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\alpha \qquad (z \in U), \tag{1.5}$$

for some α $(0 < \alpha \leq 1)$, then we say that f is strongly convex of order α in U, and this class denoted by $\overline{C}(\alpha)$ the class of all such functions.

The purpose of the present paper is to introduce class of analytic function $\mu(\alpha)$ and to study various properties for functions belonging to this class. The subclass $\mu(\alpha)$ is defined as the following:

Definition 1.1 *A function f in $\mathcal{A}$ is said to be a member of the class $\mu(\alpha)$ if and only if*

$$\left|\frac{4z^2f'(z)}{[f(z) - f(-z)]^2} - 1\right| < 1 - \alpha, \tag{1.6}$$

for some α $(0 \leq \alpha < 1)$ and all $z \in U$.

Note that the condition (1.6) implies

$$\mathrm{Re}\left(\frac{4z^2f'(z)}{[f(z) - f(-z)]^2}\right) > \alpha. \tag{1.7}$$

In order to derive our main results, we first recall some preliminary lemmas which shall be used in our proof.

2 Preliminary Results

To establish our results, we have to recall the following:

Lemma 2.1 *(see [1]). Let the function $f \in \mathcal{A}$ satisfy the condition*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 \qquad (z \in U), \tag{2.1}$$

then f is univalent in U.

Lemma 2.2 *(see [2]). Let $w(z)$ be analytic in U and such that $w(0) = 0$. Then if $|w(z)|$ attains its maximum value on circle $|z| = r < 1$ at a point $z_0 \in U$, we have*

$$z_0 w'(z_0) = k w(z_0), \tag{2.2}$$

where $k \geq 1$ is a real number.

Lemma 2.3 *(see [3]). Let a function $p(z)$ be analytic in U, $p(0) = 1$, and $p(z) \neq 0$ $(z \in U)$.If there exists a point $z_0 \in U$ such that*

$$|\arg(p(z))| < \frac{\pi}{2}\alpha, \qquad \mathit{for} \quad |z| < |z_0|, \qquad |\arg(p(z_0))| = \frac{\pi}{2}\alpha, \tag{2.3}$$

with $(0 < \alpha \leq 1)$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha, \tag{2.4}$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1, \quad \mathit{when} \quad \arg(p(z_0)) = \frac{\pi}{2}\alpha,$$

$$k \leq \frac{-1}{2}\left(a + \frac{1}{a}\right) \leq -1 \quad \mathit{when} \quad \arg(p(z_0)) = -\frac{\pi}{2}\alpha,$$

$$(p(z_0))^{\frac{1}{\alpha}} = \pm ai, \quad (a > 0).$$

We begin with the statement and the proof of the following result.

3 Main Results

Our main results, are the following:

Theorem 3.1 *If $f \in \mathcal{A}$ satisfies*

$$\left| \frac{(zf(z))''}{f'(z)} - \frac{2z\left[f(z) - f(-z)\right]'}{f(z) - f(-z)} \right| < \frac{1-\alpha}{2-\alpha} \qquad (z \in U), \tag{3.1}$$

for some α $(0 \leq \alpha < 1)$, then $f(z) \in \mu(\alpha)$.

Proof. Assume $f \notin \mu(\alpha)$, so by Definition 1.1

$$\left| \frac{4z^2 f'(z)}{\left(f(z) - f(-z)\right)^2} - 1 \right| \geq 1 - \alpha.$$

we define the function $w(z)$ by

$$\frac{4z^2 f'(z)}{\left(f(z) - f(-z)\right)^2} = 1 + (1-\alpha)w(z). \tag{3.2}$$

Then $w(z)$ is analytic in U and $w(0) = 0$. By the logarithmic differentiations, we get from (3.2) that

$$\frac{(zf(z))''}{f'(z)} - \frac{2z\left[f(z) - f(-z)\right]'}{f(z) - f(-z)} = \frac{(1-\alpha)zw'(z)}{1 + (1-\alpha)w(z)}. \tag{3.3}$$

