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A First Order Differential Subordination and Its Applications

Sukhwinder Singh, Sushma Gupta and Sukhjit Singh

Department of Applied Sciences
Baba Banda Singh Bahadur Engineering College
Fatehgarh Sahib-140 407, Punjab, India
e-mail: ssbilling@gmail.com
Department of Mathematics
Sant Longowal Institute of Engineering & Technology
Deemed University, Longowal-148 106, Punjab, India
e-mail: sushmagupta1@yahoo.com
e-mail: sukhjit_d@yahoo.com

Abstract

In this paper, we obtain a first order differential subordination and discuss its applications to univalent and multivalent functions. We show that our results generalize and improve certain known results.

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1 Introduction

A function f is said to be analytic at a point z in a domain $\mathbb{D}$ if it is differentiable not only at z but also in some neighborhood of point z. A function f is said to be analytic on a domain $\mathbb{D}$ if it is analytic at each point of $\mathbb{D}$.

Let $\mathcal{A}$ be the class of all functions f which are analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$ and normalized by the conditions that $f(0) = f'(0) - 1 = 0$. Thus, $f \in \mathcal{A}$ has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

A function f is said to be univalent in a domain $\mathbb{D}$ in the extended complex plane $\mathbb{C}$ if and only if it is regular (analytic) in $\mathbb{D}$ except for at most one simple pole and $f(z_1) \neq f(z_2)$ for $z_1 \neq z_2$ $(z_1, z_2 \in \mathbb{D})$.

In this case, the equation $f(z) = w$ has at most one root in $\mathbb{D}$ for any complex number w. Such functions map $\mathbb{D}$ conformally onto a domain in the w-plane.

Let $\mathcal{S}$ denote the class of all analytic univalent functions f defined on the unit disk $\mathbb{E}$ which are normalized by the conditions $f(0) = f'(0) - 1 = 0$.

The function, for which the equation $f(z) = w$ has p roots in $\mathbb{D}$ for every complex number w, is said to be p-valent (or multivalent) function.

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \ p \in \mathbb{N} = \{1, 2, \cdots\},$$

which are analytic and p-valent (or multivalent) in the open unit disk $\mathbb{E}$. We note that $\mathcal{A}_1 = \mathcal{A}$.

Let f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g in $\mathbb{E}$, written as $f(z) \prec g(z)$, if g is univalent in $\mathbb{E}$, $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Let $\psi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ and let h be univalent in $\mathbb{E}$. If p is analytic in $\mathbb{E}$ and satisfies the differential subordination

$$\psi(p(z), zp'(z); z) \prec h(z), \ \psi(p(0), 0; 0) = h(0), \tag{1}$$

then p is called a solution of the differential subordination (1). The univalent function q is called a dominant of the differential subordination (1) if $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1), is said to be the best dominant of (1).

A function $f \in \mathcal{A}_p$ is said to be p-valent starlike of order $\alpha (0 \leq \alpha < p)$ in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ z \in \mathbb{E}.$$

We denote by $\mathcal{S}_p^*(\alpha)$, the class of all such functions.

A function $f \in \mathcal{A}_p$ is said to be p-valent convex of order $\alpha (0 \leq \alpha < p)$ in $\mathbb{E}$ if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ z \in \mathbb{E}.$$

Let $\mathcal{K}_p(\alpha)$ denote the class of all those functions $f \in \mathcal{A}_p$ which are p-valent convex of order α in $\mathbb{E}$.

We write $\mathcal{S}_p^*(0) = \mathcal{S}_p^*$ and $\mathcal{K}_p(0) = \mathcal{K}_p$, which are, respectively, the classes of p-valent starlike and p-valent convex functions.

