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A Class of Multivalent Functions Involving a Generalized Linear Operator and Subordination

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Abstract

In this manuscript, a class of multivalent functions defined in terms of a linear multiplier operator containing the generalized Komatu integral operator is introduced and investigated. The main results of inclusion relation, integral preserving property, argument estimate and subordination property are proved by making use of subordination formulas between analytic functions.

Keywords: *Multivalent functions, Komatu integral operator, multiplier operator, subordination.*
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1 Introduction

Let $\mathcal{A}(p)$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in \mathbb{N}) \tag{1}$$

that are analytic and $p-$valent in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C}, |\, z \,|< 1\}$.

The generalized Komatu integral operator $\mathcal{K}_{c,p}^{\delta} : \mathcal{A}(p) \to \mathcal{A}(p)$ is defined for $\delta > 0$ and $c > -p$ as (see, e.g. [7,14,15])

$$\mathcal{K}_{c,p}^{\delta} f(z) = \frac{(c+p)^{\delta}}{\Gamma(\delta) z^{c}} \int_{0}^{z} t^{c-1} \left(\log \frac{z}{t}\right)^{\delta-1} f(t)dt \qquad (2)$$

and $\mathcal{K}_{c,p}^{0} f(z) = f(z)$.
For $f(z) \in \mathcal{A}(p)$, it can be easily verified that

$$\mathcal{K}_{c,p}^{\delta} f(z) = z^{p} + \sum_{k=1}^{\infty} \left(\frac{c+p}{c+p+k}\right)^{\delta} a_{p+k} z^{p+k}. \qquad (3)$$

Now, in terms of $\mathcal{K}_{c,p}^{\delta}$, we introduce the linear multiplier operator $\mathcal{J}_{c,p,\lambda}^{m,\delta} : \mathcal{A}(p) \to \mathcal{A}(p)$ as follows:

$$\begin{aligned} \mathcal{J}_{c,p,\lambda}^{0,0} f(z) &= f(z) \\ \mathcal{J}_{c,p,\lambda}^{1,\delta} f(z) &= (1-\lambda)\mathcal{K}_{c,p}^{\delta} f(z) + \frac{\lambda z}{p}(\mathcal{K}_{c,p}^{\delta} f(z))' = \mathcal{J}_{c,p,\lambda}^{\delta} f(z) \\ \mathcal{J}_{c,p,\lambda}^{2,\delta} f(z) &= \mathcal{J}_{c,p,\lambda}^{\delta}(\mathcal{J}_{c,p,\lambda}^{1,\delta} f(z)) \\ &\;\;. \\ &\;\;. \\ &\;\;. \\ \mathcal{J}_{c,p,\lambda}^{m,\delta} f(z) &= \mathcal{J}_{c,p,\lambda}^{\delta}(\mathcal{J}_{c,p,\lambda}^{m-1,\delta} f(z)), \end{aligned} \qquad (4)$$

for $\delta > 0$, $c > -p$, $\lambda \geq 0$ and $m \in \mathbb{N}$.

If $f \in \mathcal{A}(p)$ is given by (1), then making use of (3) and (4) we conclude that

$$\mathcal{J}_{c,p,\lambda}^{m,\delta} f(z) = z^{p} + \sum_{k=1}^{\infty} B_{k,m}(c,p,\lambda,\delta) a_{p+k} z^{p+k}, \qquad (5)$$

where

$$B_{k,m}(c,p,\lambda,\delta) = \left[\left(\frac{c+p}{c+p+k}\right)^{\delta} (1 + \frac{\lambda k}{p})\right]^{m}. \qquad (6)$$

Remark 1:
(i) $\mathcal{J}_{c,p,0}^{1,\delta} \equiv \mathcal{K}_{c,p}^{\delta}$ which is the generalized Komatu operator [7,14,15].
(ii) $\mathcal{J}_{c,1,0}^{1,\delta} \equiv \mathcal{P}_{c}^{\delta}$ which is the integral operator studied by Komatu [8] and Raina and Bapna [11].

