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Application of Differential Subordinations to Some Properties of Linear Operators

F. M. Al-Oboudi and Z. M. Al- Qahtani

Mathematical Sciences Department, Faculty of Science,
Princess Nora University, Riyadh, Saudi Arabia.
e mail: fma34@yahoo.com
e mail: zhr_math@yahoo.com

Abstract

Using the techniques of differential subordination.we study some properties of Al-Oboudi differential operator $D_\lambda^m f(z)$ and the operator $I_\lambda^m f(z)$ where $I_\lambda^m\left(D_\lambda^m f(z)\right) = D_\lambda^m\left(I_\lambda^m f(z)\right) = f(z)$.

Keywords: Differential Subordination, Differential operator, Integral operator.

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1 Introduction.

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

We denote by $ST(\alpha)$, the class of starlike functions of order α, where

$$ST(\alpha) = \left\{ f \in A, \operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha,\ 0 \le \alpha < 1,\ z \in U \right\}.$$

An analytic function $f(z)$ is said to be subordinate to an analytic function $g(z)$ on U (written $f(z) \prec g(z)$) [4], if $g(z)$ is univalent, $f(0) = g(0)$ and $f(U) \subset g(U)$. If $g(z)$ is not univalent we say that $f(z)$ is subordinate to $g(z)$, if $f(z) = g(w(z))$, $z \in U$, for some analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1,\ z \in U$.

For an analytic function $f(z)$ given by (1.1), AL-Oboudi [2] defined the differential operator $D_\lambda^m, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda \geq 0$, by

$$\begin{aligned}
&D_\lambda^0 f(z) = f(z)\\
&D_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z)\\
&\vdots\\
&D_\lambda^m f(z) = D_\lambda \left(D_\lambda^{m-1} f^{\cdot}(z)\right),\ m \in \mathbb{N}.
\end{aligned}$$

Thus

$$D_\lambda^m f(z) = z + \sum_{k=2}^{\infty} (1+\lambda(k-1))^m a_k z^k,\ m \in \mathbb{N}_0. \tag{1.2}$$

If we put $\lambda = 1$, we get Sãlãgean differential operator [14].

Remark 1.1 If $f(z) \in A$ and the differential operator D_λ^m is given by (1.2), then

$$D_\lambda^{m+1} f(z) = (1-\lambda) D_\lambda^m f(z) + \lambda z (D_\lambda^m f(z))'. \tag{1.3}$$

For an analytic function $f(z)$ given by (1.1), the authors defined [3] an integral operator $I_\lambda^m, m \in \mathbb{N}_0,\ \lambda > 0$, by

$$\begin{aligned}
&I_\lambda^0 f(z) = f(z)\\
&I_\lambda^1 f(z) = \tfrac{1}{\lambda z^{\frac{1}{\lambda}-1}} \int_0^z t^{\frac{1}{\lambda}-2} f(t) dt = I_\lambda f(z),\\
&I_\lambda^2 f(z) = I_\lambda (I'_\lambda f),\\
&\vdots\\
&I_\lambda^m f(z) = I_\lambda \left(I_\lambda^{m-1} f\right), m \in \mathbb{N}.
\end{aligned}$$

Thus

$$I_\lambda^m f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{(1+\lambda(k-1))^m} z^k,\ m \in \mathbb{N}_0. \tag{1.4}$$

If we put $\lambda = 1$, we get Sãlãgean integral operator [14].

Remark 1.2 If $f(z) \in A$ and the integral operator I_λ^m is given by (1.4), then

$$I_\lambda^m f(z) = (1-\lambda) I_\lambda^{m+1} f(z) + \lambda z \left(I_\lambda^{m+1} f(z)\right)'. \tag{1.5}$$

Remark 1.3 If $f(z) \in A$, then $I_\lambda^m (D_\lambda^m f(z)) = D_\lambda^m (I_\lambda^m f(z)) = f(z)$.