Suppose there exists $z_0 \in U$ such that

$$\max |w(z)| = |w(z_0)| = 1, \qquad |z| < |z_0|, \tag{3.4}$$

then from Lemma 2.2, we have

$$z_0 w'(z_0) = k w(z_0), \quad k \geq 1. \tag{3.5}$$

Letting $w(z_0) = e^{i\theta}$, and substitution z_0 in (3.3), we have

$$\begin{aligned} \left| \frac{(z_0 f(z_0))''}{f'(z_0)} - \frac{2z_0\left[f(z_0) - f(-z_0)\right]'}{f(z_0) - f(-z_0)} \right| &= \left| \frac{(1-\alpha)z_0 w'(z_0)}{1 + (1-\alpha)w(z_0)} \right|, \\ &= \left| \frac{(1-\alpha)k e^{i\theta}}{1 + (1-\alpha)e^{i\theta}} \right|, \\ &\geq \frac{(1-\alpha)k\left|e^{i\theta}\right|}{1 + (1-\alpha)\left|e^{i\theta}\right|}, \\ &\geq \frac{1-\alpha}{2-\alpha}. \end{aligned} \tag{3.6}$$

Which contradicts our assumption (3.1). Therefore $|w(z)| < 1$ holds for all $z \in U$. Ultimately, from (3.2) we have

$$\left| \frac{4z^2 f'(z)}{(f(z) - f(-z))^2} - 1 \right| = (1-\alpha)\,|w(z)| < 1 - \alpha \quad (z \in U), \tag{3.7}$$

that is, $f \in \mu(\alpha)$. The proof of Theorem 3.1 is complete.

Taking $\alpha = 0$ and f in the form $f(z) = z + \sum\limits_{k=2}^{\infty} a_{2k-1} z^{2k-1}$ in Theorem 3.1 and applying Lemma 2.1 we have the following corollary.

Corollary 3.2 *If $f \in \mathcal{A}$ satisfies*

$$\left| \frac{(zf(z))''}{f'(z)} - \frac{2z\,[f(z) - f(-z)]'}{f(z) - f(-z)} \right| < \frac{1}{2} \qquad (z \in U). \tag{3.8}$$

Then f is univalent in U.

Next, we prove the following theorem.

Theorem 3.3 *Let $f \in \mathcal{A}$ and the form*

$$f(z) = z + \sum_{k=2}^{\infty} a_{2k-1} z^{2k-1}.$$

If $f \in \mu(\alpha)$, then

$$\left| \arg \left(\frac{f(z) - f(-z)}{2z} \right) \right| < \frac{\pi}{2}\alpha \qquad (z \in U), \tag{3.9}$$

for some α $(0 < \alpha < 1)$ and $\frac{2}{\pi}\tan^{-1}\alpha - \alpha = 1$.

Proof. Assume that (3.9) incorrect, that is

$$\left| \arg \left(\frac{f(z) - f(-z)}{2z} \right) \right| \geq \frac{\pi}{2}\alpha,$$

we define the function $p(z)$ by

$$\frac{f(z) - f(-z)}{2z} = \frac{2f(z)}{2z} = p(z) = 1 + a_3 z^2 + a_5 z^4 + \dots\ . \tag{3.10}$$

Then we see that $p(z)$ is analytic in U, $p(0) = 1$, and $p(z) \neq 0$ $(z \in U)$. It follows from (3.10) that

$$\begin{aligned} \frac{4z^2 f'(z)}{(f(z) - f(-z))^2} &= \frac{z^2 f'(z)}{f^2(z)}, \\ &= \frac{p(z) + zp'(z)}{p^2(z)}, \\ &= \frac{1}{p(z)} \left(1 + \frac{zp'(z)}{p(z)}\right). \end{aligned} \tag{3.11}$$

Now, assume there exists a point $z_0 \in U$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\alpha, \qquad \text{for} \quad |z| < |z_0|, \qquad |\arg(p(z_0))| = \frac{\pi}{2}\alpha. \tag{3.12}$$

Then, applying Lemma 2.3, we can write that

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha, \tag{3.13}$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1, \quad \text{when} \quad \arg(p(z_0)) = \frac{\pi}{2}\alpha,$$

$$k \leq \frac{-1}{2}\left(a + \frac{1}{a}\right) \leq -1 \quad \text{when} \quad \arg(p(z_0)) = -\frac{\pi}{2}\alpha,$$

$$(p(z_0))^{\frac{1}{\alpha}} = \pm ai, \quad (a > 0). \tag{3.14}$$

Substitution z_0 in (3.11), and take the case $\arg(p(z_0)) = \frac{\pi}{2}\alpha$, we get

$$\begin{aligned} \frac{4z_0^2 f'(z_0)}{(f(z_0) - f(-z_0))^2} &= \frac{z_0^2 f'(z_0)}{f^2(z_0)}, \\ &= \frac{1}{p(z_0)}\left(1 + \frac{z_0 p'(z_0)}{p(z_0)}\right), \\ &= \frac{1}{(ai)^\alpha}(1 + ik\alpha), \\ &= a^{-\alpha} e^{i\frac{-\pi\alpha}{2}}(1 + ik\alpha). \end{aligned} \tag{3.15}$$