Note that $\mathcal{S}_1^*(\alpha)$ and $\mathcal{K}_1(\alpha)$ are, respectively, the usual classes of univalent starlike functions of order α and univalent convex functions of order $\alpha, 0 \leq$

$\alpha < 1$, and will be denoted here by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively. We shall use $\mathcal{S}^*$ and $\mathcal{K}$ to denote $\mathcal{S}^*(0)$ and $\mathcal{K}(0)$, respectively which are the classes of univalent starlike (w.r.t. the origin) and univalent convex functions.

Denote by $\mathcal{S}^*[A,B]$, $-1 \leq B < A \leq 1$, the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz},\ z \in \mathbb{E}.$$

Note that $\mathcal{S}^*[1-2\alpha,-1] = \mathcal{S}^*(\alpha),\ 0 \leq \alpha < 1$ and $\mathcal{S}^*[1,-1] = \mathcal{S}^*$.

Let $\mathcal{S}_p(k)$ denote the subclass of functions $f \in \mathcal{A}_p$ which satisfy the differential inequality

$$\Re\left(1 + \frac{zf''(z)}{f'(z)} + k\frac{zf'(z)}{f(z)}\right) \geq 0,\ z \in \mathbb{E}, \tag{2}$$

where $k+1 \geq 0$.

In 1962, Sakaguchi [2], proved that if $k = -1$, $f(z) \equiv z^p$, is the only function that satisfies (2). The members of $\mathcal{S}_p(k)$ are p-valent convex for $-1 < k \leq 0$ and p-valent starlike for $k > 0$.

When we divide (2) by $k+1 > 0$ and then select $\dfrac{1}{k+1} = \alpha$, we notice that (2) reduces to

$$\Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \geq 0,\ z \in \mathbb{E}. \tag{3}$$

The functions $f \in \mathcal{A}$ satisfying (3), are called α-convex functions. The class of α-convex functions was introduced by Mocanu [5] in 1969.

A function f $(f'(0) \neq 0)$ is said to be close-to-convex in $\mathbb{E}$ if and only if there is a starlike function h (not necessarily normalized) such that

$$\Re\left(\frac{zf'(z)}{h(z)}\right) > 0,\ z \in \mathbb{E}.$$

It is well-known that every close-to-convex function is univalent.

Let ϕ be analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0$ and $\Re\left(\phi'(0)\right) > 0$, then, the function $f \in \mathcal{A}$ is said to be ϕ-like in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > 0,\ z \in \mathbb{E}.$$

This concept was introduced by Brickman [1]. He proved that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is ϕ-like for some ϕ.

Later, Ruscheweyh [12] investigated the following general class of ϕ-like functions:

Let ϕ be analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0, \phi'(0) = 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$, then the function $f \in \mathcal{A}$ is called ϕ-like with respect to a univalent function $q, q(0) = 1$, if

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z),\ z \in \mathbb{E}.$$

In 1999, Silverman [13], defined the class $\mathcal{G}_b$ as

$$\mathcal{G}_b = \left\{ f \in \mathcal{A} : \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b,\ z \in \mathbb{E} \right\}$$

and proved that the functions in the class $\mathcal{G}_b$ are starlike in $\mathbb{E}$. Later on, this class was studied by Obradovič and Tuneski [10] and Tuneski [14].

In fact the results of starlikeness expressed in terms of the quotient of convex and starlike factors were available in literature before the introduction of the class $\mathcal{G}_b$.

In 1989, Obradovič and Owa [9], obtained a sufficient condition for starlikeness of $f \in \mathcal{A}$. They proved the following result.

Theorem 1.1 *If $f \in \mathcal{A}$ satisfies the condition*

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| < K \left| \frac{zf'(z)}{f(z)} \right|,\ z \in \mathbb{E},$$

then $f \in \mathcal{S}^*$, where $K = 1.2849\cdots$.

Later on Nunokawa [7], improved the above result in Theorem 1.1 by proving the following result for more general class.

Theorem 1.2 *If $f \in \mathcal{A}_p$ satisfies the condition*

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| < \left| \frac{zf'(z)}{f(z)} \right| \frac{1}{p}\ log(4e^{p-1}),\ z \in \mathbb{E}$$

then $f \in \mathcal{S}_p^*$.

