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Extremal Function and Coefficient Inequalities For Certain Analytic Functions

K. Hamai, T. Hayami, K. Kuroki and S. Owa

Department of Mathematics Kinki University
Higashi-Osaka, Osaka 577-8502, Japan
e mail: 0933310146v@kindai.ac.jp
e mail: ha_ya_to112@hotmail.com
e mail: freedom@sakai.zaq.ne.jp
e mail: owa@math.kindai.ac.jp(corresponding author)

Abstract

For analytic functions $f(z)$ in the open unit disk $\mathbb{U}$, an interesting subclass $\mathcal{R}_\alpha$ with $|2\alpha - 1| < \frac{\mathrm{Re}(\alpha)}{|\alpha|}$ of analytic functions is introduced. The object of the present paper is to discuss an extremal function and some coefficient inequalities for the class $\mathcal{R}_\alpha$.

Keywords: analytic, extremal function, coefficient inequality.

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1 Introduction and Definitions

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C}\,;\, |z| < 1\}$.

If $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\mathrm{Re}(f'(z)) > \alpha \qquad (z \in \mathbb{U})$$

for some real α ($0 \leqq \alpha < 1$), then we say that $f(z) \in \mathcal{R}(\alpha)$. This class was investigated by Hayami and Owa (1). In this paper, we consider the new subclass $\mathcal{R}_\alpha$ of $\mathcal{A}$ defined by some complex number α.

Definition 1.1 *If $f(z) \in \mathcal{A}$ satisfies the following inequality*

$$\left| \frac{1}{f'(z)} - \frac{1}{2\alpha} \right| < \mathrm{Re}\left(\frac{1}{2\alpha}\right) \qquad (z \in \mathbb{U})$$

for some complex number α $\left(|2\alpha - 1| < \dfrac{\mathrm{Re}(\alpha)}{|\alpha|} \right)$, then we say that $f(z) \in \mathcal{R}_\alpha$. If $0 < \alpha < 1$, then the class $\mathcal{R}_\alpha$ is equivalent to the class $\mathcal{R}(\alpha)$.

We first introduce the following remark to think about the extremal function for the class $\mathcal{R}_\alpha$.

Remark 1.2 *Let $M(z)$ be defined by*

$$M(z) = \frac{a - mz}{1 - \dfrac{\bar{a}}{m} z} \qquad (a \in \mathbb{C} \text{ and } m > 0).$$

Then, we know that $M(0) = a$ and $M(z)$ maps the open unit disk $\mathbb{U}$ onto the following entire circular domain

$$\mathbb{D} = \{w \in \mathbb{C} \; ; |w| < m\}.$$

This assertion has been investigated by Miller and Mocanu (2). Using this result, we consider the extremal function for the class $\mathcal{R}_\alpha$.

Theorem 1.3 *The extremal function for the class $\mathcal{R}_\alpha$ is $f(z)$ defined by*

$$f(z) = \frac{B}{A} z + \left(1 - \frac{B}{A}\right) \frac{1}{A} \log(1 + Az)$$

where $A = \dfrac{(\mathrm{Im}(\alpha))^2 - 2\bar{\alpha}|\alpha|^2}{2\mathrm{Re}(\alpha)|\alpha|^2}$, $B = \dfrac{\alpha - 2|\alpha|^2}{\mathrm{Re}(\alpha)}$.

Proof : Noting that $f(z) \in \mathcal{R}_\alpha$ satisfies

$$\left| \frac{1}{f'(z)} - \frac{1}{2\alpha} \right| < \mathrm{Re}\left(\frac{1}{2\alpha} \right),$$

If we define the function $M(z)$ by

$$M(z) = \frac{1}{f'(z)} - \frac{1}{2\alpha},$$

then it is clear that $M(0) = 1 - \dfrac{1}{2\alpha}$ and $|M(z)| < \mathrm{Re}\left(\dfrac{1}{2\alpha}\right)$. Hence, from Remark 1.2 , we can write

$$M(z) = \frac{\left(1 - \dfrac{1}{2\alpha}\right) - \mathrm{Re}\left(\dfrac{1}{2\alpha}\right) z}{1 - \dfrac{1 - \dfrac{1}{2\bar{\alpha}}}{\mathrm{Re}\left(\dfrac{1}{2\alpha}\right)} z}.$$

A simple computation gives us that

$$f'(z) = \frac{1 + \dfrac{\alpha - 2|\alpha|^2}{\mathrm{Re}(\alpha)} z}{1 + \dfrac{(\mathrm{Im}(\alpha))^2 - 2\bar{\alpha}|\alpha|^2}{2\mathrm{Re}(\alpha)|\alpha|^2} z}.$$

Integrating both sides from 0 to 2π on θ, we have that

$$f(z) = \frac{B}{A} z + \left(1 - \frac{B}{A}\right) \frac{1}{A} \log\left(1 + Az\right).$$

Thus, the above function $f(z)$ is the extremal function for the class $\mathcal{R}_\alpha$.