For an analytic function $f(z)$ given by (1.1), the function $F_\mu(z)$ is defined by

$$F_\mu(z) = \frac{\mu+1}{z^\mu} \int_0^z t^{\mu-1} f(t) dt,\ \mu+1 > 0,\ z \in U, \tag{1.6}$$

hence

$$F_\mu(z) = I^1_{\frac{1}{\mu+1}} f(z)$$
$$= z + \sum_{k=2}^{\infty} \frac{\mu+1}{\mu+k} a_k z^k.$$

It is easy to see that

$$z\left(D^m_\lambda F_\mu(z)\right)' = (\mu+1)D^m_\lambda f(z) - \mu D^m_\lambda F_\mu(z), \tag{1.7}$$

and from Remark 1.3, we have

$$z\left(I^m_\lambda F_\mu(z)\right)' = (\mu+1)I^m_\lambda f(z) - \mu I^m_\lambda F_\mu(z). \tag{1.8}$$

For real or complex numbers a, b and $c(c \neq 0, -1, -2, \cdots)$ the hypergeometric series

$${}_2F_1(a, b; c; z) = 1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1) \cdot b(b+1)}{2!c(c+1)} z^2 + \cdots .$$

For an analytic function $p(z) = 1 + p_1 z^1 + p_2 z^2 + \cdots$, the condition

$$p(z) \prec \frac{1+Az}{1+Bz}, \; -1 \leq B < A \leq 1, \; z \in U,$$

means that the image of U under $p(z)$ is inside the open disc centred on the real axis whose diameter has and end points $(1-A)/(1-B)$ and $(1+A)/(1+B)$. From this we conclude that $p(z)$ has a positive real part and hence univalent in U [7].

2 preliminary lemmas.

Lemma 2.1 [6] *Let $h(z)$ be a convex (univalent) function in U, with $h(0) = 1$ and let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, be analytic in U, if*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z), (z \in U),$$

for $\gamma \neq 0$ and $\mathrm{Re}\gamma \geq 0$, then

$$p(z) \prec q(z) = \frac{\gamma}{z^\gamma} \int_0^z t^{\gamma-1} h(t) dt \prec h(z), z \in U,$$

and $q(z)$ is the best dominant.

Lemma 2.2 [15] *For real or complex numbers* a, b *and* $c\,(c \neq 0, -1, -2, \cdots)$, *and* $\mathrm{Re}c > \mathrm{Re}b > 0$, *we have*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt = \frac{\Gamma(b)\,\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z) \tag{2.1}$$

$${}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z) \tag{2.2}$$

$${}_2F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) \tag{2.3}$$

and

$$(b+1){}_2F_1(1,b;b+1;z) = (b+1) + bz{}_2F_1(1,b+1;b+2;z). \tag{2.4}$$

3 Main Results

Theorem 3.1 *Let* $f(z) \in A$, *satisfies*

$$(1-\lambda)\frac{D_\lambda^m f(z)}{z} + \lambda\frac{D_\lambda^{m+1} f(z)}{z} \prec \frac{1+Az}{1+Bz},\ \lambda > 0, \tag{3.1}$$

then

$$\frac{D_\lambda^m f(z)}{z} \prec q(z) \prec \frac{1+Az}{1+Bz},$$

where

$$q(z) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1}{}_2F_1\left(1,1;\frac{1}{\lambda^2}+1,\frac{Bz}{1+Bz}\right), & B \neq 0 \\ 1 + \frac{Az}{1+\lambda^2}, & B = 0, \end{cases} \tag{3.2}$$

and $q(z)$ *is the best dominant.*

Furthermore,

$$\mathrm{Re}\left(\frac{D_\lambda^m f(z)}{z}\right) > \rho(A,B,\lambda), \tag{3.3}$$

where

$$\rho(A,B,\lambda) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1,1;\frac{1}{\lambda^2}+1;\frac{B}{B-1}\right), & B \neq 0 \\ 1 - \frac{A}{1+\lambda^2}, & B = 0, \end{cases} \tag{3.4}$$

this result is sharp.