Nunokawa [8], also proved the following result.

Theorem 1.3 *Let q be analytic in $\mathbb{E}$ with $q(0) = p$ and suppose that*

$$\Re \left(\frac{zq'(z)}{(q(z))^2} \right) < \frac{1}{2p},$$

then $\Re\,(q(z)) > 0$ in $\mathbb{E}$.

Nunokawa [8] and also Muhammet [6], proved the following result.

Theorem 1.4 *Let* $f \in \mathcal{A}_p$ $(f(z) \neq 0)$ *in* $0 < |z| < 1$ *and suppose that*

$$\Re \left(\frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \right) < 1 + \frac{1}{2p},$$

then $f \in \mathcal{S}_p^*$.

In 2003, Muhammet Kamali [6], studied the differential inequality (2) and proved the following result.

Theorem 1.5 *Let* $f \in \mathcal{A}_p$ $(f(z) \neq 0)$ *in* $0 < |z| < 1$ *and suppose that*

$$\Re \left[1 + \frac{z \left(1 + \frac{zf''(z)}{f'(z)} + k\frac{zf'(z)}{f(z)} \right)'}{\left(1 + \frac{zf''(z)}{f'(z)} + k\frac{zf'(z)}{f(z)} \right)^2} \right] < 1 + \frac{1}{2p(k+1)}, \;\; k > 0$$

then $f \in \mathcal{S}_p(k)$.

In this paper, we obtain a first order differential subordination and discuss its applications to univalent and multivalent functions. As an application to multivalent functions, we obtain the sufficient conditions for a function $f \in \mathcal{A}_p$ to be a member of the class $\mathcal{S}_p(k)$. We show that our results generalize and improve certain known results in this direction.

2 Preliminaries

We shall need the following definition and lemmas to prove our results.

Definition 2.1 *A function* $L(z,t), z \in \mathbb{E}$ *and* $t \geq 0$ *is said to be a subordination chain if* $L(.,t)$ *is analytic and univalent in* $\mathbb{E}$ *for all* $t \geq 0$, $L(z,.)$ *is continuously differentiable on* $[0,\infty)$ *for all* $z \in \mathbb{E}$ *and* $L(z,t_1) \prec L(z,t_2)$ *for all* $0 \leq t_1 \leq t_2$.

Lemma 2.1 *([11, p.159]). The function* $L(z,t) : \mathbb{E} \times [0,\infty) \to \mathbb{C}$, *of the form* $L(z,t) = a_1(t)z + \cdots$ *with* $a_1(t) \neq 0$ *for all* $t \geq 0$, *and* $\lim_{t \to \infty} |a_1(t)| = \infty$, *is a subordination chain if and only if* $\Re \left(\frac{z\partial L/\partial z}{\partial L/\partial t} \right) > 0$ *for all* $z \in \mathbb{E}$ *and* $t \geq 0$.

Lemma 2.2 *([4]). Let* F *be analytic in* $\mathbb{E}$ *and let* G *be analytic and univalent in* $\overline{\mathbb{E}}$ *except for points* ζ_0 *such that* $\lim_{z \to \zeta_0} G(z) = \infty$, *with* $F(0) = G(0)$. *If* $F \not\prec G$ *in* $\mathbb{E}$, *then there is a point* $z_0 \in \mathbb{E}$ *and* $\zeta_0 \in \partial\mathbb{E}$ *(boundary of* $\mathbb{E}$*) such that* $F(|z| < |z_0|) \subset G(\mathbb{E})$, $F(z_0) = G(\zeta_0)$ *and* $z_0 F'(z_0) = m\zeta_0 G'(\zeta_0)$ *for some* $m \geq 1$.

