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A Subclass of Close to Convex Functions

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Abstract

In this paper, close-to-convexity and univalence of analytic functions in terms of certain differential inequalities, have been obtained. As a special case, it has been shown that analytic functions satisfying a differential inequality are strongly starlike of order $\mu,\ 0 < \mu < 1$.

Keywords: *Analytic Function, Close-to-convex function, Differential subordination, Univalent function.*

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1 Introduction

A function f is said to be analytic at a point z in a domain $\mathbb{D}$ if it is differentiable not only at z but also in some neighborhood of point z. A function f is said to be analytic on a domain $\mathbb{D}$ if it is analytic at each point of $\mathbb{D}$.

Let $\mathcal{A}$ be the class of all functions f which are analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$ and normalized by the conditions that $f(0) = f'(0) - 1 = 0$. Thus, $f \in \mathcal{A}$ has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

For two analytic functions f and g in the open unit disk $\mathbb{E}$, we say that f is subordinate to g in $\mathbb{E}$ and write as $f \prec g$ if there exists a Schwartz function w analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{E}$ such that $f(z) = g(w(z))$.

In case the function g is univalent, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

A function $f \in \mathcal{A}$ is said to be close-to-convex if there is a real number $\alpha, -\pi/2 < \alpha < \pi/2$, and a convex function g (not necessarily normalized) such that

$$\Re\left(e^{i\alpha}\frac{f'(z)}{g'(z)}\right) > 0,\ z \in \mathbb{E}.$$

It is well-known that every close-to-convex function is univalent. In 1934/35, Noshiro [3] and Warchawski [5] obtained a simple but interesting criterion for univalence of analytic functions. They proved that if an analytic function f satisfies the condition $\Re\ f'(z) > 0$ for all z in $\mathbb{E}$, then f is close-to-convex and hence univalent in $\mathbb{E}$.

A function $f \in \mathcal{A}$ is said to be strongly starlike of order α, $0 < \alpha \le 1$, if

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2},$$

equivalently

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}.$$

The main objective of this paper is to obtain close-to-convexity and univalence of analytic functions in terms of differential inequalities involving real part and modulus of the operator $(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)$ in the open unit disk $\mathbb{E}$, where α is a pre-assigned real number. Some results of subordination involving the said operator are also obtained.

To prove the main results, the following lemmas are used.

Lemma 1.1 *([2]). Let $\mathbb{D}$ be a subset of $\mathbb{C} \times \mathbb{C}$ ($\mathbb{C}$ is the complex plane) and let $\phi : \mathbb{D} \to \mathbb{C}$ be a complex function. For $u = u_1 + iu_2,\ v = v_1 + iv_2$ (u_1, u_2, v_1, v_2 are real), let ϕ satisfy the following conditions:*

(i) $\phi(u, v)$ is continuous in $\mathbb{D}$;

(ii) $(1, 0) \in \mathbb{D}$ and $\Re\ \phi(1, 0) > 0$; and

(iii) $\Re\{\phi(iu_2, v_1)\} \le 0$ for all $(iu_2, v_1) \in \mathbb{D}$ such that $v_1 \le -(1 + u_2^2)/2$.

Let $p(z) = 1 + p_1z + p_2z^2 + \cdots$ be regular in the unit disk $\mathbb{E}$, such that $(p(z), zp'(z)) \in \mathbb{D}$ for all $z \in \mathbb{E}$. If

$$\Re[\phi(p(z), zp'(z))] > 0,\ z \in \mathbb{E},$$

then $\Re\ p(z) > 0$ in $\mathbb{E}$.

Lemma 1.2 *([1]). Let G be a convex function in $\mathbb{E}$, with $G(0) = a$ and let γ be a complex number, with $\Re\,\gamma > 0$. If $F(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$, is analytic in $\mathbb{E}$ and $F \prec G$, then*

$$\frac{1}{z^{\gamma}}\int_0^z F(w)w^{\gamma-1}dw \prec \frac{1}{nz^{\gamma/n}}\int_0^z G(w)w^{\frac{\gamma}{n}-1}dw.$$

Lemma 1.3 *([4]). Suppose $f \in \mathcal{A}$ is such that $f'(z) \prec 1 + az$ in $\mathbb{E}$, where $0 < a \leq 1$, then*

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\mu}, \; z \in \mathbb{E},$$

where $0 < a \leq \dfrac{2\sin\left(\frac{\pi\mu}{2}\right)}{\sqrt{5+4\cos\left(\frac{\pi\mu}{2}\right)}}$, $0 < \mu < 1$.