Remark 1.4 *The extremal function $f(z)$ for the class $\mathcal{R}_\alpha$ has the following Taylor expansion of the form*

$$f(z) = \frac{B}{A} z + \left(1 - \frac{B}{A}\right) \frac{1}{A} \log(1 + Az) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} A^{n-2}(A - B)}{n} z^n$$

where $A = \dfrac{(\mathrm{Im}(\alpha))^2 - 2\bar{\alpha}|\alpha|^2}{2\mathrm{Re}(\alpha)|\alpha|^2},\ B = \dfrac{\alpha - 2|\alpha|^2}{\mathrm{Re}(\alpha)}.$

2 Coefficient inequalities

We now consider the coefficient inequalities for $f(z)$ belonging to the class $\mathcal{R}_\alpha$.

Theorem 2.1 *If a function $f(z) \in \mathcal{A}$ satisfies the following inequality*

$$\sum_{n=2}^{\infty} n|a_n| \leqq \frac{\mathrm{Re}(\alpha) - |\alpha||2\alpha - 1|}{\mathrm{Re}(\alpha) + |\alpha|}$$

for some complex number α $\left(|2\alpha - 1| < \dfrac{\mathrm{Re}(\alpha)}{|\alpha|}\right)$, then $f(z) \in \mathcal{R}_\alpha$.

Proof : Noting that

$$\left|\frac{1}{f'(z)} - \frac{1}{2\alpha}\right| = \frac{1}{2|\alpha|}\left|\frac{2\alpha - 1 - \sum_{n=2}^{\infty} n a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}}\right| \leqq \frac{1}{2|\alpha|}\frac{|2\alpha - 1| + \sum_{n=2}^{\infty} n|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1}}$$

$$< \frac{1}{2|\alpha|}\frac{|2\alpha - 1| + \sum_{n=2}^{\infty} n|a_n|}{1 - \sum_{n=2}^{\infty} n|a_n|},$$

if $f(z)$ satisfies the following inequality

$$|2\alpha - 1| + \sum_{n=2}^{\infty} n|a_n| \leqq \frac{\mathrm{Re}(\alpha)}{|\alpha|}\left(1 - \sum_{n=2}^{\infty} n|a_n|\right),$$

that is,

$$\sum_{n=2}^{\infty} n|a_n| \leqq \frac{\mathrm{Re}(\alpha) - |\alpha||2\alpha - 1|}{\mathrm{Re}(\alpha) + |\alpha|},$$

then we see

$$\left|\frac{1}{f'(z)} - \frac{1}{2\alpha}\right| < \frac{\mathrm{Re}(\alpha)}{2|\alpha|^2}.$$

This completes the proof of the theorem.

Letting $0 < \alpha < 1$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.2 *If a function $f(z) \in \mathcal{A}$ satisfies the following inequality*

$$\sum_{n=2}^{\infty} n|a_n| \leqq \begin{cases} \alpha & \left(0 < \alpha \leqq \dfrac{1}{2}\right) \\ 1-\alpha & \left(\dfrac{1}{2} < \alpha < 1\right) \end{cases}$$

for some real number α $(0 < \alpha < 1)$, then $f(z) \in \mathcal{R}(\alpha)$.

Next we derive the following necessary condition for the class $\mathcal{R}_\alpha$.

Theorem 2.3 *If a function $f(z) \in \mathcal{R}_\alpha$ with $a_n = |a_n|e^{i((n-1)\theta+\pi)}$ $(n = 2, 3, 4 \cdots)$, then*

$$\sum_{n=2}^{\infty} n|a_n| \leqq \begin{cases} 1-\alpha & (0 < \alpha < 1) \\ 1 - \dfrac{2|\alpha|^2\left(\mathrm{Re}(\alpha) - \sqrt{(\mathrm{Re}(\alpha))^2 - (\mathrm{Im}(\alpha))^2}\right)}{(\mathrm{Im}(\alpha))^2} & (\alpha \notin \mathbb{R}). \end{cases}$$

Proof : By using the same method with Theorem 2.1, we obtain that

$$\left| \frac{2\alpha - 1 - \sum_{n=2}^{\infty} n a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}} \right| < \frac{\mathrm{Re}(\alpha)}{|\alpha|} \qquad (z \in \mathbb{U})$$

for $f(z) \in \mathcal{R}_\alpha$. Since $a_n = |a_n|e^{i((n-1)\theta+\pi)}$, if we take $z = |z|e^{-i\theta}$, then we know that

$$\left| \frac{2\mathrm{Re}(\alpha) - 1 + \sum_{n=2}^{\infty} n|a_n||z|^{n-1} + 2i\mathrm{Im}(\alpha)}{1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1}} \right| < \frac{\mathrm{Re}(\alpha)}{|\alpha|} \qquad (z \in \mathbb{U}).$$