3 Main Results

Theorem 3.1 *Let q $(q(z) \neq 0)$ be a univalent function such that either $\frac{zq'(z)}{(q(z))^2}$ is starlike in $\mathbb{E}$ or $\frac{1}{q(z)}$ is convex in $\mathbb{E}$. If an analytic function $P(P(z) \neq 0)$ satisfies the differential subordination*

$$1 + \alpha \frac{zP'(z)}{(P(z))^2} \prec 1 + \alpha \frac{zq'(z)}{(q(z))^2}, \tag{4}$$

where $\alpha > 0$, is a real number, then $P(z) \prec q(z)$ and q is the best dominant.

Proof. Let us define a function

$$h(z) = 1 + \alpha \frac{zq'(z)}{(q(z))^2}, \ z \in \mathbb{E}. \tag{5}$$

Differentiating (5) and simplifying a little, we get

$$\frac{zh'(z)}{Q(z)} = \alpha \left(1 + \frac{zq''(z)}{q'(z)} - 2\frac{zq'(z)}{q(z)}\right) = \alpha \frac{zQ'(z)}{Q(z)}, \ z \in \mathbb{E},$$

where $Q(z) = \frac{zq'(z)}{(q(z))^2}$.

Since Q is starlike and $\alpha > 0$, is a real number, therefore, we obtain

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0, \ z \in \mathbb{E}.$$

Thus, h is close-to-convex and hence univalent in $\mathbb{E}$. The subordination in (4) is, therefore, well-defined in $\mathbb{E}$.

We need to show that $P(z) \prec q(z)$. Suppose to the contrary that $P(z) \not\prec q(z)$ in $\mathbb{E}$. Then by Lemma 2.2, there exist points $z_0 \in \mathbb{E}$ and $\zeta_0 \in \partial\mathbb{E}$ such that $P(z_0) = q(\zeta_0)$ and $z_0 P'(z_0) = m\zeta_0 q'(\zeta_0), \ m \geq 1$. Then

$$1 + \alpha \frac{z_0 P'(z_0)}{(P(z_0))^2} = 1 + \alpha \frac{m\zeta_0 q'(\zeta_0)}{(q(\zeta_0))^2}, \ z \in \mathbb{E}. \tag{6}$$

Consider a function

$$L(z,t) = 1 + \alpha(1+t)\frac{zq'(z)}{(q(z))^2}, \ z \in \mathbb{E}. \tag{7}$$

The function $L(z,t)$ is analytic in $\mathbb{E}$ for all $t \geq 0$ and is continuously differentiable on $[0, \infty)$ for all $z \in \mathbb{E}$. Now,

$$a_1(t) = \left(\frac{\partial L(z,t)}{\partial z}\right)_{(0,t)} = \alpha(1+t)\frac{q'(0)}{(q(0))^2}.$$

As q is univalent in $\mathbb{E}$, so, $q'(0) \neq 0$. Therefore, it follows that $a_1(t) \neq 0$ and $\lim_{t \to \infty} |a_1(t)| = \infty$.

A simple calculation yields

$$z\frac{\partial L/\partial z}{\partial L/\partial t} = (1+t)\frac{zQ'(z)}{Q(z)}, \; z \in \mathbb{E}.$$

In view of the given conditions, we obtain

$$\Re\left(z\frac{\partial L/\partial z}{\partial L/\partial t}\right) > 0, \; z \in \mathbb{E}.$$

Hence, in view of Lemma 2.1, $L(z,t)$ is a subordination chain. Therefore, $L(z,t_1) \prec L(z,t_2)$ for $0 \leq t_1 \leq t_2$.

From (7), we have $L(z,0) = h(z)$, thus we deduce that $L(\zeta_0,t) \notin h(\mathbb{E})$ for $|\zeta_0| = 1$ and $t \geq 0$. In view of (6) and (7), we can write

$$1 + \alpha\frac{z_0P'(z_0)}{(P(z_0))^2} = L(\zeta_0, m-1) \notin h(\mathbb{E}),$$

where $z_0 \in \mathbb{E}$, $|\zeta_0| = 1$ and $m \geq 1$ which is a contradiction to (4). Hence, $P(z) \prec q(z)$. This completes the proof of the theorem.

