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On Univalence of a General Integral Operator

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Abstract

In this paper, we obtain univalence of a certain general integral operator and some interesting properties involving the integral operators defined by Cho-Kwon-Srivastava Operator. Relevant connections of the results, which are presented in this paper, with various other interesting results are also pointed out.

Keywords: *univalent functions, integral operators,*

1 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \tag{1.1}$$

which are analytic in the open disc $U = \{z \in C; |z| < 1\}$ and S be the subclass of function $f \in A$, which are univalent in U.

For $g_1(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g_2(z) = z + \sum_{n=2}^{\infty} b_n z^n$ of the class A, the Hadamard product (or convolution) is defined by

$$g_1(z) * g_2(z) = (g_1 * g_2)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \quad . \tag{1.2}$$

Let

$$q_{\beta,\lambda}(z)=z+\sum_{k=2}^{\infty}\left(\frac{\lambda}{k+\lambda-1}\right)^{\beta}z^{k}\quad(\beta\geq 0,\lambda>0),\tag{1.3}$$

$$p_{\mu}(z)=z+\sum_{k=2}^{\infty}\frac{(\mu)_{k-1}}{(k-1)!}z^{k}\quad(\mu\geq 0).\tag{1.4}$$

Analogously N.E. Cho, O.S. Kwon and H.M. Srivastava [4] operator, for $\beta\geq 0,\lambda>0,\mu>0$, we define a linear operator $I_{\lambda,\mu}^{\beta}:A\rightarrow A$as follows:

$$I_{\lambda,\mu}^{\beta}f(z)=(f*p_{\mu}*q_{\beta,\lambda})(z)\ =z+\sum_{k=2}^{\infty}\left(\frac{\lambda}{k+\lambda-1}\right)^{\beta}\frac{(\mu)_{k-1}}{(k-1)!}a_{k}z^{k}\ .\tag{1.5}$$

Definition: For $m\in N,i\in\{1,2,...,m\},\alpha_i\in C$, let us define the integral operator $R_{\mu}^{\beta}(f_1,f_2,...,f_m):A^m\rightarrow A$,

$$R_{\mu}^{\beta}(f_1,f_2,...,f_m)(z)=\int_0^z\left[\frac{I_{\lambda,\mu}^{\beta}f_1(t)}{t}\right]^{\alpha_1}...\left[\frac{I_{\lambda,\mu}^{\beta}f_m(t)}{t}\right]^{\alpha_m}dt\quad,z\in U\tag{1.6}$$

where $f_i(z)\in A$and $I_{\lambda,\mu}^{\beta}$is defined in (1.5).

Obviously, putting $\beta\rightarrow 0,\mu=1$in the operator, we get an operator studied by Breaz and Breaz [1].

Recently many authors (see for example [1], [2], [3] and [4]) have studied and obtained univalence conditions for the analytic function. In the present paper, we also obtain univalence conditions for integral operator which is defined by (1.6). To prove our main results we need followings Lemmas.

Lemma 1 [6]. If the function f is regular in the unit disc U, $f(z)=z+a_2z^2+...,$

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right|\leq 1,\quad\forall z\in U\ .\tag{1.7}$$

Then the function f is univalent.

Lemma 2 [7]. (Schwarz's Lemma) If the analytic function f(z) $f(z)$ is regular in U, with $f(0)=0$ and $|f(z)|<1$for all $z\in U$, then

$$|f(z)|\leq|z|,\forall z\in U\text{ and }|f'(0)|\leq 1\tag{1.8}$$

The equality holds if and only if $f(z)=cz,z\in U,|c|=1$.

2 Main Results

Theorem 1. Let $m \in N, i \in \{1,2,...,m\}, \alpha_i \in C, f_i \in A$. If

$$\left| \frac{z\left(I_{\lambda,\mu}^{\beta} f_i(z)\right)'}{I_{\lambda,\mu}^{\beta} f_i(z)} - 1 \right| \le 1,$$

$$|\alpha_1| + |\alpha_2| + ... + |\alpha_m| \le 1, \; z \in U, \tag{2.1}$$

then $R_{\mu}^{\beta}(f_1, f_2, ..., f_m)(z)$ given by (1.6) is univalent.