(iii) $\mathcal{J}^{1,\delta}_{1,p,0} \equiv \mathcal{I}^{\delta}_{p}$ which is the integral operator studied by Shams *et al.* [16] and Ebadian *et al.* [5].
(iv) $\mathcal{J}^{1,1}_{c,1,0} \equiv \mathcal{L}_c$ which is the Bernardi-Libra-Livingston integral operator [2].
(v) $\mathcal{J}^{1,\delta}_{1,1,0} \equiv \mathcal{I}^{\delta}$ which is the integral operator studied by Ebadian and Najafzadeh [4].
(vi) $\mathcal{J}^{m,0}_{c,1,\lambda} \equiv \mathcal{D}^{m}_{\lambda}$ which is the generalized Sălăgean operator studied by Al-Oboudi [1].
(vii) $\mathcal{J}^{m,0}_{c,1,0} \equiv \mathcal{D}^{m}$ which is the Sălăgean operator [13].

It can be easily verified for $\delta \geq \frac{1}{m}$, $m \in \mathbb{N}$ that

$$z\left(\mathcal{J}^{m,\delta}_{c,p,\lambda} f(z)\right)' = (c+p)\mathcal{J}^{m,\delta-\frac{1}{m}}_{c,p,\lambda} f(z) - c\mathcal{J}^{m,\delta}_{c,p,\lambda} f(z). \tag{7}$$

Let P denote the class of functions $h(z)$ of the form

$$h(z) = 1 + \sum_{k=1}^{\infty} h_k z^k \qquad (z \in \mathcal{U}),$$

which are analytic and convex in $\mathcal{U}$ and satisfy the condition

$$\mathrm{Re}\,(h(z)) > 0 \qquad (z \in \mathcal{U}).$$

For two functions f and g analytic in $\mathcal{U}$, we say that $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$ written as $f(z) \prec g(z)$ if there exists a Schwarz function $\omega(z)$, analytic in $\mathcal{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$, $z \in \mathcal{U}$.

By making use of the multiplier operator $\mathcal{J}^{m,\delta}_{c,p,\lambda}$ and the above mentioned principle of subordination, we define and investigate the following subclass of $p-$valent functions.

Definition 1. *A function $f(z) \in \mathcal{A}(p)$ is in the class $\mathcal{S}(m, \delta, c, p, \lambda, \alpha; h)$ if it satisfies the differential subordination*

$$\frac{1}{p-\alpha}\left(\frac{z\left(\mathcal{J}^{m,\delta}_{c,p,\lambda} f(z)\right)'}{\mathcal{J}^{m,\delta}_{c,p,\lambda} f(z)} - \alpha\right) \prec h(z) \qquad (z \in \mathcal{U}), \tag{8}$$

for $p, m \in \mathbb{N}$, $c > -p$, $\delta > 0$, $\lambda > 0$, $0 \leq \alpha < p$ and $h \in P$.

Remark 2: The class $\mathcal{S}(m, \delta, c, p, \lambda, \alpha; h)$ contains several well-known as well as new classes of analytic functions. For example:
(i) $\mathcal{S}(1, \delta, 1, p, 0, \alpha; h) \equiv \mathcal{S}^{\delta}_{p}(\alpha; h)$ which is the class studied by Ebadian *et al.*

[5].
(ii) $\mathcal{S}(1,0,1,p,0,\alpha;\frac{1+Az}{1+Bz}) \equiv \mathcal{S}_p^*(\alpha;A,B)$ which is the class studied by Cho *et al.* [3].
(iii) $\mathcal{S}(1,0,1,p,0,\alpha;\frac{1+z}{1-z}) \equiv \mathcal{S}_p^*(\alpha)$,where $\mathcal{S}_p^*(\alpha)$ consists of all $p-$valent starlike functions of order α.
(iv) $\mathcal{S}(1,0,1,p,1,\alpha;\frac{1+z}{1-z}) \equiv \mathcal{C}_p^*(\alpha)$,where $\mathcal{C}_p^*(\alpha)$ consists of all $p-$valent convex functions of order α.

The purpose of this investigation is to establish results as inclusion relation, subordination properties, integral preserving property and argument estimate for the class $\mathcal{S}(m,\delta,c,p,\lambda,\alpha;h)$ by making use of the subordination relations included in the Lemmas of the following section.

2 Required Results

The following Lemmas are needed to prove our main results.