Letting $\alpha \to \infty$ in Theorem 3.1, we obtain the following result.

Theorem 3.2 *Let q $(q(z) \neq 0)$ be a univalent function such that either $\dfrac{zq'(z)}{(q(z))^2}$ is starlike in $\mathbb{E}$ or $\dfrac{1}{q(z)}$ is convex in $\mathbb{E}$. If an analytic function P $(P(z) \neq 0)$ satisfies the differential subordination*

$$\frac{zP'(z)}{(P(z))^2} \prec \frac{zq'(z)}{(q(z))^2},$$

then $P(z) \prec q(z)$ and q is the best dominant.

4 Applications to Univalent Functions

Let the dominant be $q(z) = \dfrac{1+Az}{1+Bz}$. We observe that q is univalent in $\mathbb{E}$ and $\dfrac{1}{q(z)}$ is convex in $\mathbb{E}$ where $-1 \leq B < A \leq 1$. Thus q satisfies all the conditions of Theorem 3.1 and Theorem 3.2.

On writing $P(z) = \dfrac{zf'(z)}{f(z)}$ and $q(z) = \dfrac{1+Az}{1+Bz}$ in Theorem 3.1, we have the following result.

Theorem 4.1 *Let A and B be real numbers $-1 \leq B < A \leq 1$. If an analytic function $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0, z \in \mathbb{E}$, satisfies the differential subordination*

$$\frac{(1-\alpha)zf'(z)/f(z) + \alpha(1 + zf''(z)/f'(z))}{zf'(z)/f(z)} \prec 1 + \alpha\frac{(A-B)z}{(1+Az)^2},$$

where $\alpha > 0$ is a real number, then $f \in \mathcal{S}^*[A, B]$.

Taking $P(z) = \dfrac{zf'(z)}{\phi(f(z))}$ and $q(z) = \dfrac{1+Az}{1+Bz}$ in Theorem 3.2, we obtain the following result.

Theorem 4.2 *Let A and B be real numbers $-1 \leq B < A \leq 1$. Let $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0, z \in \mathbb{E}$, satisfy the differential subordination*

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/\phi(f(z))} - \frac{(\phi(f(z)))'}{f'(z)} \prec \frac{(A-B)z}{(1+Az)^2},\ z \in \mathbb{E},$$

for some function ϕ, analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0, \phi'(0) = 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$, then $\dfrac{zf'(z)}{\phi(f(z))} \prec \dfrac{1+Az}{1+Bz}$.

By taking $A = 0$ and $B = -1$ in Theorem 4.2, we obtain the following result.

Corollary 4.1 *Let $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0, z \in \mathbb{E}$, satisfy the condition*

$$\left|\frac{1 + zf''(z)/f'(z)}{zf'(z)/\phi(f(z))} - \frac{(\phi(f(z)))'}{f'(z)}\right| < 1,$$

for some function ϕ same as in Theorem 4.2, then $\dfrac{zf'(z)}{\phi(f(z))} \prec \dfrac{1}{1-z}$.

In particular, when $A = 1$ and $B = -1$, Theorem 4.2 reduces to the following result of Gupta et al. [3].

Corollary 4.2 *Let $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0, z \in \mathbb{E}$, satisfy the differential subordination*

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/\phi(f(z))} - \frac{(\phi(f(z)))'}{f'(z)} \prec \frac{2z}{(1+z)^2},\ z \in \mathbb{E},$$

for some function ϕ same as in Theorem 4.2, then $\Re\left(\dfrac{zf'(z)}{\phi(f(z))}\right) > 0$, i.e. f is ϕ-like in $\mathbb{E}$.

When we select $q(z) = \dfrac{p(1+z)}{1-z}$ in Theorem 3.2, we obtain the following result.

