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# A Multiplier Transformation Defined by Convolution Involving a Differential Operator

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**Abstract**

***The object of this paper is to introduce a multiplier transformation defined by convolution involving differential operator given by Al-Oboudi. A new subclass of strongly close-to-convex functions in the open unit disk using this operator will be discussed. Our results include several previous known results as special cases.***

**Keywords:** Analytic function, Starlike and Strongly close-to-convex functions.

**AMS Mathematics Subject Classification (2000):** 30C45.

## 1 Introduction

Let $H$ be the class of analytic functions in the open unit disk $U = \{z : |z| < 1\}$ and $H[a, n]$ be the subclasses of $H$ consisting of functions of the form :

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$$

Let $A$ be the subclass of $H$ consisting of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U \tag{1}$$

which are analytic in the unit disk $U$. Let $F$ and $G$ be analytic functions in the unit disk $U$, the function $F$ is said to be subordinate to $G$ or $G$ is said to be superordinate to $F$, if there exists a function $w$ analytic in $U$ with $w(0) = 0$ and $|w| < 1$ for $z \in U$ and such that $F(z) = G(w(z))$, $z \in U$ in such a case, we write $F \prec G$ or $F(z) \prec G(z)$ if the function $G$ is univalent in $U$, then

$$F \prec G \Leftrightarrow F(0) = G(0), \ F(U) \subset G(U).$$

For functions $f$ given by (1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $z \in U$. let $(f * g)(z)$ denote the Hadamard product (convolution) of $f(z)$ and $g(z)$, defined by :

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$$

For $f \in A$, Al- Oboudi [2] introduced the following operator :

$$D^0 f(z) = f(z) \tag{2}$$

$$D^1_\lambda f(z) = D_\lambda f(z) = (1-\lambda) f(z) + \lambda z f'(z) \tag{3}$$

$$D^m_\lambda f(z) = D_\lambda \left( D^{m-1}_\lambda f(z) \right), \quad \lambda > 0 \tag{4}$$

if $f$ is given by (1), then from (3) and (4) we see that

$$D^m_\lambda f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda]^m a_n z^n, \quad m \geq 0, \ \lambda > 0$$

when $\lambda = 1$, we get Salagean differential operator [16] .

For any complex number $s$, we define the multiplier transformation $I^s_\delta$ of functions $f \in A$ by :

$$I^s_\delta f(z) = z + \sum_{n=2}^{\infty} \left( \frac{n+\delta}{1+\delta} \right)^s z^n \ , \ (\delta > -1)$$

By Hadamard product we get $D^{m,s}_{\lambda,\delta} f(z)$ defined by :

$$D^{m,s}_{\lambda,\delta} f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda]^m \left( \frac{n+\delta}{1+\delta} \right)^s a_n z^n \ ,$$

$$(s\in C,\lambda>0,\delta>-1,m\geq 0,z\in U).$$

Obviously, we observe that

$$D_{\lambda,\delta}^{m,s}\left(D_{\lambda,\delta}^{l,k}f(z)\right)=D_{\lambda,\delta}^{m+l,s+k}f(z),\quad (s,k\in C,\delta>-1,l,m\geq 0,z\in U).$$

For $s\in Z,\ \delta=1$ and $m=0$ the operator $D_{\lambda,\delta}^{m,s}$ was studied by Uralegaddi and Somanatha [19], and for $s\in Z,\ m=0$ the operator $D_{\lambda,\delta}^{m,s}$ was closely related to multiplier transformations studied by Flett [6], also, for $s=-1,\ m=0$ the operator $D_{\lambda,\delta}^{m,s}$ belongs to integral operator studied by Owa and Srivastava [14]. And for any negative real number $s$ and $\delta=1,\ m=0$, the operator $D_{\lambda,\delta}^{m,s}$ was a multiplier transformation studied by Jung et, al.[7], and for any nonnegative integer $s$ and $\delta=m=0$, the operator $D_{\lambda,\delta}^{m,s}$ was the differential operator given by Salagean [16]. Finally, for different choices of $s,\delta$ and $m$, several operators investigated earlier by other authors (see for example Ahuja [1], Cho and Kim [4], and Lin and Owa [9]) are obtained .

