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Subordination and Superordination Results for Analytic Functions Associated With Convolution Structure

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Abstract

In the present investigation, we obtain some subordination and superordination results involving Hadamard product for certain normalized analytic functions in the open unit disk. Relevant connections of the results, which are presented in this paper, with various other known results are also pointed out.

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1 Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\mathcal{U} := \{z : |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots .$$

Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + a_2 z^2 + \dots . \tag{1}$$

Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathcal{C}^3 \times \mathcal{U} \to \mathcal{C}$. If p and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z), \tag{2}$$

then p is a solution of the differential superordination (2). (If f is subordinate to F, then F is superordinate to f.) An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (2). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (2) is said to be the best subordinant. Recently Miller and Mocanu[10] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

For two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{3}$$

Recently Bulboacă [2] (see also [1]) considered certain classes of first order differential superordinations as well as superordination-preserving integral operators by using the results of Miller and Mocanu[10]. Further, using the results in [1] and [10], Magesh and Murugusundaramoorthy [7], Magesh et al., [8], Murugusundaramoorthy and Magesh [11, 12, 13] and Shanmugam et al., [18] have obtained sandwich results for certain classes of analytic functions.

The main object of the present paper is to find sufficient condition for certain normalized analytic functions f in $\mathcal{U}$ to satisfy

$$q_1(z) \prec \left(\frac{\alpha(f * \Phi)(z) + \beta(f * \Psi)(z)}{z} \right)^{\mu} \prec q_2(z), \tag{4}$$

where q_1, q_2 are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$, $q_2(0) = 1$ and $\Phi(z) = z + \sum\limits_{n=2}^{\infty} \lambda_n z^n$, $\Psi(z) = z + \sum\limits_{n=2}^{\infty} \mu_n z^n$ are analytic functions in $\mathcal{U}$ with $\lambda_n \geq 0$, $\mu_n \geq 0$ and $\lambda_n \geq \mu_n$. Also, we obtain the number of known results as their special cases.

2 Preliminary Results

For our present investigation, we shall need the following:

Lemma 2.1 *[15, p.159, Theorem 6.2] The function* $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots$ *with* $a_1(t) \neq 0$ *for* $t \geq 0$ *and* $\lim_{t\to\infty} |a_1(t)| = +\infty$, *is a subordination chain if*

$$Re \left\{ z \frac{\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} \right\} > 0, z \in \mathcal{U},\ t \geq 0.$$

Definition 2.2 *[10, p.817, Definition 2] Denote by* Q, *the set of all functions* f *that are analytic and injective on* $\overline{\mathcal{U}} - \mathcal{E}(f)$, *where*

$$\mathcal{E}(f) = \{\zeta \in \partial\mathcal{U} : \lim_{z\to\zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ *for* $\zeta \in \partial\mathcal{U} - \mathcal{E}(f)$.

Lemma 2.3 *[9, p.132, Theorem 3.4h] Let* q *be univalent in the unit disk* $\mathcal{U}$ *and* θ *and* ϕ *be analytic in a domain* $\mathcal{D}$ *containing* $q(\mathcal{U})$ *with* $\phi(w) \neq 0$ *when* $w \in q(\mathcal{U})$. *Set*

$$Q(z) := zq'(z)\phi(q(z)) \quad \text{and} \quad h(z) := \theta(q(z)) + Q(z).$$

Suppose that

1. $Q(z)$ *is starlike univalent in* $\mathcal{U}$ *and*
2. $Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ *for* $z \in \mathcal{U}$.

If p *is analytic with* $p(0) = q(0)$, $p(\mathcal{U}) \subseteq \mathcal{D}$ *and*

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{5}$$

then

$$p(z) \prec q(z)$$

and q *is the best dominant.*

Lemma 2.4 *[2, p.289, Corollary 3.2] Let* q *be convex univalent in the unit disk* $\mathcal{U}$ *and* ϑ *and* φ *be analytic in a domain* $\mathcal{D}$ *containing* $q(\mathcal{U})$. *Suppose that*

1. $Re\{\vartheta'(q(z))/\varphi(q(z))\} > 0 \text{for} z \in \mathcal{U}$ *and*
2. $\psi(z) = zq'(z)\varphi(q(z))$ *is starlike univalent in* $\mathcal{U}$.