Proof. Let

$$p(z) = \frac{D_\lambda^m f(z)}{z}, \tag{3.5}$$

then $p(z)$ is analytic in U with $p(0)=1$. Using (1.3), (3.1) and (3.5), we have

$$p(z)+\lambda^2 zp'(z) \prec \frac{1+Az}{1+Bz}.$$

Using Lemma 2.1 for $\gamma=\frac{1}{\lambda^2}$, we deduce that

$$\frac{D_\lambda^m f(z)}{z} \prec \frac{1}{\lambda^2 z^{\frac{1}{\lambda^2}}}\int\limits_0^z t^{\frac{1}{\lambda^2}-1}\frac{1+At}{1+Bt}dt = q(z) \prec \frac{1+Az}{1+Bz}.$$

Now, rewriting $q(z)$ by changing the variables, we have

$$q(z)=\frac{1}{\lambda^2}\int\limits_0^1 t^{\frac{1}{\lambda^2}}\left(1+Bzt\right)^{-1}dt+\frac{1}{\lambda^2}Az\int\limits_0^1 t^{\frac{1}{\lambda^2}}\left(1+Bzt\right)^{-1}dt,$$

by using (2.1), (2.3) and (2.4) from Lemma 2.2, we see that $q(z)$ is given by (3.2).

To prove (3.3), we show that

$$\inf_{|z|<1} \operatorname{Re}\{q(z)\} = q(-1).$$

Since $\frac{1+Az}{1+Bz}$ for $-1\le B<A\le 1$, is convex (univalent) in U $|z|\le r<1$, then

$$\operatorname{Re}\left(\frac{1+Az}{1+Bz}\right) \ge \frac{1-Ar}{1-Br}.$$

Setting

$$g(t,z)=\frac{1+Atz}{1+Btz}, \quad 0\le t\le 1,\ z\in U,$$

and

$$d\mu(t)=\frac{1}{\lambda^2}t^{\frac{1}{\lambda^2}-1}dt,$$

we get

$$q(z)=\int\limits_0^1 g(t,z)d\mu(t),$$

so that

$$\operatorname{Re}q(z)=\int\limits_0^1 \operatorname{Re}\left(\frac{1+Atz}{1+Btz}\right)d\mu(t) \ge \int\limits_0^1 \operatorname{Re}\left(\frac{1-Atr}{1-Btr}\right)d\mu(t) = q(-r),\ |z|\le r<1.$$

Now, letting $r\to 1^-$ in the above inequality, we obtain

$$\operatorname{Re}q(z)\ge q(-1), z\in U,$$

which implies (3.3). $\square$

Special Cases:

(i) If we put $\lambda = 1$ in Theorem 3.1, we get a better result for Sãlãgean differential operator proved by Oros [10].

(ii) If we put $\lambda = 1$ and $m = 0$ in Theorem 3.1, we have the result of Obradovic [9].

(iii) If we put $A = 1 - 2\alpha, 0 \leq \alpha < 1$ and $B = -1$ in (ii), we obtain an improvement of the result of Owa and Obradovic [11].

If we put $\lambda = 1,\ m = 1$ in Theorem 3.1, we obtain

Corollary 3.1 *If $f(z) \in A$, satisfies*

$$zf''(z) + f'(z) \prec \frac{1 + Az}{1 + Bz},$$

then

$$f'(z) \prec q(z) \prec \frac{1 + Az}{1 + Bz},$$

where

$$q(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right) \frac{\ln(1+Bz)}{Bz}, & B \neq 0 \\ 1 + \frac{A}{2} z, & B = 0, \end{cases}$$

Furthermore,

$$\mathrm{Re} f'(z) > \rho(A, B),$$

where

$$\rho(A, B) = \begin{cases} \frac{A}{B} - \left(1 - \frac{A}{B}\right) \frac{\ln(1-B)}{B}, & B \neq 0 \\ 1 - \frac{A}{2}, & B = 0, \end{cases}$$

this result is sharp.

If we put $A = 1 - 2\alpha,\ 0 \leq \alpha < 1,\ B = -1$ in the above corollary, we have the following.