2 Main Results

Theorem 2.1 *Let α and β be real numbers such that $\alpha + 2\beta \geq 0$ and $\beta < 1$. If $f \in \mathcal{A}$ satisfies the differential inequality*

$$\Re\left[(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right] > \beta, \; z \in \mathbb{E}, \tag{1}$$

then $\Re\,\dfrac{f(z)}{z} > 0$ *in* $\mathbb{E}$.

Proof. Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be an analytic function in $\mathbb{E}$ such that for all $z \in \mathbb{E}$,

$$\frac{f(z)}{z} = p(z). \tag{2}$$

Then,

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = p(z) + \alpha z p'(z).$$

Therefore, the condition (1) is equivalent to

$$\Re\left[\frac{p(z) + \alpha z p'(z) - \beta}{1-\beta}\right] > 0, \; z \in \mathbb{E}. \tag{3}$$

If $\mathbb{D} = \mathbb{C} \times \mathbb{C}$, define $\Phi(u, v) : \mathbb{D} \to \mathbb{C}$ as below:

$$\Phi(u, v) = \frac{1}{1-\beta}[u + \alpha v - \beta].$$

Then $\Phi(u,v)$ is continuous in $\mathbb{D}$, $(1,0)\in\mathbb{D}$ and $\Re\,\Phi(1,0)=1>0$. Further, in view of (3), we get $\Re\,\Phi(p(z),zp'(z))>0,\ z\in\mathbb{E}$. Let $u=u_1+iu_2,\ v=v_1+iv_2$ where u_1,u_2,v_1 and v_2 are real. Then, for $(iu_2,v_1)\in\mathbb{D}$ with $v_1\le -\dfrac{1+u_2^2}{2}$, we have

$$\Re\,\Phi(iu_2,v_1)=\Re\left[\frac{iu_2+\alpha v_1-\beta}{1-\beta}\right]=-\frac{\alpha+2\beta}{2(1-\beta)}\le 0.$$

In view of (2) and Lemma1.1, proof now follows.

Theorem 2.2 *Let α and β be real numbers such that $\alpha+2\beta\le 0$ and $\beta>1$. If $f\in\mathcal{A}$ satisfies the differential inequality*

$$\Re\left[(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)\right]<\beta,\ \ z\in\mathbb{E}, \tag{4}$$

then $\Re\,\dfrac{f(z)}{z}>0$ in $\mathbb{E}$.

Proof. Let $p(z)=1+p_1z+p_2z^2+\cdots$ be an analytic function in $\mathbb{E}$ such that for all $z\in\mathbb{E}$,

$$\frac{f(z)}{z}=p(z). \tag{5}$$

Then,

$$(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)=p(z)+\alpha zp'(z).$$

Therefore, the condition (4) is equivalent to

$$\Re\left[\frac{p(z)+\alpha zp'(z)-\beta}{1-\beta}\right]>0,\ \ z\in\mathbb{E}. \tag{6}$$

Now, the proof can be completed proceeding same as in case of Theorem 2.1.

Theorem 2.3 *Let α and β be real numbers such that $\alpha>1$ and $0\le\beta<1$. If $f\in\mathcal{A}$ satisfies the differential inequality*

$$\Re\left[(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)\right]>\beta,\ \ z\in\mathbb{E},$$

then $\Re\,f'(z)>\dfrac{\beta}{\alpha}$ in $\mathbb{E}$ and therefore, f is close-to-convex and hence univalent in $\mathbb{E}$.

Proof. The use of Theorem 2.1 for $0\le\beta<1$ implies $\Re\,\dfrac{f(z)}{z}>0,\ z\in\mathbb{E}$. Write

$$(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)=P(z).$$

Therefore,

$$f'(z) = \frac{1}{\alpha}P(z) + \left(1 - \frac{1}{\alpha}\right)\frac{f(z)}{z}.$$

Hence $\Re\, f'(z) > \dfrac{\beta}{\alpha}$.