Letting $|z| \to 1$ and squaring both sides, we obtain that

$$\frac{(2\mathrm{Re}(\alpha) - 1)^2 + 2(2\mathrm{Re}(\alpha) - 1)\beta + \beta^2 + 4(\mathrm{Im}(\alpha))^2}{1 - 2\beta + \beta^2} \leqq \frac{(\mathrm{Re}(\alpha))^2}{|\alpha|^2},$$

that is, that

$$(\mathrm{Im}(\alpha))^2\beta^2 + 2(2\mathrm{Re}(\alpha)|\alpha|^2 - (\mathrm{Im}(\alpha))^2)\beta$$
$$+4|\alpha|^4 - 4\mathrm{Re}(\alpha)|\alpha|^2 + (\mathrm{Im}(\alpha))^2 \leqq 0. \qquad (2.1)$$

where $\beta = \sum_{n=2}^{\infty} n|a_n|$. If $0 < \alpha < 1$, then we see that the inequality (2.1) is equivalent to

$$\beta \leqq 1 - \alpha.$$

If $\alpha \notin \mathbb{R}$, then solving the inequality (2.1), we have that

$$\beta \leqq \frac{-(2\mathrm{Re}(\alpha)|\alpha|^2 - (\mathrm{Im}(\alpha))^2) + 2|\alpha|^2\sqrt{(\mathrm{Re}(\alpha))^2 - (\mathrm{Im}(\alpha))^2}}{(\mathrm{Im}(\alpha))^2},$$

which is the desired our result.

Furthermore, we state about the following coefficient inequality.

Theorem 2.4 *If a function $f(z) \in \mathcal{R}_\alpha$ $(\alpha \notin \mathbb{R})$, then*

$$\sum_{n=2}^{\infty} n^2|a_n|^2 \leqq \frac{(\mathrm{Re}(\alpha))^2 - |\alpha(2\alpha-1)|^2}{(\mathrm{Im}(\alpha))^2}.$$

Proof : From the definition of the class $\mathcal{R}_\alpha$, we note that

$$|\alpha|^2|2\alpha - f'(z)|^2 < (\mathrm{Re}(\alpha))^2|f'(z)|^2.$$

Setting $z = re^{i\theta}$ $(0 \leqq r < 1,\ 0 \leqq \theta < 2\pi)$ and integrating both sides from 0 to 2π on θ, we have that

$$|\alpha|^2 \int_0^{2\pi} |2\alpha - f'(re^{i\theta})|^2 d\theta < (\mathrm{Re}(\alpha))^2 \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta.$$

A simple calculation gives us that

$$2\pi|\alpha|^2 \left(|2\alpha-1|^2 + \sum_{n=2}^{\infty} n^2|a_n|^2 r^{2(n-1)} \right) < 2\pi(\mathrm{Re}(\alpha))^2 \left(1 + \sum_{n=2}^{\infty} n^2|a_n|^2 r^{2(n-1)} \right).$$

Therefore, letting $r \to 1$, we obtain that

$$\sum_{n=2}^{\infty} n^2|a_n|^2 \leqq \frac{(\mathrm{Re}(\alpha))^2 - |\alpha(2\alpha-1)|^2}{(\mathrm{Im}(\alpha))^2},$$

which completes the proof of the theorem.

3 Open problem

In view of Theorem 2.3, we have that

$$|a_n| \leqq \frac{1-\alpha}{n} \qquad (0 < \alpha < 1; n = 2, 3, 4, \cdots)$$

and

$$|a_n| \leqq \frac{1}{n}\left(1 - \frac{2|\alpha|^2\left(\mathrm{Re}(\alpha) - \sqrt{(\mathrm{Re}(\alpha))^2 - (\mathrm{Im}(\alpha))^2}\right)}{(\mathrm{Im}(\alpha))^2}\right)$$

where $\alpha \notin \mathbb{R}; n = 2, 3, 4, \cdots$.

Also, from Theorem 2.4, we have that

$$|a_n| \leqq \frac{1}{n}\left(\frac{(\mathrm{Re}(\alpha))^2 - |\alpha(2\alpha-1)|^2}{(\mathrm{Im}(\alpha))^2}\right)^{\frac{1}{2}}.$$

But, we know that the extremal function $f(z)$ for the class $\mathcal{R}_\alpha$ in Theorem 1.3 satisfies

$$|a_n| = \frac{A^{n-2}(A-B)}{n} \qquad (n = 2, 3, 4, \cdots).$$

Therefore, we guess that the function $f(z) \in \mathcal{R}_\alpha$ satisfies

$$|a_n| \leqq \frac{A^{n-2}(A-B)}{n} \qquad (n = 2, 3, 4, \cdots). \tag{3.1}$$

How can we prove the coefficient inequality (3.1) for $f(z) \in \mathcal{R}_\alpha$?

References

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