Corollary 3.2 *If $f(z) \in A$, satisfies*

$$\mathrm{Re}(f'(z) + zf''(z)) > \alpha,$$

then

$$\mathrm{Re} f'(z) > (2\alpha - 1) + 2(1 - \alpha) \ln 2,$$

the result is sharp.

Thie result is an improvement the result of saitoh (at $\lambda = 1$) [13], he obtained a lower bound of the form $\mathrm{Re} f'(z) > \frac{2\alpha+1}{3}$.

Remark 3.1 If $zf''(z) + f'(z) \prec \frac{1+A'z}{1+Bz}$, $B \neq 0$, $A' = \frac{B\ln(1-B)}{B+\ln(1-B)}$, then $\mathrm{Re}f'(z) > 0$ in U and hence $f(z)$ is univalent in U. This gives a new criteria for univalency. If we take $B = -1$, we note that if $\mathrm{Re}(zf''(z)+f'(z)) > \frac{\log 4-1}{\log 4-2}$, then $\mathrm{Re}f'(z) > 0$ in U.

By using Remark 1.3, we obtain the following.

Theorem 3.2 *Let $f(z) \in A$, satisfies*

$$(1-\lambda)\frac{I_\lambda^{m+1}f(z)}{z} + \lambda\frac{I_\lambda^m f(z)}{z} \prec \frac{1+Az}{1+Bz},$$

then

$$\frac{I_\lambda^{m+1}f(z)}{z} \prec q(z) \prec \frac{1+Az}{1+Bz},$$

where $q(z)$ is given by (3.2) *and $q(z)$ is the best dominant.*

Funthermore,

$$\mathrm{Re}\left(\frac{I_\lambda^{m+1}f(z)}{z}\right) > \rho(A,B,\lambda),$$

where $\rho(A,B,\lambda)$ given by (3.4) *and this result is sharp.*

If we put $\lambda = 1$ in Theorem 3.2, we obtain a new property of Sãlãgean integral operator.

Now we prove the following.

Theorem 3.3 *Let $f(z) \in A$ satisfies*

$$\frac{D_\lambda^m f(z)}{z} \prec \frac{1+Az}{1+Bz},\ z \in U,\ -1 \leq B < A \leq 1,$$

then

$$\frac{D_\lambda^m F_\mu(z)}{z} \prec q(z) \prec \frac{1+Az}{1+Bz},$$

where $F_\mu(z)$ is defined by (1.6) *and $q(z)$ is given by*

$$q(z) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1}{}_2F_1\left(1,1;\mu+2;\frac{Bz}{1+Bz}\right), & B \neq 0 \\ 1 + \frac{\mu+1}{\mu+2}Az\,, & B = 0, \end{cases} \tag{3.6}$$

and $q(z)$ is the best dominant.

Furthermore

$$\mathrm{Re}\left(\frac{D_\lambda^m F_\mu(z)}{z}\right) > \rho(A,B,\mu),$$

where

$$\rho(A,B,\mu)=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1,1;\mu+2;\frac{B}{B-1}\right), & B\neq 0\\ 1-\frac{\mu+1}{\mu+2}A\,, & B=0,\end{cases}$$

this result is sharp.

Proof. Let

$$p(z)=\frac{D_\lambda^m F_\mu(z)}{z}, \tag{3.8}$$

we see that $p(z)=1+p_1z+p_2z^2+\cdots$ is analytic in U. Differentiating both sides of (3.8), and using (1.7), we have

$$p(z)+\frac{zp'(z)}{\mu+1}\prec\frac{1+Az}{1+Bz}.$$

Using Lemma 2.1 for $\gamma=\mu+1$, we obtain,

$$p(z)\prec q(z)=\frac{\mu+1}{z^{\mu+1}}\int_0^z t^\mu\left(\frac{1+At}{1+Bt}\right)dt\prec\frac{1+Az}{1+Bz},$$

by changing the variables in $q(z)$ and using (2.1), (2.3) and (2.4), we obtain

$$q(z)=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}{}_2F_1\left(1,1;\mu+2;\frac{Bz}{1+Bz}\right), & B\neq 0\\ 1+\frac{\mu+1}{\mu+2}Az\,, & B=0,\end{cases}$$

proceeding as in Theorem 3.1, the Second part follows. □

If we put $\lambda=1$ in Theorem 3.3, we obtain a new property of Sãlãgean differential operator.