Corollary 4.3 *Let P be an analytic function in $\mathbb{E}$ with $P(0) = p$ and suppose that P satisfies the differential subordination*

$$\frac{zP'(z)}{(P(z))^2} \prec \frac{2}{p}\frac{z}{(1+z)^2} = F_1(z),$$

then $\Re\left(P(z)\right) > 0,\ z \in \mathbb{E}$.

Remark 4.1 *We observe that $F_1(z)$ is a conformal mapping of $\mathbb{E}$ with $F_1(0) = 0$ and*

$$F_1(\mathbb{E}) = \mathbb{C} \setminus \left\{w \in \mathbb{C} : \frac{1}{2p} \leq \Re(w) < \infty,\ \Im(w) = 0\right\}.$$

Therefore, the result in Corollary 4.3, improves the result of Nunokawa [8], stated in Theorem 1.3, as is evident from the fact that the region of variability of the operator $\dfrac{zP'(z)}{(P(z))^2}$, is extended substantially. In Figure 4.1, we have plotted the graph of $F_1(\mathbb{E})$ for $p = 2$. Different colour on the right hand side of the plot depicts the extension of the region of variability of the functional $\dfrac{zP'(z)}{(P(z))^2}$ in comparison to Theorem 1.3.

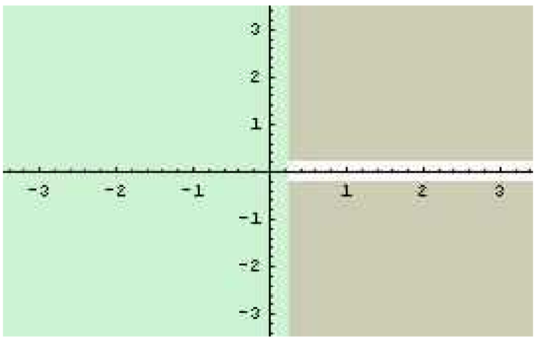

Figure 4.1 ($p = 2$)

When we select $q(z) = \dfrac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 3.2, we obtain the following result of Tuneski [14].

Corollary 4.4 *Let A and B be real numbers $-1 \leq B < A \leq 1$. If an analytic function P $(P(z) \neq 0)$ satisfies the differential subordination*

$$\frac{zP'(z)}{(P(z))^2} \prec \frac{(A-B)z}{(1+Az)^2},\ z \in \mathbb{E},$$

then $P(z) \prec \dfrac{1+Az}{1+Bz}$.

On writing $P(z) = \dfrac{zf'(z)}{f(z)}$ and $q(z) = \dfrac{1+Az}{1+Bz},\ -1 \leq B < A \leq 1$ in Theorem 3.2, we have the following result of Tuneski [14].

Corollary 4.5 *Let A and B be real numbers $-1 \leq B < A \leq 1$. If an analytic function $f \in \mathcal{A}, \dfrac{zf'(z)}{f(z)} \neq 0, z \in \mathbb{E}$, satisfies the differential subordination*

$$\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \frac{(A-B)z}{(1+Az)^2},$$

then $f \in \mathcal{S}^*[A,B]$.

On writing $P(z) = \dfrac{zf'(z)}{f(z)}$ and $q(z) = \dfrac{1}{1-z}$, in Theorem 3.2, we have the following result.

Corollary 4.6 *Let $f \in \mathcal{A}, \dfrac{zf'(z)}{f(z)} \neq 0, z \in \mathbb{E}$, satisfy the condition*

$$\left| \frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < 1,$$

then $f \in \mathcal{S}^*(1/2)$.

On writing $P(z) = \dfrac{zf'(z)}{f(z)}$ and $q(z) = 1 + Az,\ 0 < A \leq 1$ in Theorem 3.2, we have the following result of Gupta et al. [3].