Now, by using $D_{\lambda,\delta}^{m,s}$, new classes of analytic functions are defined as follows: For $s\in C,\ \delta>-1$ and $m\geq 0$, let $K_{\lambda,\delta}^{m,s}(\gamma,\alpha,\beta,A,B)$ be the class of functions $f\in A$ satisfying the condition :

$$\left|\arg\left(\frac{z\left(D_{\lambda,\delta}^{m,s}f(z)\right)'}{D_{\lambda,\delta}^{m,s}g(z)}-\gamma\right)\right|<\frac{\pi}{2}\alpha\quad (0\leq\gamma<1\,;0<\alpha\leq 1\,;z\in U)$$

for some $g\in S_{\lambda,\delta}^{m,s}(\beta,A,B)$, where

$$S_{\lambda,\delta}^{m,s}(\beta,A,B)=\left\{g\in A:\frac{1}{1-\beta}\left(\frac{z\left(D_{\lambda,\delta}^{m,s}g(z)\right)'}{D_{\lambda,\delta}^{m,s}g(z)}-\beta\right)\prec\frac{1+Az}{1+Bz}\right\},$$

$$(0\leq\beta<1\,;-1\leq B<A\leq 1\,;z\in U)$$

Note that $K_{1,0}^{1,0}(\gamma,1,\beta,1,-1)$ and $K_{0,0}^{0,0}(\gamma,1,\beta,1,-1)$ are the classes of quasi-convex and close-to-convex functions of order $\gamma$ and type $\beta$, respectively introduced and studied by Noor and Alkhora sani [11] and Silverman [17]. Further $K_{0,0}^{0,1}(0,\alpha,0,1,-1)=K_{1,0}^{1,0}(0,\alpha,0,1,-1)$ is the class of strongly close-to-convex functions of order $\alpha$ in the sense of Pommerenke [15]. Finally, notice that for integer $s$ and $m=0$, the class $K_{0,\delta}^{0,s}(\gamma,\alpha,\beta,A,B)$ was studied by Cho and Kim [4].

We need the following lemmas to prove our main results:

**Lemma 1.1** *[5] Let $h$ be convex univalent in $U$ with $h(0)=1$ and $\Re(\beta h(z)+\gamma)>0,\ (\beta,\gamma\in C)$. If $p$ is analytic in $U$ with $p(0)=1$, then*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec h(z),\quad (z\in U)$$

*implies*

$$p(z)\prec h(z)$$

**Lemma 1.2** *[10] Let $h$ be convex univalent in $U$ and $w$ be analytic in $U$ with $\Re w(z)\geq 0$. If $p$ is analytic in $U$ and $p(0)=h(0)$, then*

$$p(z)+w(z)\,zp'(z)\prec h(z)$$

*implies*

$$p(z)\prec h(z)$$

**Lemma 1.3** *[13] Let $p$ be analytic in $U$ with $p(0)=1$ and $p(z)\neq 0$ in $U$. suppose that there exists a point $z_0\in U$ such that :*

$$|\arg p(z)|<\frac{\pi}{2}\eta\quad for\quad |z|<|z_0| \tag{5}$$

*and*

$$|\arg p(z_0)|=\frac{\pi}{2}\eta\qquad (0<\eta\leq 1). \tag{6}$$

*then we have*

$$\frac{zp'(z_0)}{p(z_0)}=ik\eta, \tag{7}$$

*where*

$$k\geq\frac{1}{2}\left(a+\frac{1}{a}\right)\qquad when\quad \arg p(z_0)=\frac{\pi}{2}\eta \tag{8}$$

*and*

$$k\leq-\frac{1}{2}\left(a+\frac{1}{a}\right)\qquad when\quad \arg p(z_0)=-\frac{\pi}{2}\eta \tag{9}$$

*and*

$$p(z_0)^{\frac{1}{\eta}}=\pm\, ia\qquad (a>0). \tag{10}$$

At first, with the help of Lemma 1.1, we obtain the following theorem :

**Theorem 1.4** *Let $h$ be convex univalent in $U$ with $h(0)=1$ and $\Re((1-\beta)h(z)+\beta+\delta)>0$. If a function $f\in A$ satisfies the condition*

$$\frac{1}{1-\beta}\left(\frac{z\left(D_{\lambda,\delta}^{m,s+1}f(z)\right)'}{D_{\lambda,\delta}^{m,s+1}f(z)}-\beta\right)\prec h(z),\quad (0\leq\beta<1\,;z\in U)$$