Lemma A. (cf,e.g. Eenigenburg *et al.* [6])
Let $\zeta,\nu\in\mathbb{C}$ and suppose that $p(z)$ is convex and univalent in $\mathcal{U}$ with $p(0)=1$ and $\mathrm{Re}(\zeta p(z)+\nu)>0$, $(z\in\mathcal{U})$.If $q(z)$ is analytic in $\mathcal{U}$ with $q(0)=1$, then the subordination
$$q(z)+\frac{zq'(z)}{\zeta q(z)+\nu}\prec p(z)$$
implies $q(z)\prec p(z)$, $\quad (z\in\mathcal{U})$.

Lemma B. (cf,e.g. Miller and Mocanu [9])
Let $h(z)$ be convex and univalent in $\mathcal{U}$ and $g(z)$ be analytic in $\mathcal{U}$ with $\mathrm{Re}(g(z))\geq 0 \quad (z\in\mathcal{U})$. If $\varphi(z)$ is analytic in $\mathcal{U}$ with $\varphi(0)=h(0)$, then the subordination $\varphi(z)+g(z)\varphi'(z)\prec h(z) \quad (z\in\mathcal{U})$ implies that $\varphi(z)\prec h(z) \quad (z\in\mathcal{U})$.

Lemma C. (cf,e.g. Ebadian *et al.* [5])
Let q be analytic in $\mathcal{U}$ with $q(0)=1$ and $q(z)\neq 0$ for all $z\in\mathcal{U}$. If there exists two points $z_1,z_2\in\mathcal{U}$ such that
$$-\frac{\pi}{2}\alpha_1=\arg(q(z_1))<\arg(q(z))<\arg(q(z_2))=\frac{\pi}{2}\alpha_2$$
for some α_1 and α_2 $\quad(\alpha_1,\alpha_2>0)$ and for all z $\quad(|z|<|z_1|=|z_2|)$, then
$$\frac{z_1q'(z_1)}{q(z_1)}=-i\left(\frac{\alpha_1+\alpha_2}{2}\right)l \quad \textit{and} \quad \frac{z_2q'(z_2)}{q(z_2)}=i\left(\frac{\alpha_1+\alpha_2}{2}\right)l$$
where
$$l\geq\frac{1-|b|}{1+|b|} \quad \text{and} \quad b=i\tan\frac{\pi}{4}\left(\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\right).$$

Lemma D. (cf,e.g. Robertson [12])
The function

$$(1-z)^{\gamma} \equiv \exp\left(\gamma \log(1-z)\right) \quad (\gamma \neq 0)$$

is univalent if and only if γ *is either in the closed disk* $|\gamma - 1| \leq 1$ *or in the closed disk* $|\gamma + 1| \leq 1$.

Lemma E. (cf,e.g. Miller and Mocanu [10])
Let $q(z)$ *be univalent in* $\mathcal{U}$ *and* $\theta(\omega)$ *and* $\varphi(\omega)$ *be analytic in a domain* $\mathcal{D}$ *containing* $q(\mathcal{U})$ *with* $\varphi(\omega) \neq 0$ *when* $\omega \in q(\mathcal{U})$. *Set*

$$Q(z) = zq'(z)\varphi(q(z)), \qquad h(z) = \theta(q(z)) + Q(z)$$

and suppose that

1. $Q(z)$ *is starlike(univalent) in* $\mathcal{U}$*;*

2. $\text{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \text{Re}\left(\frac{\theta'(q(z))}{\varphi(q(z))} + \frac{zQ'(z)}{Q(z)}\right) > 0 \quad (z \in \mathcal{U})$

If $p(z)$ *is analytic in* $\mathcal{U}$ *with* $p(0) = q(0)$ *and* $p(\mathcal{U}) \subset \mathcal{D}$, *and*

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)) = h(z),$$

then $p(z) \prec q(z)$, *and* q *is the best dominant.*

3 Main Results

We begin our main results by proving the following theorem in which inclusion relationship for the class $\mathcal{S}(m,\delta,c,p,\lambda,\alpha;h)$ is obtained.