Theorem 2.4 *Let h be a convex function in $\mathbb{E}$, with $h(0) = 1$ and let α be a real number, with $\alpha > 1$. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec h(z),\ \ z \in \mathbb{E}, \tag{7}$$

then

$$\frac{f(z)}{z} \prec \frac{1}{\alpha z^{\frac{1}{\alpha}}}\int_0^z h(w)w^{\frac{1}{\alpha}-1}\,dw.$$

Proof. Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be an analytic function in $\mathbb{E}$ such that for all $z \in \mathbb{E}$,

$$\frac{f(z)}{z} = p(z).$$

Then, (7) reduces to

$$p(z) + \alpha z p'(z) \prec h(z). \tag{8}$$

Using Lemma 1.2 for $\gamma = \dfrac{1}{\alpha}$, from (8), we get

$$p(z) \prec \frac{1}{\alpha z^{\frac{1}{\alpha}}}\int_0^z h(w)w^{\frac{1}{\alpha}-1}\,dw.$$

Therefore,

$$\frac{f(z)}{z} \prec \frac{1}{\alpha z^{\frac{1}{\alpha}}}\int_0^z h(w)w^{\frac{1}{\alpha}-1}\,dw.$$

3 Deductions

Theorem 3.1 *Let α be a real number, with $\alpha > 1$. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec 1 + \lambda z,\ \ \lambda > 0,\ \ z \in \mathbb{E}, \tag{9}$$

then

$$\frac{f(z)}{z} \prec 1 + \frac{\lambda z}{\alpha + 1}, \tag{10}$$

and

$$f'(z) \prec 1 + \frac{2\lambda z}{\alpha + 1}.$$

Proof. The use of Theorem 2.4 for $h(z) = 1 + \lambda z$ gives (10), whenever (9) holds. Write

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) - 1 = Q(z).$$

Therefore,

$$f'(z) - 1 = \frac{1}{\alpha}Q(z) + \left(1 - \frac{1}{\alpha}\right)\left(\frac{f(z)}{z} - 1\right).$$

In view of (9) and (10), from the above equation, we obtain

$$|f'(z) - 1| < \frac{2\lambda}{\alpha+1} \text{ or } f'(z) \prec 1 + \frac{2\lambda z}{\alpha+1}, \ z \in \mathbb{E}.$$

From the above theorem, we immediately get the following result.

Corollary 3.1 *Let α and λ be real numbers, with $\alpha > 1$ and $0 < \lambda \leq \dfrac{\alpha+1}{2}$. If $f \in \mathcal{A}$ satisfies*

$$\left|(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) - 1\right| < \lambda, \ z \in \mathbb{E},$$

then $|f'(z) - 1| < \dfrac{2\lambda}{\alpha+1} \leq 1$, therefore f is close-to-convex and hence univalent in $\mathbb{E}$.

In view of Lemma 1.3 and Theorem 3.1, we obtain the following result.

Corollary 3.2 *Let α and λ be real numbers, with $\alpha > 1$ and $0 < \dfrac{2\lambda}{\alpha+1} \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$\left|(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) - 1\right| < \lambda, \ z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\mu}, \ z \in \mathbb{E},$$

where $0 < \dfrac{2\lambda}{\alpha+1} \leq \dfrac{2\sin\left(\frac{\pi\mu}{2}\right)}{\sqrt{5 + 4\cos\left(\frac{\pi\mu}{2}\right)}}$, $0 < \mu < 1$. Hence f is strongly starlike of order μ.

Theorem 3.2 *Let α be a real number, with $\alpha > 1$. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec \frac{1-(1+\alpha)z}{(1-z)^2}, \ z \in \mathbb{E}, \tag{11}$$

then

$$\frac{f(z)}{z} \prec \frac{1}{1-z} \tag{12}$$

and

$$\Re\, f'(z) > \frac{3}{4},\ z \in \mathbb{E}$$

and therefore, f is close-to-convex and hence univalent in $\mathbb{E}$.

Proof. It can be easily verified that $h(z) = \dfrac{1-(1+\alpha)z}{(1-z)^2}$, $z \in \mathbb{E}$ is convex in $\mathbb{E}$ for $\alpha > 1$. Therefore the use of Theorem 2.4 for $h(z) = \dfrac{1-(1+\alpha)z}{(1-z)^2}$ ensures the existence of (12), whenever (11) holds. Write

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = R(z).$$

Therefore,

$$f'(z) = \frac{1}{\alpha}R(z) + \left(1 - \frac{1}{\alpha}\right)\frac{f(z)}{z}.$$

In view of (11) and (12), from the above equation, we obtain

$$\Re\, f'(z) > \frac{3}{4},\ z \in \mathbb{E}.$$

4 Open Problem

Most of the results proved in this paper give close-to-convexity and univalence of normalized analytic functions for $\alpha > 1$, so it would be interesting to settle the question of close-to-convexity and univalence of functions $f \in \mathcal{A}$, implied by the operator $(1-\alpha)\dfrac{f(z)}{z} + \alpha f'(z)$ in case $\alpha < 1$.

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