Proof: Since $f_i \in A, i \in \{1,2,...,m\}$, we have

$$\frac{I_{\lambda,\mu}^{\beta} f_i(z)}{z} = \frac{z + \sum_{k=2}^{\infty} \left(\frac{\lambda}{k+\lambda-1}\right)^{\beta} \frac{(\mu)_{k-1}}{(k-1)!} a_k z^k}{z}$$

$$= 1 + \sum_{k=2}^{\infty} \left(\frac{\lambda}{k+\lambda-1}\right)^{\beta} \frac{(\mu)_{k-1}}{(k-1)!} a_k z^{k-1}, \tag{2.2}$$

$$\frac{I_{\lambda,\mu}^{\beta} f_i(z)}{z} \ne 0, \quad z \in U.$$

For z = 0, we have

$$\left\{ \left[\frac{I_{\lambda,\mu}^{\beta} f_1(z)}{z}\right]^{\alpha_1} ... \left[\frac{I_{\lambda,\mu}^{\beta} f_m(z)}{z}\right]^{\alpha_m} \right\}_{z=0} = 1. \tag{2.3}$$

By differentiating (1.6), we obtain

$$\left[R_{\mu}^{\beta}(f_1, f_2, ..., f_m)(z)\right]' = \left[\frac{I_{\lambda,\mu}^{\beta} f_1(z)}{z}\right]^{\alpha_1} ... \left[\frac{I_{\lambda,\mu}^{\beta} f_m(z)}{z}\right]^{\alpha_m}, \; z \in U. \tag{2.4}$$

$$\left[R_{\mu}^{\beta}(f_1, f_2, ..., f_m)(0)\right]' = 1.$$

Using (2.4), we obtain

$$\log\left[R_{\mu}^{\beta}(f_1, f_2, ..., f_m)(z)\right]' = \alpha_1\left[\log I_{\lambda,\mu}^{\beta} f_1(z) - \log z\right] + ... + \alpha_m\left[\log I_{\lambda,\mu}^{\beta} f_m(z) - \log z\right], z \in U \tag{2.5}$$

By differentiating (2.5), we have

$$\frac{\left[R_{\mu}^{\beta}(f_1, f_2, ..., f_m)(z)\right]''}{\left[R_{\mu}^{\beta}(f_1, f_2, ..., f_m)(z)\right]'} = \alpha_1\left[\frac{\left(I_{\lambda,\mu}^{\beta} f_1(z)\right)'}{I_{\lambda,\mu}^{\beta} f_1(z)} - \frac{1}{z}\right] + ... + \alpha_m\left[\frac{\left(I_{\lambda,\mu}^{\beta} f_m(z)\right)'}{I_{\lambda,\mu}^{\beta} f_m(z)} - \frac{1}{z}\right], z \in U \tag{2.6}$$

Simple computation, we get

$$\frac{z\left[R_{\mu}^{\beta}\left(f_1,f_2,...,f_m\right)(z)\right]''}{\left[R_{\mu}^{\beta}\left(f_1,f_2,...,f_m\right)(z)\right]'}=\alpha_1\left[\frac{z\left(I_{\lambda,\mu}^{\beta}f_1(z)\right)'}{I_{\lambda,\mu}^{\beta}f_1(z)}-1\right]+...+\alpha_m\left[\frac{z\left(I_{\lambda,\mu}^{\beta}f_m(z)\right)'}{I_{\lambda,\mu}^{\beta}f_m(z)}-1\right],z\in U \tag{2.7}$$

multiplying (2.7) by $\left(1-|z|^2\right)$ and obtain

$$\left(1-|z|^2\right)\left|\frac{z\left[R_{\mu}^{\beta}\left(f_1,f_2,...,f_m\right)(z)\right]''}{\left[R_{\mu}^{\beta}\left(f_1,f_2,...,f_m\right)(z)\right]'}\right|=\left(1-|z|^2\right)\left|\left[\frac{z\left(I_{\lambda,\mu}^{\beta}f_1(z)\right)'}{I_{\lambda,\mu}^{\beta}f_1(z)}-1\right]+...+\alpha_m\left[\frac{z\left(I_{\lambda,\mu}^{\beta}f_m(z)\right)'}{I_{\lambda,\mu}^{\beta}f_m(z)}-1\right]\right| \tag{2.8}$$

$$\le\left(1-|z|^2\right)\left[|\alpha_1|\left|\frac{z\left(I_{\lambda,\mu}^{\beta}f_1(z)\right)'}{I_{\lambda,\mu}^{\beta}f_1(z)}-1\right|+...+|\alpha_m|\left|\frac{z\left(I_{\lambda,\mu}^{\beta}f_m(z)\right)'}{I_{\lambda,\mu}^{\beta}f_m(z)}-1\right|\right]$$

$$\le\left(1-|z|^2\right)\left[|\alpha_1|+...+|\alpha_m|\right]\le|\alpha_1|+...+|\alpha_m|\le 1.$$

Thus by Lemma 1, we have $R_{\mu}^{\beta}\left(f_1,f_2,...,f_m\right)(z)\in S$.