By using Remark 1.3 in the above theorem, we obtain the following.

Theorem 3.4 *let $f(z)\in A$ satisfies*

$$\frac{I_\lambda^m f(z)}{z}\prec\frac{1+Az}{1+Bz},\ z\in U,\ -1\leq B<A\leq 1,$$

then

$$\frac{I_\lambda^m F_\mu(z)}{z}\prec q(z)\prec\frac{1+Az}{1+Bz},$$

where $q(z)$ is given by (3.6) *and $q(z)$ is the best dominant. Furthermore*

$$\operatorname{Re}\left(\frac{I_\lambda^m F_\mu}{z}\right)>\rho(A,B,\mu),$$

where $\rho(A,B,\mu)$ is given by (3.7) *and this result is sharp.*

If we put $\lambda = 1$ in the above theorem , we obtain a property of Sãlãgean integral operator which was proved by Patel and sahoo [12].

If we put $A = 1 - 2\alpha,\ 0 \leq \alpha < 1, B = -1$ and $m = 0$ in Theorem 3.3 and Theorem 3.4, we obtain the following.

Corollary 3.3 *If* $f(z) \in A$ *satisfies* $\mathrm{Re}\frac{f(z)}{z} > \alpha, 0 \leq \alpha < 1,$ *then*

$$\mathrm{Re}\left(\frac{\mu+1}{z^{\mu+1}}\int_0^z t^{\mu-1}f(t)dt\right) > \alpha + (1-\alpha)\left[{}_2F_1\left(1,1;\mu+2;\frac{1}{2}\right) - 1\right].$$

The result is sharp.

This result is an improvement the result of Obradovic [8], he obtained a lower bound of the form $\mathrm{Re}\left[\frac{\mu+1}{z^{\mu+1}}\int_0^z t^{\mu-1}f(t)dt\right] > \alpha + \frac{1-\alpha}{3+2\mu}$.

Now we prove the partial converse of Theorem 3.3, for $A = 1-2\alpha, 0 \leq \alpha < 1$ and $B = -1$.

Theorem 3.5 *Let* $f(z) \in A$, *satisfies*

$$\mathrm{Re}\left[\frac{D_\lambda^m F_\mu(z)}{z}\right] > \alpha,\ 0 \leq \alpha < 1, \tag{3.9}$$

then

$$\mathrm{Re}\left[\frac{D_\lambda^m f(z)}{z}\right] > \alpha,\ |z| < R_1,$$

where

$$R_1 = \frac{\sqrt{(\mu+1)^2+1} - 1}{\mu+1}, \tag{3.10}$$

this result is sharp.

Proof. From (3.9), we have

$$\frac{D_\lambda^m F_\mu(z)}{z} = \alpha + (1-\alpha)p(z), \tag{3.11}$$

we see that $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic and $\mathrm{Re}p(z) > 0,\ z \in U$. Differentiating both sides of (3.11) and using (1.7) in the resulting equation, we get

$$\begin{aligned} \mathrm{Re}\left[\frac{D_\lambda^m f(z)}{z} - \alpha\right] &= (1-\alpha)\mathrm{Re}\left[p(z) + \frac{zp'(z)}{\mu+1}\right] \\ &\geq (1-\alpha)\left(\mathrm{Re}\,p(z) - \frac{|zp'(z)|}{\mu+1}\right). \end{aligned} \tag{3.12}$$

Using the well-known estimate

$$\frac{|zp'(z)|}{\operatorname{Re} p(z)} \leq \frac{2r}{1-r^2}, \ (|z| = r < 1)\,, \tag{3.13}$$

in (3.12), we deduce that

$$\operatorname{Re}\left[\frac{D_\lambda^m f(z)}{z} - \alpha\right] \geq (1-\alpha)\operatorname{Re}p(z)\left(1 - \frac{2r}{(\mu+1)(1-r^2)}\right),$$

which is certainly positive if $r < R_1$, where R_1 given by (3.10). □

If we put $\lambda = 1$ in Theorem 3.5, we obtain a new property of Sãlãgean differential operator.