*then*

$$\frac{1}{1-\beta}\left(\frac{z\left(D_{\lambda,\delta}^{m,s}f(z)\right)'}{D_{\lambda,\delta}^{m,s}f(z)}-\beta\right)\prec h(z)\,,\quad (0\le\beta<1\,;z\in U)$$

**Proof** : Let

$$p(z)=\frac{1}{1-\beta}\left(\frac{z\left(D_{\lambda,\delta}^{m,s}f(z)\right)'}{D_{\lambda,\delta}^{m,s}f(z)}-\beta\right)$$

where $p$ is analytic function with $p(0)=1$. By using the equation :

$$z\left(D_{\lambda,\delta}^{m,s}f(z)\right)'=(\delta+1)\,D_{\lambda,\delta}^{m,s+1}f(z)-\delta D_{\lambda,\delta}^{m,s}f(z) \tag{11}$$

we get :

$$\delta+\beta+(1-\beta)\,p(z)=\frac{(\delta+1)\,D_{\lambda,\delta}^{m,s+1}f(z)}{D_{\lambda,\delta}^{m,s}f(z)} \tag{12}$$

taking logarithmic derivatives in both sides of (12) and multiplying by $z$, we have

$$p(z)+\frac{zp'(z)}{\delta+\beta+(1-\beta)\,p(z)}=\frac{1}{1-\beta}\left(\frac{z\left(D_{\lambda,\delta}^{m,s+1}f(z)\right)'}{D_{\lambda,\delta}^{m,s+1}f(z)}-\beta\right),\qquad z\in U.$$

Applying Lemma 1.1, it follows that $p\prec h$, that is

$$\frac{1}{1-\beta}\left(\frac{z\left(D_{\lambda,\delta}^{m,s}f(z)\right)'}{D_{\lambda,\delta}^{m,s}f(z)}-\beta\right)\prec h(z)\,.$$

Taking $h(z)=(1+Az)/(1+Bz)\,,\ (-1\le B<A\le1)$ in Theorem 1.4 we have

**Corollary 1.5** *The inclusion relation* $S_{\lambda,\delta}^{m,s+1}(\beta,A,B)\subset S_{\lambda,\delta}^{m,s}(\beta,A,B)$ *holds for* $s\in C,\ \delta>-1,\ m\ge0.$

Letting $s=\delta=0,\ m=0$ and $h(z)=((1+z)/(1-z))^{\mu}\,,\quad(0<\mu\le1)$ in Theorem 1.4 we have the following inclusion relation:

**Corollary 1.6** *For* $s\in C,\ \delta>-1,\ m\ge0$ *and* $h(z)=((1+z)/(1-z))^{\mu}\,,$ $(0<\mu\le1)$ *then we have* $C(\mu,\beta)\subset S^{*}(\mu,\beta)\,.$

**Theorem 1.7** *Let $h$ be convex univalent in $U$ with $h(0) = 1$ and $\Re\left((1-\beta)h(z) + \beta + \frac{1}{\lambda} - 1\right) > 0$. If a function $f \in A$ satisfies the condition*

$$\frac{1}{1-\beta}\left(\frac{z\left(D_{\lambda,\delta}^{m+1,s} f(z)\right)'}{D_{\lambda,\delta}^{m+1,s} f(z)} - \beta\right) \prec h(z), \quad (0 \le \beta < 1; z \in U)$$

*then*

$$\frac{1}{1-\beta}\left(\frac{z\left(D_{\lambda,\delta}^{m,s} f(z)\right)'}{D_{\lambda,\delta}^{m,s} f(z)} - \beta\right) \prec h(z), \quad (0 \le \beta < 1; z \in U)$$