Corollary 1. Let $m\in N, i\in\{1,2,...,m\}, \alpha_i\in C, f_i\in A$ .Putting $\beta=0,\mu=1$ in Theorem 1, we obtain if

$$\left|\frac{z\left(R\left(f_i(z)\right)\right)'}{R\left(f_i(z)\right)}-1\right|\le 1,$$

$$|\alpha_1|+|\alpha_2|+...+|\alpha_m|\le 1,\quad z\in U.$$

Then $R\left(f_1,f_2,...,f_m\right)(z)$ is univalent, where $R\left(f_1,f_2,...,f_m\right)(z)$ is given by Breaz and Breaz[1].

Theorem 2. Let $m\in N, i\in\{1,2,...,m\}, \alpha_i\in C$. If $f_i\in A$ satisfy and

(i) $|\alpha_1|+|\alpha_2|+...+|\alpha_m|\le\frac{1}{3}$,

(ii) $\left|I_{\lambda,\mu}^{\beta}f_i(z)\right|\le 1$,

(iii) $\left|\frac{z^2\left(I_{\lambda,\mu}^{\beta}f_i(z)\right)'}{I_{\lambda,\mu}^{\beta}f_i(z)}-1\right|\le 1.$

For all $z \in U$, then the integral operator given by (1.6) is univalent.

Proof: Using (2.6), we obtain

$$\left|\frac{z\left[R_{\mu}^{\beta}\left(f_1, f_2, ..., f_m\right)(z)\right]''}{\left[R_{\mu}^{\beta}\left(f_1, f_2, ..., f_m\right)(z)\right]'}\right| = |\alpha_1|\left|\frac{z\left(I_{\lambda,\mu}^{\beta} f_1(z)\right)'}{I_{\lambda,\mu}^{\beta} f_1(z)} - 1\right| + ... + |\alpha_m|\left|\frac{z\left(I_{\lambda,\mu}^{\beta} f_m(z)\right)'}{I_{\lambda,\mu}^{\beta} f_m(z)} - 1\right|. \tag{2.9}$$

Multiply (2.9) by $\left(1-|z|^2\right)$, using Schwarz's Lemma and obtain

$$\left(1-|z|^2\right)\left|\frac{z\left[R_{\mu}^{\beta}(z)\right]''}{\left[R_{\mu}^{\beta}(z)\right]'}\right| = \left(1-|z|^2\right)|\alpha_1|\left|\frac{z\left(I_{\lambda,\mu}^{\beta} f_1(z)\right)'}{I_{\lambda,\mu}^{\beta} f_1(z)} - 1\right| + ... + |\alpha_m|\left|\frac{z\left(I_{\lambda,\mu}^{\beta} f_m(z)\right)'}{I_{\lambda,\mu}^{\beta} f_m(z)} - 1\right|$$

$$\leq \left(1-|z|^2\right)|\alpha_1|\left|\frac{z\left(I_{\lambda,\mu}^{\beta} f_1(z)\right)'}{I_{\lambda,\mu}^{\beta} f_1(z)}\right| + \left(1-|z|^2\right)|\alpha_1| + ... + \left(1-|z|^2\right)|\alpha_m|\left|\frac{z\left(I_{\lambda,\mu}^{\beta} f_m(z)\right)'}{I_{\lambda,\mu}^{\beta} f_m(z)}\right|$$

$$+ \left(1-|z|^2\right)|\alpha_m|$$

$$\leq \left(1-|z|^2\right)\left[|\alpha_1|\left|\frac{z\left(I_{\lambda,\mu}^{\beta} f_1(z)\right)'}{I_{\lambda,\mu}^{\beta} f_1(z)}\right| + ... + |\alpha_m|\left|\frac{z\left(I_{\lambda,\mu}^{\beta} f_m(z)\right)'}{I_{\lambda,\mu}^{\beta} f_m(z)}\right|\right] + \left(1-|z|^2\right)\left[|\alpha_1| + ... + |\alpha_m|\right]$$