Theorem 1. *Let* $f\in\mathcal{S}(m,\delta-\frac{1}{m},c,p,\lambda,\alpha;h)$ *with* $\mathrm{Re}((p-\alpha)h(z)+c+p)>0$. *Then*

$$\mathcal{S}(m,\delta-\frac{1}{m},c,p,\lambda,\alpha;h)\subset\mathcal{S}(m,\delta,c,p,\lambda,\alpha;h).$$

Proof. Let $f\in\mathcal{S}(m,\delta-\frac{1}{m},c,p,\lambda,\alpha;h)$ and assume that

$$q(z)=\frac{1}{p-\alpha}\left(\frac{z\left(\mathcal{J}^{m,\delta}_{c,p,\lambda}f(z)\right)'}{\mathcal{J}^{m,\delta}_{c,p,\lambda}f(z)}-\alpha\right). \tag{9}$$

Then $q(z)$ is analytic in $\mathcal{U}$ with $q(0)=1$.
Making use of (7), then (9) implies

$$(c+p)\left(\frac{\mathcal{J}^{m,\delta-\frac{1}{m}}_{c,p,\lambda}f(z)}{\mathcal{J}^{m,\delta}_{c,p,\lambda}f(z)}\right)=(p-\alpha)q(z)+c+\alpha. \tag{10}$$

Now, by logarithmic differentiation of (9) and applying (7), we get

$$\frac{1}{p-\alpha}\left(\frac{z\left(\mathcal{J}^{m,\delta-\frac{1}{m}}_{c,p,\lambda}f(z)\right)'}{\mathcal{J}^{m,\delta-\frac{1}{m}}_{c,p,\lambda}f(z)}-\alpha\right)=q(z)+\frac{zq'(z)}{(p-\alpha)q(z)+c+\alpha}\prec h(z). \tag{11}$$

Since $\mathrm{Re}((p-\alpha)h(z)+c+p)>0$ by hypothesis above, then applying Lemma A to (11) yields

$$q(z)\prec h(z)\qquad(z\in\mathcal{U})$$

which implies $f\in\mathcal{S}(m,\delta,c,p,\lambda,\alpha;h)$. □

Theorem 2. *Let* $1<\beta<2$ *and* $\gamma\neq0$ *be a real number satisfying either* $|2\gamma(\beta-1)(p+c)-1|\leq1$ *or* $|2\gamma(\beta-1)(p+c)+1|\leq1$. *If* $f\in\mathcal{A}(p)$ *satisfies the condition*

$$\mathrm{Re}\left(1+\frac{\mathcal{J}^{m,\delta-\frac{1}{m}}_{c,p,\lambda}f(z)}{\mathcal{J}^{m,\delta}_{c,p,\lambda}f(z)}\right)>2-\beta+\frac{c-1}{c+p}\,. \tag{12}$$

Then

$$\left(z\mathcal{J}^{m,\delta}_{c,p,\lambda}f(z)\right)^{\gamma}\prec q(z)=\frac{1}{(1-z)^{2\gamma(\beta-1)(c+p)}}\,. \tag{13}$$

Proof. Let

$$k(z) = \left(z\mathcal{J}^{m,\delta}_{c,p,\lambda} f(z)\right)^{\gamma}.$$

Hence by logarithmic differentiation and using (7), we obtain

$$\frac{zk'(z)}{k(z)} = \gamma(1-c)+\gamma(c+p)\frac{\mathcal{J}^{m,\delta-\frac{1}{m}}_{c,p,\lambda} f(z)}{\mathcal{J}^{m,\delta}_{c,p,\lambda} f(z)}. \tag{14}$$

So, by virtue of the condition (12), we get

$$1+\frac{zk'(z)}{\gamma(c+p)k(z)} \prec \frac{1+(2\gamma-3)z}{1-z}. \tag{15}$$

Now, suppose that

$$\theta(\omega) = 1, \quad q(z) = \frac{1}{(1-z)^{2\gamma(\beta-1)(c+p)}} \quad \text{and} \quad \varphi(\omega) = \frac{1}{\gamma\omega(c+p)},$$

then making use of Lemma D, $q(z)$ is univalent in $\mathcal{U}$. It follows that

$$Q(z) = zq'(z)\varphi(q(z)) = \frac{2(\beta-1)z}{1-z}$$

and

$$\theta(q(z)) + Q(z) = \frac{1+(2\beta-3)z}{1-z} = h(z).$$

If the domain $\mathcal{D}$ is defined by

$$q(\mathcal{U}) = \left\{\omega : \left|\omega^{\frac{1}{\eta}} - 1\right| < \left|\omega^{\frac{1}{\eta}}\right|,\ \eta = 2\gamma(\beta-1)(c+p)\right\} \subset \mathcal{D},$$

then it is easily verified that the conditions of Lemma E are satisfied. Hence $k(z) \prec q(z)$. □