*for $s \in C$, $\delta > -1$, $m \ge 0$*

**Proof** : Let

$$p(z) = \frac{1}{1-\beta}\left(\frac{z\left(D_{\lambda,\delta}^{m,s} f(z)\right)'}{D_{\lambda,\delta}^{m,s} f(z)} - \beta\right), \qquad (0 \le \beta < 1; z \in U)$$

where $p$ is analytic function with $p(0) = 1$. By using the equation

$$\lambda z\left(D_{\lambda,\delta}^{m,s} f(z)\right)' = D_{\lambda,\delta}^{m+1,s} f(z) - (1-\lambda) D_{\lambda,\delta}^{m,s} f(z)$$

we get

$$\beta + \frac{1}{\lambda} - 1 + (1-\beta)p(z) = \frac{D_{\lambda,\delta}^{m+1,s} f(z)}{\lambda D_{\lambda,\delta}^{m,s} f(z)} \tag{13}$$

and taking logarithmic derivatives in both sides of (13) and multiplying by $z$ we get

$$p(z) + \frac{zp'(z)}{\beta + \frac{1}{\lambda} - 1 + (1-\beta)p(z)} = \frac{1}{1-\beta}\left(\frac{z\left(D_{\lambda,\delta}^{m+1,s} f(z)\right)'}{D_{\lambda,\delta}^{m+1,s} f(z)} - \beta\right).$$

Applying Lemma 1.1 it follows that $p \prec h$, that is

$$\frac{1}{1-\beta}\left(\frac{z\left(D_{\lambda,\delta}^{m,s} f(z)\right)'}{D_{\lambda,\delta}^{m,s} f(z)} - \beta\right) \prec h(z).$$

Taking $h(z) = (1+Az)/(1+Bz)$, $(-1 \le B < A \le 1)$ in Theorem 1.7 we have

**Corollary 1.8** *The inclusion relation* $S^{m+1,\,s}_{\lambda,\,\delta}(\beta, A,\, B) \subset S^{m,\,s}_{\lambda,\,\delta}(\beta, A,\, B)$ *holds for* $s \in C,\ \delta > -1,\ m \geq 0$.

**Theorem 1.9** *Let* $h$ *be convex univalent in* $U$, *with* $h(0) = 1$ *and* $\Re\left((1-\beta)\, h(z) + \beta + c\right) > 0$. *If a function* $f \in A$ *satisfies the condition*

$$\frac{1}{1-\beta}\left(\frac{z\left(D^{m,\,s}_{\lambda,\,\delta} f(z)\right)'}{D^{m,\,s}_{\lambda,\,\delta} f(z)} - \beta\right) \prec h(z)\,, \quad (0 \leq \beta < 1\,; z \in U)$$

*then*

$$\frac{1}{1-\beta}\left(\frac{z\left(D^{m,\,s}_{\lambda,\,\delta} F_c(f)(z)\right)'}{D^{m,\,s}_{\lambda,\,\delta} F_c(f)(z)} - \beta\right) \prec h(z)\,, \quad (0 \leq \beta < 1\,; z \in U)$$

*where* $F_c$ *be the integral operator defined by*

$$F_c(f) := F_c(f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\, dt, \quad (c \geq 0) \tag{14}$$

**Proof** : From (14) we have

$$z\left(D^{m,\,s}_{\lambda,\,\delta} F_c(f)(z)\right)' = (c+1)\, D^{m,\,s}_{\lambda,\,\delta} f(z) \;-\; c\, D^{m,\,s}_{\lambda,\,\delta} F_c(f)(z) \tag{15}$$

By using the same technique as in the proof of the Theorem 1.4 and Lemma 1.1 the required result is obtained.

Letting $h(z) = (1+Az)/(1+Bz)\,,\ (-1 \leq B < A \leq 1)$ in Theorem 1.9 we have immediately the following

**Corollary 1.10** *If* $f \in S^{m,\,s}_{\lambda,\,\delta}(\beta, A, B)\,$, *then* $F_c(f)(z) \in S^{m,\,s}_{\lambda,\,\delta}(\beta, A, B)$ *where* $F_c$ *is the integral defined by (14).*

Now, we obtain the following:

**Theorem 1.11** *Let* $f \in A$ *and* $0 < \alpha \leq 1, \quad 0 \leq \gamma < 1$. *If*

$$\left|\arg\left(\frac{z\left(D^{m,\,s+1}_{\lambda,\,\delta} f(z)\right)'}{D^{m,\,s+1}_{\lambda,\,\delta} g(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha$$