That implies

$$\begin{aligned}
\arg\left(\frac{4z_0^2 f'(z_0)}{(f(z_0)-f(-z_0))^2}\right) &= \arg\left[\frac{1}{p(z_0)}\left(1+\frac{z_0 p'(z_0)}{p(z_0)}\right)\right], \\
&= \arg\left(\frac{1}{p(z_0)}\right)+\arg\left(1+\frac{z_0 p'(z_0)}{p(z_0)}\right), \\
&= \frac{-\pi}{2}\alpha+\arg\left(1+ik\alpha\right), \\
&= \frac{-\pi}{2}\alpha+\tan^{-1}k\alpha. \qquad (3.16)
\end{aligned}$$

Now, since $\tan^{-1}k\alpha \geq \tan^{-1}\alpha$, where $(k \geq 1)$. So from (3.16) we obtain

$$\begin{aligned}
\arg\left(\frac{4z_0^2 f'(z_0)}{(f(z_0)-f(-z_0))^2}\right) &\geq \frac{-\pi}{2}\alpha+\tan^{-1}\alpha, \\
&= \frac{\pi}{2}\left(\frac{2}{\pi}\tan^{-1}\alpha-\alpha\right)=\frac{\pi}{2}, \qquad (3.17)
\end{aligned}$$

if

$$\frac{2}{\pi}\tan^{-1}\alpha-\alpha=1.$$

Also, if we take the case $\arg(p(z_0)) = -\frac{\pi}{2}\alpha$, and applying the same arguments we get

$$\arg\left(\frac{4z_0^2 f'(z_0)}{(f(z_0)-f(-z_0))^2}\right) \leq -\frac{\pi}{2}. \qquad (3.18)$$

Provided by

$$\frac{\pi}{2}\alpha+\tan^{-1}k\alpha \leq \frac{\pi}{2}\alpha-\tan^{-1}\alpha, \quad \text{where } k \leq -1,$$

and if

$$\frac{2}{\pi}\tan^{-1}\alpha-\alpha=1.$$

Now (3.17) and (3.18) contradict the assumption of the theorem. Hence, (3.9) is correct. This completes the proof.

Now, we prove the following theorem.

Theorem 3.4 *Let $p(z)$ be analytic in U, $p(z) \neq 0$ in U and suppose that*

$$\left|\arg\left(p(z)+\frac{4z^3 f'(z)}{[f(z)-f(-z)]^2}p'(z)\right)\right| < \frac{\pi}{2}\alpha \quad (z \in U), \qquad (3.19)$$

where α $(0 < \alpha < 1)$ and $f \in \mu(\alpha)$. Then we have

$$|\arg(p(z))| < \frac{\pi}{2}\alpha \qquad (z \in U). \qquad (3.20)$$

Proof. Assume that (3.20) incorrect, that is

$$|\arg(p(z))| \geq \frac{\pi}{2}\alpha.$$

Let $p(z)$ be analytic in U, $p(z) \neq 0$ in U and suppose there exists a point $z_0 \in U$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\alpha, \qquad \text{for } \ |z| < |z_0|, \qquad |\arg(p(z_0))| = \frac{\pi}{2}\alpha. \tag{3.21}$$

Using, Lemma 2.3 we can write that

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha, \tag{3.22}$$

where

$$\begin{gathered} k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1, \quad \text{when} \quad \arg(p(z_0)) = \frac{\pi}{2}\alpha, \\ k \leq \frac{-1}{2}\left(a + \frac{1}{a}\right) \leq -1 \quad \text{when} \quad \arg(p(z_0)) = -\frac{\pi}{2}\alpha, \\ (p(z_0))^{\frac{1}{\alpha}} = \pm ai, \quad (a > 0). \end{gathered} \tag{3.23}$$

