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A Note on a Subclass of Analytic Functions Defined by Multiplier Transformations

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Abstract

Let $A(p,n) = \{f \in \mathcal{H}(U) : \ f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, \quad z \in U\}$, ***with*** $A(1,n) = \mathcal{A}_n$, $n \in N$***. In this paper, we consider multiplier transformations***

$$I(m,\lambda,l) f(z) := z + \sum_{j=n+1}^{\infty} \left(\frac{\lambda(j-1)+l+1}{l+1} \right)^m a_j z^j,$$

where $p, n \in N$***,*** $m \in N \cup \{0\}$***,*** $\lambda, l \geq 0$***.***
By making use of the multiplier transformation we define a new class $\mathcal{BI}(m,n,\mu,\alpha,\lambda,l)$ ***involving functions*** $f \in \mathcal{A}_n$***. Parallel results, for some related classes including the class of starlike and convex functions respectively, are also obtained.***

Keywords: *Analytic function, starlike function, convex function, multiplier transformations.*
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1 Introduction and definitions

Denote by U the unit disc of the complex plane, $U = \{z \in C : \ |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let

$$\mathcal{A}(p,n) = \{f \in \mathcal{H}(U) : \ f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, \quad z \in U\},$$

with $\mathcal{A}(1,n) = \mathcal{A}_n$ and

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U):\ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots,\ z \in U\},$$

where $p, n \in N$, $a \in C$.

Let $\mathcal{S}$ denote the subclass of functions that are univalent in U.

By $\mathcal{S}^*(\alpha)$ we denote a subclass of $\mathcal{A}_n$ consisting of starlike univalent functions of order α, $0 \leq \alpha < 1$ which satisfies

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U. \tag{1.1}$$

Further, a function f belonging to $\mathcal{S}$ is said to be convex of order α in U, if and only if

$$Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha, \quad z \in U \tag{1.2}$$

for some α, $(0 \leq \alpha < 1)$. We denote by $\mathcal{K}(\alpha)$ the class of functions in $\mathcal{S}$ which are convex of order α in U and denote by $\mathcal{R}(\alpha)$ the class of functions in $\mathcal{A}_n$ which satisfy

$$Ref'(z) > \alpha, \quad z \in U. \tag{1.3}$$

It is well known that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}$.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Definition 1.1 *[5] For $f \in \mathcal{A}(p,n)$, $p, n \in N$, $m \in N \cup \{0\}$, $\lambda, l \geq 0$, the operator $I_p(m,\lambda,l)f(z)$ is defined by the following infinite series*

$$I_p(m,\lambda,l)f(z) := z^p + \sum_{j=p+n}^{\infty}\left(\frac{p + \lambda(j-1) + l}{p+l}\right)^m a_j z^j.$$

Remark 1.2 *It follows from the above definition that*

$$I_p(0,\lambda,l)f(z) = f(z),$$

$$(p+l)\,I_p(m+1,\lambda,l)f(z) = [p(1-\lambda)+l]\,I_p(m,\lambda,l)f(z) + \lambda z\left(I_p(m,\lambda,l)f(z)\right)',$$

for $z \in U$.

Remark 1.3 *If $p = 1$ we have $I_1(m,\lambda,l)f(z) = I(m,\lambda,l)$ and*

$$(l+1)\,I(m+1,\lambda,l)f(z) = [l+1-\lambda]\,I(m,\lambda,l)f(z) + \lambda z\left(I(m,\lambda,l)f(z)\right)',$$

for $z \in U$.

Remark 1.4 *If* $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, *then*

$$I(m,\lambda,l) f(z) = z + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m a_j z^j,$$

for $z \in U$.

Remark 1.5 *For* $l = 0$, $\lambda \geq 0$, *the operator* $D_\lambda^m = I(m,\lambda,0)$ *was introduced and studied by Al-Oboudi , which is reduced to the Sălăgean differential operator* $S^m = I(m,1,0)$ *for* $\lambda = 1$. *The operator* $I_l^m = I(m,1,l)$ *was studied recently by Cho and Srivastava [8] and Cho and Kim [9]. The operator* $I_m = I(m,1,1)$ *was studied by Uralegaddi and Somanatha [13], the operator* $D_\lambda^\delta = I(\delta,\lambda,0)$, *with* $\delta \in R$, $\delta \geq 0$, *was introduced by Acu and Owa [1].*

To prove our main theorem we shall need the following lemma.