*for some* $g \in S^{m,\,s+1}_{\lambda,\,\delta}(\beta, A, B)\,$, *then*

$$\left|\arg\left(\frac{z\left(D^{m,\,s}_{\lambda,\,\delta} f(z)\right)'}{D^{m,\,s}_{\lambda,\,\delta} g(z)} - \gamma\right)\right| < \frac{\pi}{2}\eta$$

*where* $\eta$ $(0 < \eta \leq 1)$ *is the solution of the equation :*

$$\alpha = \left\{ \begin{array}{ll} \eta + \frac{2}{\pi}\,\tan^{-1}\left(\frac{\eta\cos\frac{\pi}{2}t_1}{\frac{(1-\beta)(1+A)}{1+B}+\beta+\delta+\eta\sin\frac{\pi}{2}t_1}\right) & for\ B \neq -1 \\ \eta & for\ \ B = -1 \end{array} \right\} \tag{16}$$

*and*

$$t_1 = \frac{2}{\pi}\,\sin^{-1}\left(\frac{(1-\beta)\,(A-B)}{(1-\beta)\,(1-AB) + (\beta+\delta)\,(1-B^2)}\right). \tag{17}$$

**Proof** : Let

$$p\,(z) = \frac{1}{1-\gamma}\left(\frac{z\left(D_{\lambda,\,\delta}^{m,\,s} f\,(z)\right)'}{D_{\lambda,\,\delta}^{m,\,s} g\,(z)} - \gamma\right).$$

Using (11) and simplifying, we have

$$((1-\gamma)\,p\,(z)+\gamma)\,D_{\lambda,\,\delta}^{m,\,s} g\,(z) = (\delta+1)\,D_{\lambda,\,\delta}^{m,\,s+1} f\,(z) - \delta D_{\lambda,\,\delta}^{m,\,s} f\,(z)\,. \tag{18}$$

Differentiating (18) and multiplying by $z$, we obtain

$$(1-\gamma)\,zp'\,(z)\,D_{\lambda,\,\delta}^{m,\,s} g\,(z) + ((1-\gamma)\,p\,(z)+\gamma)\,z\left(D_{\lambda,\,\delta}^{m,\,s} g\,(z)\right)'$$

$$= (\delta+1)\,z\left(D_{\lambda,\,\delta}^{m,\,s+1} f\,(z)\right)' - \delta z\left(D_{\lambda,\,\delta}^{m,\,s} f\,(z)\right)' \tag{19}$$

Since $g \in S_{\lambda,\,\delta}^{m,\,s+1}(\beta,\ A,\ B)$, by Corollary 1.5, we know that $g \in S_{\lambda,\,\delta}^{m,\,s}(\beta,\ A,\ B)$. Let

$$q\,(z) = \frac{1}{1-\beta}\left(\frac{z\left(D_{\lambda,\,\delta}^{m,\,s} g\,(z)\right)'}{D_{\lambda,\,\delta}^{m,\,s} g\,(z)} - \beta\right).$$

Then using (11) once again, we have

$$(1-\beta)\,q\,(z) + \beta + \delta = (\delta+1)\,\frac{D_{\lambda,\,\delta}^{m,\,s+1} g\,(z)}{D_{\lambda,\,\delta}^{m,\,s} g\,(z)\,.} \tag{20}$$

From (19) and (20) we obtain

$$\frac{1}{1-\gamma}\left(\frac{z\left(D_{\lambda,\,\delta}^{m,\,s+1} f\,(z)\right)'}{D_{\lambda,\,\delta}^{m,\,s+1} g\,(z)} - \gamma\right) = p\,(z) + \frac{zp'\,(z)}{(1-\beta)\,q\,(z)+\beta+\delta}.$$

While, by using the result of Silverman and Silvia [18], we have

$$\left|q\,(z) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}, \qquad (z \in U\,;\ B \neq -1) \tag{21}$$

and

$$\Re\left\{q\left(z\right)\right\} > \frac{1-A}{2}, \qquad \left(z \in U\,;\; B \neq -1\right) \tag{22}$$

Then from (21) and (22), we obtain

$$\left(1-\beta\right)q\left(z\right)+\beta+\delta=\rho e^{i\frac{\pi\phi}{2}},$$

where

$$\begin{cases} \frac{(1-\beta)(1-A)}{1-B}+\beta+\delta< \ \rho \ <\frac{(1-\beta)(1+A)}{1+B}+\beta+\delta \\ -t_1<\phi<t_2 \ \ for \ B\neq -1, \end{cases}$$

When $t_1$ is given by (17), and

$$\begin{cases} \frac{(1-\beta)(1-A)}{2}+\beta+\delta< \ \rho \ <\infty \\ -1<\phi<1 \quad for \ B=-1. \end{cases}$$

We note that $p$ is analytic in $U$, by applying the assumption and Lemma 1.2 with $w\left(z\right)=1/((1-\beta)\,q\left(z\right)+\beta+\delta)$. Hence $p\left(z\right)\neq 0$ in $U$.