Corollary 4.7 *Let $f \in \mathcal{A}, \dfrac{zf'(z)}{f(z)} \neq 0, z \in \mathbb{E}$, satisfy*

$$\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \frac{Az}{(1+Az)^2},\ z \in \mathbb{E},\ 0 < A \leq 1,$$

then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < A.$$

On writing $P(z) = \dfrac{zf'(z)}{f(z)}$ and $q(z) = \dfrac{1+z}{1-z}$ in Theorem 3.2, we have the following result of Obradovič and Tuneski [10].

Corollary 4.8 *If an analytic function* $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0, z \in \mathbb{E}$, *satisfies the differential subordination*

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \frac{2z}{(1+z)^2},$$

then $f \in \mathcal{S}^*$.

5 Applications to Multivalent Functions

Setting $P(z) = \dfrac{1}{p(k+1)}\left(1 + \dfrac{zf''(z)}{f'(z)} + k\dfrac{zf'(z)}{f(z)}\right)$, where $k+1>0$ and $q(z) = \dfrac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 3.1, we obtain the following result.

Theorem 5.1 *Let* $\alpha,\ k,\ A$ *and* B *be real numbers with* $\alpha > 0,\ k+1>0$ *and* $-1 \leq B < A \leq 1$. *If an analytic function* $f \in \mathcal{A}_p$, $\dfrac{1}{p(k+1)}\left(1 + \dfrac{zf''(z)}{f'(z)} + k\dfrac{zf'(z)}{f(z)}\right) \neq 0,\ z \in \mathbb{E}$, *satisfies the differential subordination*

$$1 + \alpha p(k+1)\frac{z\left(1 + \frac{zf''(z)}{f'(z)} + k\frac{zf'(z)}{f(z)}\right)'}{\left(1 + \frac{zf''(z)}{f'(z)} + k\frac{zf'(z)}{f(z)}\right)^2} \prec 1 + \alpha\frac{(A-B)z}{(1+Az)^2},\ z \in \mathbb{E},$$

then $\dfrac{1}{p(k+1)}\left(1 + \dfrac{zf''(z)}{f'(z)} + k\dfrac{zf'(z)}{f(z)}\right) \prec \dfrac{1+Az}{1+Bz}$.

On writing $P(z) = \dfrac{1}{p}\dfrac{zf'(z)}{f(z)}$, $f \in \mathcal{A}_p$ and $q(z) = \dfrac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in Theorem 3.2, we obtain the following result.

Theorem 5.2 *Let* A *and* B *be real numbers* $-1 \leq B < A \leq 1$. *If an analytic function* $f \in \mathcal{A}_p$, $\dfrac{1}{p}\dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, *satisfies the differential subordination*

$$\frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \prec 1 + \frac{1}{p}\frac{(A-B)z}{(1+Az)^2},\ z \in \mathbb{E},$$

then $\dfrac{1}{p}\ \dfrac{zf'(z)}{f(z)} \prec \dfrac{1+Az}{1+Bz}$.

By taking $A = 1,\ B = -1$ and $\alpha = \dfrac{1}{p(k+1)}$ in Theorem 5.1, we have the following result.

Corollary 5.1 *If an analytic function* $f \in \mathcal{A}_p,\ \dfrac{1}{p(k+1)}\left(1 + \dfrac{zf''(z)}{f'(z)} + k\dfrac{zf'(z)}{f(z)}\right) \neq 0,\ z \in \mathbb{E}$, *satisfies the differential subordination*

$$1 + \frac{z\left(1 + \frac{zf''(z)}{f'(z)} + k\frac{zf'(z)}{f(z)}\right)'}{\left(1 + \frac{zf''(z)}{f'(z)} + k\frac{zf'(z)}{f(z)}\right)^2} \prec 1 + \frac{1}{p(k+1)}\,\frac{2z}{(1+z)^2} = F_2(z),\ k+1 > 0,$$

then $\dfrac{1}{p(k+1)}\left(1 + \dfrac{zf''(z)}{f'(z)} + k\dfrac{zf'(z)}{f(z)}\right) \prec \dfrac{1+z}{1-z},\ z \in \mathbb{E}$ *i.e.* $f \in S_p(k)$.