Lemma 1.6 [11] *Let* u *be analytic in* U *with* $u(0) = 1$ *and suppose that*

$$Re\left(1 + \frac{zu'(z)}{u(z)}\right) > \frac{3\alpha - 1}{2\alpha}, \quad z \in U. \tag{1.4}$$

Then $Reu(z) > \alpha$ *for* $z \in U$ *and* $1/2 \leq \alpha < 1$.

2 Main results

Definition 2.1 *We say that a function* $f \in \mathcal{A}_n$ *is in the class* $\mathcal{BI}(m,n,\mu,\alpha,\lambda,l)$, $m,n \in N$, $\mu \geq 0$, $\alpha \in [0,1)$ *if*

$$\left| \frac{I(m+1,\lambda,l) f(z)}{z} \left(\frac{z}{I(m,\lambda,l) f(z)} \right)^\mu - 1 \right| < 1 - \alpha, \qquad z \in U. \tag{2.5}$$

Remark 2.2 *The family* $\mathcal{BI}(m,n,\mu,\alpha,\lambda,l)$ *is a new comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example,* $\mathcal{BI}(0,1,1,\alpha,1,0) = \mathcal{S}^*(\alpha)$, $\mathcal{BI}(1,1,1,\alpha,1,0) = \mathcal{K}(\alpha)$ *and* $\mathcal{BI}(0,1,0,\alpha,1,0) = \mathcal{R}(\alpha)$. *Another interesting subclass is the special case* $\mathcal{BI}(0,1,2,\alpha,1,0) \equiv \mathcal{B}(\alpha)$ *which has been introduced by Frasin and Darus [10] and also the class* $\mathcal{BI}(0,1,\mu,\alpha,1,0) \equiv \mathcal{B}(\mu,\alpha)$ *which has been introduced by Frasin and Jahangiri [11]. Catas and Alb have been introduced the subclasses* $\mathcal{BI}(m,n,\mu,\alpha,\lambda,0) = \mathcal{BO}(m,\mu,\alpha,\lambda)$ *[6] and* $\mathcal{BI}(m,\mu,\alpha,1,0) = \mathcal{BS}(m,\mu,\alpha)$ *[7], [4].*

In this note we provide a sufficient condition for functions to be in the class $\mathcal{BI}(m,n,\mu,\alpha,\lambda,l)$. Consequently, as a special case, we show that convex functions of order 1/2 are also members of the above defined family.

Theorem 2.3 *For the function $f \in \mathcal{A}_n$, $m, n \in N$, $\mu \geq 0$, $1/2 \leq \alpha < 1$ if*

$$\frac{l+1}{\lambda}\frac{I(m+2,\lambda,l)f(z)}{I(m+1,\lambda,l)f(z)} - \frac{\mu(l+1)}{\lambda}\frac{I(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)} + \frac{(l+1)(\mu-1)}{\lambda} + 1 \prec 1+\beta z, \quad z \in U, \tag{2.6}$$

where

$$\beta = \frac{3\alpha - 1}{2\alpha},$$

then $f \in \mathcal{BI}(m,n,\mu,\alpha,\lambda,l)$.

Proof If we consider

$$u(z) = \frac{I(m+1,\lambda,l)f(z)}{z}\left(\frac{z}{I(m,\lambda,l)f(z)}\right)^{\mu}, \tag{2.7}$$

then $u(z)$ is analytic in U with $u(0) = 1$. A simple differentiation yields

$$\frac{zu'(z)}{u(z)} = \frac{l+1}{\lambda}\frac{I(m+2,\lambda,l)f(z)}{I(m+1,\lambda,l)f(z)} - \frac{\mu(l+1)}{\lambda}\frac{I(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)} + \frac{(l+1)(\mu-1)}{\lambda}. \tag{2.8}$$

Using (2.6) we get

$$Re\left(1 + \frac{zu'(z)}{u(z)}\right) > \frac{3\alpha-1}{2\alpha}.$$

Thus, from Lemma 1.6 we deduce that

$$Re\left\{\frac{I(m+1,\lambda,l)f(z)}{z}\left(\frac{z}{I(m,\lambda,l)f(z)}\right)^{\mu}\right\} > \alpha.$$

Therefore, $f \in \mathcal{BI}(m,n,\mu,\alpha,\lambda,l)$, by Definition 2.1.

As a consequence of the above theorem we have the following interesting corollaries [3].