If there exists a point $z_0\in U$ such that the conditions (5) and (6) are satisfied, then (by Lemma 1.3) we obtain (7) under the restrictions (8), (9) and (10).

At first, suppose that $p\left(z_0\right)^{\frac{1}{\eta}}=ia, \quad (a>0)$ . Then we obtain

$$\begin{aligned} &\arg\left(p\left(z_0\right)+\frac{z_0p'\left(z_0\right)}{\left(1-\beta\right)q\left(z_0\right)+\beta+\delta}\right) \\ &=\frac{\pi}{2}\eta+\arg\left(1+i\eta k\left(\rho e^{i\frac{\pi\phi}{2}}\right)^{-1}\right) \\ &\geq\frac{\pi}{2}\eta+\tan^{-1}\left(\frac{\eta k\sin\frac{\pi}{2}\left(1-\phi\right)}{\rho+\eta k\cos\frac{\pi}{2}\left(1-\phi\right)}\right) \\ &\geq\frac{\pi}{2}\eta+\tan^{-1}\left(\frac{\eta\cos\frac{\pi}{2}t_1}{\frac{(1-\beta)(1+A)}{1+B}+\beta+\delta+\eta\sin\frac{\pi}{2}t_1}\right) \\ &=\frac{\pi}{2}\,\alpha \end{aligned}$$

where $\alpha$ and $t_1$ given by (16) and (17), respectively. Similarly for the case $B=-1$ we have

$$\arg\left(p\left(z_0\right)+\frac{z_1p'\left(z_0\right)}{\left(1-\beta\right)q\left(z_0\right)+\beta+\delta}\right) \ \geq\frac{\pi}{2}\eta.$$

These evidently contradict the assumption of Theorem 1.11.

Next, suppose that $p(z_0)^{\frac{1}{\eta}} = -ia, \quad (a > 0)$ . Applying the same method as the above, we have

$$\arg\left(p(z_0) + \frac{z_0 p'(z_0)}{(1-\beta)\, q(z_0) + \beta + \delta}\right)$$

$$\leq -\frac{\pi}{2}\eta - \tan^{-1}\left(\frac{\eta \cos \frac{\pi}{2} t_1}{\frac{(1-\beta)(1+A)}{1+B} + \beta + \delta + \eta \sin \frac{\pi}{2} t_1}\right)$$

$$= -\frac{\pi}{2}\,\alpha,$$

where $\alpha$ and $t_1$ are given by (16) and(17), respectively. Similarly, for the case $B = -1$ we have

$$\arg\left(p(z_0) + \frac{z_1 p'(z_0)}{(1-\beta)\, q(z_0) + \beta + \delta}\right) \leq -\frac{\pi}{2}\eta.$$

These also contradict to the assumption of Theorem 1.11. Therefore we complete the proof of Theorem 1.11.

From Theorem 1.11, we see easily the following:

**Corollary 1.12** *The inclusion relation* $K_{\lambda,\delta}^{m,s+1}(\gamma, \alpha, \beta, A, B) \subset K_{\lambda,\delta}^{m,s}(\gamma, \alpha, \beta, A, B)$ *holds for* $s \in C,\ \delta > -1,\ m \geq 0$ *.*

Taking $s = -1,\ \delta = 0$ and $m = \lambda = 1$ in Theorem 1.11 we have

**Corollary 1.13** *Let* $f \in A$*. If*

$$\left|\arg\left(\frac{(zf'(z))'}{g'(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha, \qquad (0 \leq \gamma < 1, 0 < \alpha \leq 1)$$

*for some* $g \in S_{1,0}^{1,0}(\beta, A, B)$ *, then*

$$\left|\arg\left(\frac{zf'(z)}{g(z)} - \gamma\right)\right| < \frac{\pi}{2}\eta$$

*where* $\eta, (0 < \eta \leq 1)$ *is the solution of the equation given by (16).*

**Remark** : If we put $A = 1$, $B = 1$ and $\eta = 1$ in Corollary 1.13 then we see that every quasi-convex function of order $\gamma$ and type $\beta$ is close-to-convex function of order $\gamma$ and type $\beta$, which reduced to the result obtained by Noor [12].