Remark 5.1 *It can easily be seen that the function* $F_2(z)$ *is a conformal mapping of the unit disk* $\mathbb{E}$ *with* $F_2(0) = 1$ *and*

$$F_2(\mathbb{E}) = \mathbb{C} \setminus \left\{ w \in \mathbb{C} : 1 + \frac{1}{2p(k+1)} \leq \Re(w) < \infty,\ \Im(w) = 0 \right\}.$$

Therefore, the result in Corollary 5.1, extends the main result of Muhammet Kamali [6], stated in Theorem 1.5 of Section 1, extensively. In Figure 5.1, the fact is elaborated by plotting the region $F_2(\mathbb{E})$ *for* $p = 2$.

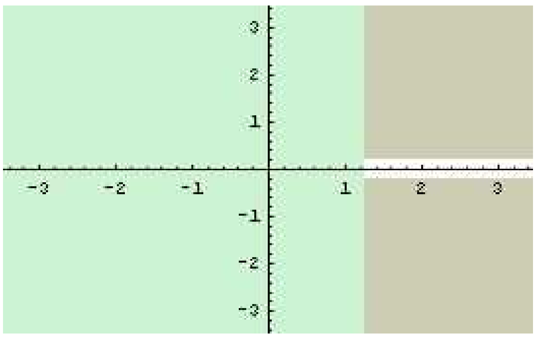

Figure 5.1 $(p = 2,\ k = 0)$

By taking $A = 0$ and $B = -1$ in Theorem 5.2 we obtain the following result.

Corollary 5.2 *If an analytic function* $f \in \mathcal{A}_p$, $\dfrac{1}{p}\dfrac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, *satisfies the inequality*

$$\left| \frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1 \right| < \frac{1}{p},$$

then $\Re\left(\dfrac{zf'(z)}{f(z)}\right) > \dfrac{p}{2}$.

By taking $A = 1$ and $B = 0$ in Theorem 5.2 we obtain the following result.

Corollary 5.3 *If an analytic function* $f \in \mathcal{A}_p$, $\dfrac{1}{p}\dfrac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, *satisfies the differential subordination*

$$\frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \prec 1 + \frac{1}{p}\,\frac{z}{(1+z)^2},$$

then $\left| \dfrac{1}{p}\,\dfrac{zf'(z)}{f(z)} - 1 \right| < 1$.

By taking $A = 1$ and $B = -1$ in Theorem 5.2, we obtain the following result.

Corollary 5.4 *If an analytic function* $f \in \mathcal{A}_p$, $\dfrac{1}{p}\dfrac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, *satisfies the differential subordination*

$$\frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \prec 1 + \frac{1}{p}\,\frac{2z}{(1+z)^2} = F_3(z),$$

then $f \in S_p^*$.

Remark 5.2 *It can easily be verified that* $F_3(z)$ *is a conformal mapping of* $\mathbb{E}$ *with* $F_3(0) = 1$ *and*

$$F_3(\mathbb{E}) = \mathbb{C} \setminus \left\{ w \in \mathbb{C} : 1 + \frac{1}{2p} \leq \Re(w) < \infty,\ \Im(w) = 0 \right\}.$$

The result in Corollary 5.4, improves the results of Obradovič and Owa [9] and Nunakawa [7], stated, respectively, in Theorem 1.1 and Theorem 1.2 of Section 1. It also extends the result of Nunokawa [8] and Muhammet [6], stated in Theorem 1.4 of Section 1. Figure 5.1, also represents $F_3(\mathbb{E})$ *for* $p = 2$. *So, it justifies our above claims.*

6 Open Problem

Some of the results obtained in this paper (e.g. Corollary 4.6, Corollary 5.2) may be extended further.

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