Theorem 3. *Let* $f \in \mathcal{S}(m,\delta,c,p,\lambda,\alpha;h)$ *with* $\mathrm{Re}((p-\alpha)h(z)+c+\alpha) > 0$ and $c > -p$. *Then the integral operator* $\mathcal{K}$ *defined by*

$$\mathcal{K}(z) \equiv \mathcal{K}^{1}_{c,p} f(z) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt \tag{16}$$

belongs to the class $\mathcal{S}(m,\delta,c,p,\lambda,\alpha;h)$.

Proof. From (16), it can be easily verified that

$$z\left(\mathcal{J}^{m,\delta}_{c,p,\lambda}\mathcal{K}(z)\right)' + c\mathcal{J}^{m,\delta}_{c,p,\lambda}\mathcal{K}(z) = (c+p)\mathcal{J}^{m,\delta}_{c,p,\lambda} f(z). \tag{17}$$

Assume that

$$Q(z)=\frac{1}{p-\alpha}\left(\frac{z\left(\mathcal{J}^{m,\delta}_{c,p,\lambda}\mathcal{K}(z)\right)'}{\mathcal{J}^{m,\delta}_{c,p,\lambda}\mathcal{K}(z)}-\alpha\right), \tag{18}$$

$Q(z)$ is analytic function in $\mathcal{U}$ with $Q(0)=0$. It follows from (17) and (18) that

$$(c+p)\frac{\mathcal{J}^{m,\delta}_{c,p,\lambda}f(z)}{\mathcal{J}^{m,\delta}_{c,p,\lambda}\mathcal{K}(z)}=c+\alpha+(p-\alpha)Q(z). \tag{19}$$

By logarithmic differentiation of (19) and making use of (18), it follows that

$$Q(z)+\frac{zQ'(z)}{c+\alpha+(p-\alpha)Q(z)}=\frac{1}{p-\alpha}\left(\frac{z\left(\mathcal{J}^{m,\delta}_{c,p,\lambda}f(z)\right)'}{\mathcal{J}^{m,\delta}_{c,p,\lambda}f(z)}-\alpha\right)\prec h(z). \tag{20}$$

Since $\mathrm{Re}((p-\alpha)h(z)+c+\alpha)>0 \quad (z\in\mathcal{U})$, then by Lemma A, we get

$$Q(z)\prec h(z)$$

which implies $\mathcal{K}(z)\in\mathcal{S}(m,\delta,c,p,\lambda,\alpha;h)$. □

Theorem 4. *Let* $f\in\mathcal{A}(p),\ 0<\delta_1,\delta_2\le 1$ *and* $0\le\alpha\le p$. *If*

$$-\frac{\pi}{2}\delta_1<\arg\left(\frac{z\left(\mathcal{J}^{m,\delta-\frac{1}{m}}_{c,p,\lambda}f(z)\right)'}{\mathcal{J}^{m,\delta-\frac{1}{m}}_{c,p,\lambda}g(z)}-\alpha\right)<\frac{\pi}{2}\delta_2$$

for some $g\in\mathcal{S}(m,\delta-1,c,p,\lambda,\alpha;\frac{1+Az}{1+Bz}),\ (-1\le B<A\le 1)$, *then*

$$-\frac{\pi}{2}\alpha_1<\arg\left(\frac{z\left(\mathcal{J}^{m,\delta}_{c,p,\lambda}f(z)\right)'}{\mathcal{J}^{m,\delta}_{c,p,\lambda}g(z)}-\alpha\right)<\frac{\pi}{2}\alpha_2$$