**Theorem 1.14** *Let $f \in A$ and $0 < \alpha \leq 1$, $0 \leq \gamma < 1$. If*

$$\left| \arg \left( \frac{z \left( D_{\lambda,\delta}^{m,s} f(z) \right)'}{D_{\lambda,\delta}^{m,s} g(z)} - \gamma \right) \right| < \frac{\pi}{2}\alpha$$

*from some $g \in S_{\lambda,\delta}^{m,s}(\beta, A, B)$, then*

$$\left| \arg \left( \frac{z \left( D_{\lambda,\delta}^{m,s} F_c(f)(z) \right)'}{D_{\lambda,\delta}^{m,s} F_c(g)(z)} - \gamma \right) \right| < \frac{\pi}{2}\eta,$$

*where $F_c$ is defined by (14), and $\eta, (0 < \eta \leq 1)$ is the solution of the equation given by (16).*

**Proof** : Let

$$p(z) = \frac{1}{1-\gamma} \left( \frac{z \left( D_{\lambda,\delta}^{m,s} F_c(f)(z) \right)'}{D_{\lambda,\delta}^{m,s} F_c(g)(z)} - \gamma \right).$$

Since $g \in S_{\lambda,\delta}^{m,s}(\beta, A, B)$, we have from Corollary 1.10 that $F_c(g)(z) \in S_{\lambda,\delta}^{m,s}(\beta, A, B)$. Using (15) we have

$$((1-\gamma)p(z) + \gamma) D_{\lambda,\delta}^{m,s} F_c(g)(z) = (c+1) D_{\lambda,\delta}^{m,s} f(z) - c D_{\lambda,\delta}^{m,s} F_c(f)(z).$$

Then, by a simple calculation, we get

$$(1-\gamma) z p'(z) + ((1-\gamma)p(z) + \gamma)((1-\beta)q(z) + c + \beta) = (c+1) \frac{z \left( D_{\lambda,\delta}^{m,s} f(z) \right)'}{D_{\lambda,\delta}^{m,s} F_c(g)(z)}$$

where

$$q(z) = \frac{1}{1-\beta} \left( \frac{z \left( D_{\lambda,\delta}^{m,s} F_c(g)(z) \right)'}{D_{\lambda,\delta}^{m,s} F_c(g)(z)} - \beta \right)$$

Hence we have

$$\frac{1}{1-\gamma} \left( \frac{z \left( D_{\lambda,\delta}^{m,s} f(z) \right)'}{D_{\lambda,\delta}^{m,s} g(z)} - \gamma \right) = p(z) + \frac{z p'(z)}{(1-\beta) q(z) + \beta + c}.$$

The remaining part of the proof in Theorem 1.14 is similar to that of Theorem 1.11 and so we omit it.

From Theorem 1.9, we see easily the following:

**Corollary 1.15** *If* $f \in K^{m,s}_{\lambda,\delta}(\gamma, \alpha, \beta, A, B)$ *then* $F_c(f) \in K^{m,s}_{\lambda,\delta}(\gamma, \alpha, \beta, A, B)$ *where* $F_c$ *is the integral operator defined by (14).*

**Remark** : If we take $s = \delta = 0$, $m = \lambda = 1$ and $s = \delta = m = 0$ with $\alpha = 1$, $A = 1$ and $B = -1$ in Corollary 1.15, respectively, then we have the corresponding results obtained by Noor and Alkhorasani [11]. Furthermore, taking $s = \delta = m = \gamma = 0$, $A = 1$ , $B = -1$ and $\alpha = 1$ in Corollary 1.15, we obtain the classical result by Bernardi [3], which implies the result studied by Libera [8].

# 2 Open Problem

The operator defined can be extended and can solve many new results and properties.

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