Corollary 2.4 *If $f \in \mathcal{A}_1$ and*

$$Re\left\{\frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)} - \frac{zf''(z)}{f'(z)}\right\} > -\frac{1}{2}, \quad z \in U, \tag{2.9}$$

then

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2}, \quad z \in U. \tag{2.10}$$

That is, f is convex of order $\frac{1}{2}$, or $f \in \mathcal{BI}\left(1,1,1,\frac{1}{2},1,0\right)$.

Corollary 2.5 *If $f \in \mathcal{A}_1$ and*

$$Re\left\{\frac{2zf'(z) + z^2 f''(z)}{f'(z) + zf''(z)}\right\} > -\frac{1}{2}, \quad z \in U, \tag{2.11}$$

then $f \in \mathcal{BI}\left(1, 1, 0, \frac{1}{2}, 1, 0\right)$, that is

$$Re\{f'(z) + zf''(z)\} > \frac{1}{2}, \quad z \in U. \tag{2.12}$$

Corollary 2.6 *If $f \in \mathcal{A}_1$ and*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2}, \quad z \in U, \tag{2.13}$$

then

$$Ref'(z) > \frac{1}{2}, \quad z \in U. \tag{2.14}$$

In another words, if the function f is convex of order $\frac{1}{2}$, then $f \in \mathcal{BI}(0, 1, 0, \frac{1}{2}, 1, 0) \equiv \mathcal{R}\left(\frac{1}{2}\right)$.

Corollary 2.7 *If $f \in \mathcal{A}_1$ and*

$$Re\left\{\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} > -\frac{3}{2}, \quad z \in U, \tag{2.15}$$

then f is starlike of order $\frac{1}{2}$, hence $f \in \mathcal{BI}(0, 1, 1, \frac{1}{2}, 1, 0)$.

3 Open Problem

The open problem is to define a generic class of analytic functions such that the class $\mathcal{BI}(m, n, \mu, \alpha, \lambda, l)$, $m, n \in N$, $\mu \geq 0, \alpha \in [0, 1)$ is contained inside and is possible to obtain. Compare the new results with the results given by [11].

References

[1] M. Acu, S. Owa, *Note on a class of starlike functions*, RIMS, Kyoto, 2006.

[2] F.M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Ind. J. Math. Math. Sci., 27 (2004), 1429-1436.

[3] A. Alb Lupaş, *A subclass of analytic functions defined by Ruscheweyh derivative*, Acta Universitatis Apulensis, nr. 19/2009, 31-34.

[4] A. Alb Lupaş, A. Cătaş, *On a subclass of analytic functions defined by differential Sălăgean operator*, Analele Universităţii din Oradea, Tom XVII, Issue no. 1, 2010, 7-10.

[5] A. Cătaş, *On certain class of p-valent functions defined by new multiplier transformations*, Adriana Catas, Proceedings Book of the International Symposium on Geometric Function Theory and Applications, August 20-24, 2007, TC Istanbul Kultur University, Turkey, 241-250.

[6] A. Cătaş, A. Alb Lupaş, *On a subclass of analytic functions defined by a generalized Sălăgean operator*, ROMAI Journal, vol. 4, nr. 2 (2008), 57-60.

[7] Adriana Cătaş, Alina Alb Lupaş, *On sufficient conditions for certain subclass of analytic functions defined by differential Sălăgean operator*, International Journal of open Problems in Computer Science and Mathematics, Vol. 1, No. 2, Nov. 2009, p. 14-18.

[8] N.E. Cho, H.M. Srivastava, *Argument estimates of certain analytic functions defined by a class of multiplier transformations* Math. Comput. Modelling, 37 (1-2) (2003), 39-49.

[9] N.E. Cho, T.H. Kim, *Multiplier transformations and strongly close-to-close functions*, Bull. Korean Math. Soc., 40 (3) (2003) 399-410.

[10] B.A. Frasin and M. Darus, *On certain analytic univalent functions*, Internat. J. Math. and Math. Sci., 25(5), 2001, 305-310.

[11] B.A. Frasin and Jay M. Jahangiri, *A new and comprehensive class of analytic functions*, Analele Universităţii din Oradea, Tom XV, 2008, 61-64.

[12] G. St. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, 1013(1983), 362-372.

[13] B.A. Uralegaddi, C. Somanatha, *Certain classes of univalent functions*, Current topics in analytic function theory, World. Sci. Publishing, River Edge, N.Y., (1992), 371-374.