where α_1 *and* α_2 $(0<\alpha_1,\alpha_2\le 1\ )$ *are the solutions of the equations*

$$\delta_1=\begin{cases}\alpha_1+\frac{2}{\pi}\tan^{-1}\left(\frac{(1-|b|)(\alpha_1+\alpha_2)\cos\frac{\pi}{2}t}{2(1+|b|)\left(\frac{(p-\alpha)(1+A)}{1+B}+\alpha+c\right)+(1-|b|)(\alpha_1+\alpha_2)\sin\frac{\pi}{2}t}\right), & B\neq -1\\ \alpha_1, & B=-1\end{cases}$$

and

$$\delta_2=\begin{cases}\alpha_2+\frac{2}{\pi}\tan^{-1}\left(\frac{(1-|b|)(\alpha_1+\alpha_2)\cos\frac{\pi}{2}t}{2(1+|b|)\left(\frac{(p-\alpha)(1+A)}{1+B}+\alpha+c\right)+(1-|b|)(\alpha_1+\alpha_2)\sin\frac{\pi}{2}t}\right), & B\neq -1\\ \alpha_2, & B=-1\end{cases}$$

with $b = i\tan\frac{\pi}{4}\left(\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\right)$ *and* $t = \frac{2}{\pi}\sin^{-1}\left(\frac{(p-\alpha)(A-B)}{(p-\alpha)(1-AB)+(\alpha+c)(1-B^2)}\right)$.

Proof. Assume that

$$q(z) = \frac{1}{p-\gamma}\left(\frac{z\left(\mathcal{J}_{c,p,\lambda}^{m,\delta}f(z)\right)'}{\mathcal{J}_{c,p,\lambda}^{m,\delta}g(z)} - \gamma\right), \tag{21}$$

with $0 \le \gamma < p$ and $g \in \mathcal{S}(m, \delta-1, c, p, \lambda, \alpha; \frac{1+Az}{1+Bz})$, $(-1 \le B < A \le 1)$. Then $q(z)$ is analytic in $\mathcal{U}$ with $q(0) = 1$. So, by virtue of (7) and (21), we get

$$[(p-\gamma)q(z)+\gamma]\,\mathcal{J}_{c,p,\lambda}^{m,\delta}g(z) = (c+p)\mathcal{J}_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z) - c\mathcal{J}_{c,p,\lambda}^{m,\delta}f(z), \tag{22}$$

differentiating both sides of (22) yields

$$\begin{aligned} & (p-\gamma)zq'(z)\mathcal{J}_{c,p,\lambda}^{m,\delta}g(z) + [(p-\gamma)q(z)+\gamma]\,z\left(\mathcal{J}_{c,p,\lambda}^{m,\delta}g(z)\right)' \\ = \; & (c+p)z\left(\mathcal{J}_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z)\right)' - cz\left(\mathcal{J}_{c,p,\lambda}^{m,\delta}f(z)\right)'. \end{aligned} \tag{23}$$

Since $g \in \mathcal{S}(m, \delta-1, c, p, \lambda, \alpha; \frac{1+Az}{1+Bz})$, then by Theorem 1, we have $g \in \mathcal{S}(m, \delta, c, p, \lambda, \alpha; \frac{1+Az}{1+Bz})$. Let

$$u(z) = \frac{1}{p-\alpha}\left(\frac{z\left(\mathcal{J}_{c,p,\lambda}^{m,\delta}g(z)\right)'}{\mathcal{J}_{c,p,\lambda}^{m,\delta}g(z)} - \alpha\right), \tag{24}$$

then making use of (7), we easily obtain

$$\frac{\mathcal{J}_{c,p,\lambda}^{m,\delta}g(z)}{\mathcal{J}_{c,p,\lambda}^{m,\delta-\frac{1}{m}}g(z)} = \frac{c+p}{(p-\alpha)u(z)+\alpha+c}, \tag{25}$$

and from (21), (24) and (25), we get

$$\frac{1}{p-\gamma}\left(\frac{z\left(\mathcal{J}_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z)\right)'}{\mathcal{J}_{c,p,\lambda}^{m,\delta-\frac{1}{m}}g(z)} - \gamma\right) = q(z)\left[1 + \frac{zq'(z)}{[(p-\alpha)u(z)+\alpha+c]q(z)}\right]. \tag{26}$$