Now, we take the first case when $\arg(p(z_0)) = \frac{\pi}{2}\alpha$, then it follows that

$$\arg\left(p(z_0) + \frac{4z_0^3 f'(z_0)}{[f(z_0) - f(-z_0)]^2} p'(z_0)\right) = \arg\left[p(z_0)\left(1 + \frac{4z_0^2 f'(z_0)}{[f(z_0) - f(-z_0)]^2}\frac{z_0 p'(z_0)}{p(z_0)}\right)\right],$$

then, by (3.22) we get

$$\begin{aligned} \arg\left(p(z_0) + \frac{4z_0^3 f'(z_0)}{[f(z_0) - f(-z_0)]^2} p'(z_0)\right) &= \arg\left[p(z_0)\left(1 + i\frac{4z_0^2 f'(z_0)}{[f(z_0) - f(-z_0)]^2}k\alpha\right)\right], \\ &= \arg(p(z_0)) + \arg\left(1 + i\frac{4z_0^2 f'(z_0)}{[f(z_0) - f(-z_0)]^2}k\alpha\right), \\ &= \frac{\pi}{2}\alpha + \arg\left(1 + i\frac{4z_0^2 f'(z_0)}{[f(z_0) - f(-z_0)]^2}k\alpha\right). \end{aligned} \tag{3.24}$$

Now, since $f(z_0) \in \mu(\alpha)$, we have

$$\operatorname{Re}\left(\frac{4z_0^2 f'(z_0)}{[f(z_0) - f(-z_0)]^2}\right) > \alpha,$$

after that, since $k \geq 1$ and $0 < \alpha < 1$ we get

$$\operatorname{Re}\left(\frac{4z_0^2 f'(z_0)}{[f(z_0) - f(-z_0)]^2}k\alpha\right) > \alpha,$$

and therefore

$$\arg\left(1+i\frac{4z_0^2f'(z_0)}{[f(z_0)-f(-z_0)]^2}k\alpha\right)>0. \tag{3.25}$$

Thus from (3.25), (3.24) becomes

$$\arg\left(p(z_0)+\frac{4z_0^3f'(z_0)}{[f(z_0)-f(-z_0)]^2}p'(z_0)\right)>\frac{\pi}{2}\alpha. \tag{3.26}$$

Similarly, if $\arg(p(z_0))=-\frac{\pi}{2}\alpha$, and by adopting same arguments we obtain that

$$\begin{aligned}\arg\left(p(z_0)+\frac{4z_0^3f'(z_0)}{[f(z_0)-f(-z_0)]^2}p'(z_0)\right) &= \arg(p(z_0))+\arg\left(1+i\frac{4z_0^2f'(z_0)}{[f(z_0)-f(-z_0)]^2}k\alpha\right),\\ &< -\frac{\pi}{2}\alpha. \end{aligned}\tag{3.27}$$

because

$$\mathrm{Re}\left(\frac{4z_0^2f'(z_0)}{[f(z_0)-f(-z_0)]^2}k\alpha\right)<0,$$

provided by $k\le -1$ and $0<\alpha<1$.

And therefore

$$\arg\left(1+i\frac{4z_0^2f'(z_0)}{[f(z_0)-f(-z_0)]^2}k\alpha\right)<0.$$

Thus we see that (3.26) and (3.27) contradict our condition (3.19). Consequently, we conclude that

$$|\arg(p(z))|<\frac{\pi}{2}\alpha \quad (z\in U).$$

This completes the proof of Theorem 3.4.

Taking $p(z)=\frac{zf'(z)}{f(z)}$ in Theorem 3.4, we have the following corollary.

Corollary 3.5 *If $f\in\mathcal{A}$ satisfying*

$$\left|\arg\left(\frac{zf'(z)}{f(z)}+\frac{4z^3f'(z)}{[f(z)-f(-z)]^2}\left(\frac{zf'(z)}{f(z)}\right)'\right)\right|<\frac{\pi}{2}\alpha \quad (z\in U),$$

where $0<\alpha<1$ and $f\in\mu(\alpha)$, then f is strongly starlike of order α in U.

Taking $p(z)=1+\frac{zf''(z)}{f'(z)}$ in Theorem 3.4, we have the following corollary.

Corollary 3.6 *If $f\in\mathcal{A}$ satisfying*

$$\left|\arg\left(\frac{(zf'(z))'}{f'(z)}+\frac{4z^3}{[f(z)-f(-z)]^2}\left((zf''(z))'-\frac{z(f''(z))^2}{f'(z)}\right)\right)\right|<\frac{\pi}{2}\alpha \quad (z\in U).$$

Where $0<\alpha<1$ and $f\in\mu(\alpha)$, then f is strongly convex of order α in U.

4 Open problem

With regards to the problems solved, the work can also be applied to other classes. For example, can the same problem be applied for certain classes defined in [4]?

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