By using Remark 1.3 in the above theorem, we obtain the partial converse of Theorem 3.4, for $A = 1-2\alpha, 0 \leq \alpha < 1$ and $B = -1$.

Theorem 3.6 *Let* $f(z) \in A$ *, satisfies*

$$\operatorname{Re}\left[\frac{I_\lambda^m F_\mu(z)}{z}\right] > \alpha, \ 0 \leq \alpha < 1,$$

then

$$\operatorname{Re}\left(\frac{I_\lambda^m f(z)}{z}\right) > \alpha, \ |z| < R_1,$$

where R_1 *given by* (3.10) *and this result is sharp.*

If we put $\lambda = 1$ in the above theorem , we obtain a property of Sãlãgean integral operator which was proved by Patel and sahoo [12].

Using the same method in [3, Theorem. 4.4], we can prove the following.

Theorem 3.7 *Let* $f(z) \in A$, *satisfies*

$$\frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} \prec \frac{1+Az}{1+Bz}$$

then

$$\frac{D_\lambda^m f(z)}{D_\lambda^m F_\mu(z)}(z) \prec \frac{1}{(\lambda+1)\,Q(z)} = \tilde{q}(z) \prec \frac{1 + \left(\frac{A+(\lambda\mu-(1-\lambda))B}{\lambda(\mu+1)}\right)z}{1+Bz} \tag{3.14}$$

where

$$Q(z) = \begin{cases} \int_0^1 \frac{t^\mu}{\lambda}\left(\frac{1+Bzt}{1+Bz}\right)^{\frac{1}{\lambda}\left(\frac{A-B}{B}\right)} dt, & B \neq 0 \\ \int_0^1 \frac{t^\mu}{\lambda} e^{A(t-1)z} dt, & B = 0, \end{cases}$$

and $\tilde{q}(z)$ *is the best dominant of* (3.14).

Now we prove the partial converse of Theorem 3.7, for $A = 2\lambda\,(\mu+1)\,(1-\alpha)-1, B = -1$.

Theorem 3.8 *Let $f(z) \in A$, satisfies*

$$\mathrm{Re}\left[\frac{D_\lambda^m f(z)}{D_\lambda^m F_\mu(z)}\right] > \alpha,\ 0 \le \alpha < 1,\ 0 < \lambda(\mu+1) \le \frac{1}{1-\alpha},\ z \in U, \tag{3.15}$$

then

$$\mathrm{Re}\left(\frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)}\right) > \beta,\ |z| < R_2,$$

where

$$\beta = \beta(\mu,\alpha,\lambda) = 1 + \lambda(\mu+1)(\alpha-1), \tag{3.16}$$

and R_2 is the smallest positive root of the equation

$$(1-\alpha)(\mu+1)\lambda r^2 + 2\lambda\left[(\mu+1)(1-\alpha)+1\right] r + (\mu+1)(1-\alpha)\lambda = 0. \tag{3.17}$$

this result is sharp.