Now, since

$$u(z) \prec \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1)$$

then, we easily obtain

$$\left|u(z) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \quad (z \in \mathcal{U}; B \ne -1) \tag{27}$$

and

$$\operatorname{Re}(u(z)) > \frac{1-A}{2} \qquad (z \in \mathcal{U}; B = -1). \tag{28}$$

(27) and (28) can be rewritten as

$$\left| (p-\alpha)u(z) + \alpha + c - \frac{(\alpha+c)(1-B^2) + (p-\alpha)(1-AB)}{1-B^2} \right| < \frac{(p-\alpha)(A-B)}{1-B^2}$$

$$(B \neq -1)$$

and

$$\operatorname{Re}\left((p-\alpha)u(z) + \alpha + c\right) > \frac{(1-A)(p-\alpha)}{2} + \alpha + c \qquad (B = -1).$$

Setting $$(p-\alpha)u(z)+\alpha+c = r\exp\left(i\frac{\pi}{2}\theta\right),$$

where $$-\rho < \theta < \rho; \quad \rho = \frac{(p-\alpha)(A-B)}{(\alpha+c)(1-B^2) + (p-\alpha)(1-AB)} \qquad (B \neq -1)$$

and $-1 < \theta < 1 \qquad (B = -1)$,

then

$$\frac{(p-\alpha)(1-A)}{1-B} + \alpha + c < r < \frac{(p-\alpha)(1+A)}{1-B} + \alpha + c \qquad (\ B = -1)$$

and

$$\frac{(p-\alpha)(1-A)}{2} + \alpha + c < r \qquad (B \neq -1).$$

Since $q(z)$ is analytic in $\mathcal{U}$ with $q(0) = 1$, then by applying Lemma B to (26) yields $q(z) \prec h(z)$.

Now, assume that

$$Q(z) = \frac{1}{p-\gamma}\left(\frac{z\left(\mathcal{J}_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z)\right)'}{\mathcal{J}_{c,p,\lambda}^{m,\delta-\frac{1}{m}} g(z)} - \gamma \right) \qquad (0 \leq \gamma < p) \ , \tag{29}$$

then by means of (26) and (29), we have

$$\arg(Q(z)) = \arg(q(z)) + \arg\left(1 + \frac{zq'(z)}{[(p-\alpha)u(z)+\alpha+c]q(z)}\right).$$

Suppose there exist $z_1, z_2 \in \mathcal{U}$ such that

$$-\frac{\pi}{2}\alpha_1 = \arg(q(z_1)) < \arg(q(z)) < \arg(q(z_2)) = \frac{\pi}{2}\alpha_2.$$

Thus by Lemma C, we have

$$\frac{z_1 q'(z_1)}{q(z_1)} = -i\left(\frac{\alpha_1+\alpha_2}{2}\right) l \quad \text{and} \quad \frac{z_2 q'(z_2)}{q(z_2)} = i\left(\frac{\alpha_1+\alpha_2}{2}\right) l,$$

where $$l \geq \frac{1-|b|}{1+|b|} \quad \text{and} \quad b = i\tan\frac{\pi}{4}\left(\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\right).$$

We have the following two cases:

(i) When $B \neq -1$,

$$\begin{aligned}
\arg(Q(z_1) &= -\tfrac{\pi}{2}\alpha_1 + \arg\left(1 - il\left(\tfrac{\alpha_1+\alpha_2}{2}\right)\tfrac{e^{-i\frac{\pi}{2}\theta}}{r}\right)\\
&= -\tfrac{\pi}{2}\alpha_1 + \arg\left(1 - \tfrac{l}{r}\left(\tfrac{\alpha_1+\alpha_2}{2}\right)\cos\tfrac{\pi}{2}(1-\theta) + \tfrac{il}{r}\left(\tfrac{\alpha_1+\alpha_2}{2}\right)\sin\tfrac{\pi}{2}(1-\theta)\right)\\
&\leq -\tfrac{\pi}{2}\alpha_1 - \tan^{-1}\left(\tfrac{l(\alpha_1+\alpha_2)\sin\frac{\pi}{2}(1-\theta)}{2r+l(\alpha_1+\alpha_2)\cos\frac{\pi}{2}(1-\theta)}\right)\\
&\leq -\tfrac{\pi}{2}\alpha_1 - \tan^{-1}\left(\tfrac{(1-|b|)(\alpha_1+\alpha_2)\cos\frac{\pi}{2}t}{2(1+|b|)\left(\frac{(p-\alpha)(1+A)}{1+B}+\alpha+c\right)+(1-|b|)(\alpha_1+\alpha_2)\sin\frac{\pi}{2}t}\right)\\
&= -\tfrac{\pi}{2}\delta_1
\end{aligned}$$