$$\leq \left(1-|z|^2\right)\left[|\alpha_1|\left|\frac{z^2\left(I_{\lambda,\mu}^{\beta} f_1(z)\right)'}{\left(I_{\lambda,\mu}^{\beta} f_1(z)\right)^2}\right|\frac{\left|\left(I_{\lambda,\mu}^{\beta} f_1(z)\right)\right|}{|z|} + ... + |\alpha_m|\left|\frac{z^2\left(I_{\lambda,\mu}^{\beta} f_m(z)\right)'}{\left(I_{\lambda,\mu}^{\beta} f_m(z)\right)^2}\right|\frac{\left|\left(I_{\lambda,\mu}^{\beta} f_m(z)\right)\right|}{|z|}\right]$$

$$+ \left(1-|z|^2\right)\left[|\alpha_1| + ... + |\alpha_m|\right]$$

$$\leq \left(1-|z|^2\right)\left[|\alpha_1|\left|\frac{z^2\left(I_{\lambda,\mu}^{\beta} f_1(z)\right)'}{\left(I_{\lambda,\mu}^{\beta} f_1(z)\right)^2}\right| + ... + |\alpha_m|\left|\frac{z^2\left(I_{\lambda,\mu}^{\beta} f_m(z)\right)'}{\left(I_{\lambda,\mu}^{\beta} f_m(z)\right)^2}\right|\right] + \left(1-|z|^2\right)\left[|\alpha_1| + ... + |\alpha_m|\right]$$

$$\leq \left(1-|z|^2\right)\left[|\alpha_1|\left|\frac{z^2\left(I_{\lambda,\mu}^{\beta} f_1(z)\right)'}{\left(I_{\lambda,\mu}^{\beta} f_1(z)\right)^2}\right| - |\alpha_1| + |\alpha_1|\right] + ...$$

$$+ \left(1-|z|^2\right)\left[|\alpha_m|\left|\frac{z^2\left(I_{\lambda,\mu}^{\beta} f_m(z)\right)'}{\left(I_{\lambda,\mu}^{\beta} f_m(z)\right)^2}\right| - |\alpha_m| + |\alpha_m|\right] + \left(1-|z|^2\right)\left[|\alpha_1| + ... + |\alpha_m|\right]$$

$$\leq\left(1-|z|^2\right)\left[|\alpha_1|\left|\frac{z^2\left(I^{\beta}_{\lambda,\mu}f_1(z)\right)'}{\left(I^{\beta}_{\lambda,\mu}f_1(z)\right)^2}-1\right|+\ldots+|\alpha_m|\left|\frac{z^2\left(I^{\beta}_{\lambda,\mu}f_m(z)\right)'}{\left(I^{\beta}_{\lambda,\mu}f_m(z)\right)^2}-1\right|\right]$$

$$+\left(1-|z|^2\right)\left[|\alpha_1|+\ldots+|\alpha_m|\right]+\left(1-|z|^2\right)\left[|\alpha_1|+\ldots+|\alpha_m|\right]$$

$$\leq\left(1-|z|^2\right)\left[|\alpha_1|+\ldots+|\alpha_m|\right]+2\left(1-|z|^2\right)\left[|\alpha_1|+\ldots+|\alpha_m|\right]=3\left(1-|z|^2\right)\left[|\alpha_1|+\ldots+|\alpha_m|\right]$$

$$\leq 3\left[|\alpha_1|+\ldots+|\alpha_m|\right] \qquad (2.10)$$

From (2.10) and condition (i), we have

$$\left(1-|z|^2\right)\left|\frac{z\left[R^{\beta}_{\mu}(z)\right]''}{\left[R^{\beta}_{\mu}(z)\right]'}\right|\leq 1, \quad \text{for all } z\in U.$$

By Lemma 1, it follows that the integral operator $R^{\beta}_{\mu}\left(f_1,f_2,\ldots,f_m\right)(z)$ is univalent. By putting $\beta\to 0,\mu=1$, Theorem 2 reduced in the following corollary.

Corollary 2. Let $m\in N, i\in\{1,2,\ldots,m\}, \alpha_i\in C$. If $f_i\in A$ and satisfy

(i) $|\alpha_1|+|\alpha_2|+\ldots+|\alpha_m|\leq\frac{1}{3}$,

(ii) $\left|f_i(z)\right|\leq 1$, and

(iii) $\left|\frac{z^2\left(f_i(z)\right)''}{\left(f_i(z)\right)'}-1\right|\leq 1$,

for all $z\in U$, then the integral operator given by (1.6) is univalent.

3 Open Problem

In this paper, we introduce a new operator $R^{\beta}_{\mu}\left(f_1,f_2,\ldots,f_m\right)(z)$. Is it possible to introduce a new operator related to this operator for meromorphic univalent functions?

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