Proof. From (3.15), we define the function $p(z)$ in U by

$$\frac{D_\lambda^m f(z)}{D_\lambda^m F_\mu(z)} = \alpha + (1-\alpha)p(z), \tag{3.18}$$

we see that $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is analytic and $\mathrm{Re}p(z) > 0$ in U. Making use of the logarithmic differentiation on both sides of (3.18) and using (1.7), we obtain.

$$\begin{aligned}\mathrm{Re}&\left[\frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} - (1+\lambda(\mu+1)(\alpha-1))\right] = \\ &(1-\alpha)\mathrm{Re}\left[\lambda(\mu+1)p(z) + \frac{\lambda z p'(z)}{(1-\alpha)p(z)}\right] \\ &\ge (1-\alpha)\left(\lambda(\mu+1)\,\mathrm{Re}\,p(z) - \frac{|\lambda z p'(z)|}{(1-\alpha)\,|p(z)|}\right).\end{aligned} \tag{3.19}$$

By using (3.13), and the estimate

$$\mathrm{Re}p(z) \ge \frac{1-r}{1+r},$$

in (3.19), we get

$$\begin{aligned}\mathrm{Re}&\left[\frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} - (1+\lambda(\mu+1)(\alpha-1))\right] \ge \\ &(1-\alpha)\mathrm{Re}p(z)\left[(\mu+1)\lambda - \frac{2r\lambda}{(1-r)^2(1-\alpha)}\right],\end{aligned}$$

which is certainly positive if $r < R_2$, where R_2 is the smallest positive root of the equation (3.17).

The bound R_2 is sharp for the function $f(z) \in A$ defind in U by

$$\frac{D_\lambda^m f(z)}{D_\lambda^m F_\mu(z)} = \frac{1+(1-2\alpha)z}{1-z}.$$

□

If we put $\lambda = 1$ in Theorem 3.8, we obtain a new property of Sãlãgean differential operator.

From (1.7) and (1.3), we can rewrite the above theorem as the following.

Remark 3.2 If $f(z) \in A$ and

$$D_\lambda^m F_\mu(z) \in ST\left(\alpha(\mu+1)-\mu\right), -1 < \mu \leq \frac{\alpha}{1-\alpha},$$

then

$$D_\lambda^m f(z) \in ST\left(\alpha(\mu+1)-\mu\right),\ |z| < R_2,$$

where R_2 is the smallest positive root of the equation (3.17).

Now by Using Remark 1.3, we prove the partial converse of the Theorem 4.4 in [3], for $A = 2\lambda\left(\mu+1\right)\left(1-\alpha\right)-1, B = -1$,

Theorem 3.9 *Let* $f(z) \in A$ *, satisfies*

$$\mathrm{Re}\left[\frac{I_\lambda^{m+1} f(z)}{I_\lambda^{m+1} F_\mu(z)}\right] > \alpha,\ 0 \leq \alpha < 1,\ 0 < \lambda(\mu+1) \leq \frac{1}{1-\alpha},\ z \in U,$$

then

$$\mathrm{Re}\left(\frac{I_\lambda^m f(z)}{I_\lambda^{m+1} f(z)}\right) > \beta,\ |z| < R_2,$$

where β *is given by* (3.16) *and* R_2 *is the smallest positive root of the equation* (3.17)*and this result is sharp.*

If we put $\lambda = 1$ in the above theorem , we obtain a property of Sãlãgean integral operator which was proved by Patel and sahoo [12].

From (1.8) and (1.5), we can rewrite the above theorem as follows.

Remark 3.3 If $f(z) \in A$ and

$$I_\lambda^{m+1} F_\mu(z) \in ST\left(\alpha(\mu+1)-\mu\right), -1 < \mu \leq \frac{\alpha}{1-\alpha},$$

then

$$I_\lambda^{m+1} f(z) \in ST\left(\alpha(\mu+1)-\mu\right), |z| < R_2,$$

where R_2 is the smallest positive root of the equation (3.17).

4 Open Problems.

It would be of interest to raise the following problems.

(i) Most of the previous theorems can be extended into p-valent functions.

(ii) For more general properties, the function $\frac{1+Az}{1+Bz}$, can be replaced by any convex function $h(z)$, with $h(0) = 1$ and Re $h(z) \geq 0$.

(iii) Analogue properties can be discussed for the generlized derivative operator $\mu^{n,m}_{\lambda_1,\lambda_2}$ which studied in [1].

(iv) The techniques of superordination can be applied to get more properties for the operators that defined above for example see [5].

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