and

$$\begin{aligned}
\arg(Q(z_2) &\geq \tfrac{\pi}{2}\alpha_2 + \tan^{-1}\left(\tfrac{(1-|b|)(\alpha_1+\alpha_2)\cos\frac{\pi}{2}t}{2(1+|b|)\left(\frac{(p-\alpha)(1+A)}{1+B}+\alpha+c\right)+(1-|b|)(\alpha_1+\alpha_2)\sin\frac{\pi}{2}t}\right)\\
&= \tfrac{\pi}{2}\delta_2.
\end{aligned}$$

(ii) When $B = -1$, similarly we have

$$\arg(Q(z_1)) = \arg\left(q(z_1) + \frac{zq'(z_1)}{[(p-\alpha)u(z_1)+\alpha+c]q(z_1)}\right) \leq -\tfrac{\pi}{2}\delta_1$$

and

$$\arg(Q(z_2)) = \arg\left(q(z_2) + \frac{zq'(z_2)}{[(p-\alpha)u(z_2)+\alpha+c]q(z_2)}\right) \geq \tfrac{\pi}{2}\delta_2.$$

The above two cases contradict the assumptions of the theorem. Hence the proof is complete. □

By suitable choice of the parameters $m, \delta, c, p, \lambda, \alpha$ and the function h, one can deduce several special cases of the above theorems. For example, when $m = 1$ and $\lambda = 0$ the above theorems reduce to the results recently obtained by Ebadian *et al.* [5].

4 Open Problem

Let the linear operator $\mathcal{D}_{p,\lambda}^{n,\alpha}:\mathcal{A}(p)\rightarrow\mathcal{A}(p)$ be defined as follows

$$\begin{aligned}
\mathcal{D}_{p,\lambda}^{0,0}f(z) &= f(z)\\
\mathcal{D}_{p,\lambda}^{1,\alpha}f(z) &= (1-\lambda)\Omega_p^{\alpha}f(z)+\frac{\lambda z}{p}(\Omega_p^{\alpha}f(z))'=\mathcal{D}_{p,\lambda}^{\alpha}f(z)\\
\mathcal{D}_{p,\lambda}^{2,\alpha}f(z) &= \mathcal{D}_{p,\lambda}^{\alpha}(\mathcal{D}_{p,\lambda}^{1,\alpha}f(z))\\
&\ \ .\\
&\ \ .\\
&\ \ .\\
\mathcal{D}_{p,\lambda}^{n,\alpha}f(z) &= \mathcal{D}_{p,\lambda}^{\alpha}(\mathcal{D}_{p,\lambda}^{n-1,\alpha}f(z))
\end{aligned}$$

for $n\in\mathbb{N}$, $\lambda\geq 0$ and $0\leq\alpha<1$, where

$$\Omega_p^{\alpha}f(z)=\frac{\Gamma(1+p-\alpha)}{\Gamma(1+p)}z^{\alpha}D_z^{\alpha}f(z)$$

and $D_z^{\alpha}f(z)$ is the well known Riemann-Liouville fractional derivative of order α $(0\leq\alpha<1)$ (see, e.g. Srivastava and Aouf [17]). The question arises that if we rewrite the class defined in Definition 1 by replacing $\mathcal{J}_{c,p,\lambda}^{m,\delta}f(z)$ by $\mathcal{D}_{p,\lambda}^{n,\alpha}f(z)$, then can the main results obtained in the above theorems be derived for this class?. More precisely, what equation can be derived instead of equation (7) in order to proceed to the main results?.

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