If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, *with* $p(\mathcal{U}) \subseteq \mathcal{D}$, *and* $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ *is univalent in* $\mathcal{U}$ *and*

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{6}$$

then $q(z) \prec p(z)$ *and* q *is the best subordinant.*

3 Subordination results

Using Lemma 2.3, we first prove the following theorem.

Theorem 3.1 *Let* $\Phi, \Psi \in \mathcal{A}$, $\gamma_i \in \mathcal{C}$ $(i = 1, 2, 3)(\gamma_3 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ *such that* $\mu \neq 0$ *and* $\alpha + \beta \neq 0$, q *be convex univalent with* $q(0) = 1$, *and assume that*

$$Re\left\{\frac{\gamma_2}{\gamma_3}q(z) + 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0 \quad (z \in \mathcal{U}). \tag{7}$$

If $f \in \mathcal{A}$ *satisfies*

$$\Delta^{(\gamma_i)_1^3}(f; \Phi, \Psi, \alpha, \beta) = \Delta(f,\ \Phi, \Psi,\ \gamma_1, \gamma_2, \gamma_3, \alpha, \beta) \prec \gamma_1 + \gamma_2 q(z) + \gamma_3 \frac{zq'(z)}{q(z)}, \tag{8}$$

where

$$\Delta^{(\gamma_i)_1^3}(f; \Phi, \Psi, \alpha, \beta) := \begin{cases} \gamma_1 + \gamma_2 \left(\frac{\alpha(f*\Phi)(z)+\beta(f*\Psi)(z)}{z}\right)^{\mu} \\ +\gamma_3\mu\left(\frac{\alpha z(f*\Phi)'(z)+\beta z(f*\psi)'(z)}{\alpha(f*\Phi)(z)+\beta(f*\psi)(z)} - 1\right), \end{cases} \tag{9}$$

then

$$\left(\frac{\alpha(f*\Phi)(z) + \beta(f*\Psi)(z)}{z}\right)^{\mu} \prec q(z)$$

and q *is the best dominant.*

Proof: Define the function p by

$$p(z) := \left(\frac{\alpha(f*\Phi)(z) + \beta(f*\Psi)(z)}{z}\right)^{\mu} \quad (z \in \mathcal{U}). \tag{10}$$

Then the function p is analytic in $\mathcal{U}$ and $p(0) = 1$. Therefore, by making use of (10), we obtain

$$\begin{aligned} \gamma_1 + \gamma_2 &\left(\frac{\alpha(f*\Phi)(z) + \beta(f*\Psi)(z)}{z}\right)^{\mu} \\ &+\gamma_3\mu\left(\frac{\alpha z(f*\Phi)'(z) + \beta z(f*\psi)'(z)}{\alpha(f*\Phi)(z) + \beta(f*\psi)(z)} - 1\right) \\ &= \gamma_1 + \gamma_2 p(z) + \gamma_3 \frac{zp'(z)}{p(z)}. \end{aligned} \tag{11}$$

By using (11) in (8), we have

$$\gamma_1 + \gamma_2 p(z) + \gamma_3 \frac{zp'(z)}{p(z)} \prec \gamma_1 + \gamma_2 q(z) + \gamma_3 \frac{zq'(z)}{q(z)}. \tag{12}$$

By setting

$$\theta(w) := \gamma_1 + \gamma_2 \omega \quad \text{and} \quad \phi(\omega) := \frac{\gamma_3}{w},$$

it can be easily observed that $\theta(w)$, $\phi(w)$ are analytic in $\mathcal{C} - \{0\}$ and $\phi(w) \neq 0$. Also we see that

$$Q(z) := zq'(z)\phi(q(z)) = \gamma_3 \frac{zq'(z)}{q(z)}$$

and

$$h(z) := \theta(q(z)) + Q(z) = \gamma_1 + \gamma_2 q(z) + \gamma_3 \frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in $\mathcal{U}$ and

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\frac{\gamma_2}{\gamma_3} q(z) + 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0.$$

By the hypothesis of Theorem 3.1, the result now follows by an application of Lemma 2.3.

By fixing $\Phi(z) = \frac{z}{1-z}$ and $\Psi(z) = \frac{z}{(1-z)^2}$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.2 *Let* $\gamma_i \in \mathcal{C}$ $(i = 1, 2, 3)(\gamma_3 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ *such that* $\mu \neq 0$ *and* $\alpha + \beta \neq 0$, q *be convex univalent with* $q(0) = 1$, *and* (7) holds true. If $f \in \mathcal{A}$ satisfies

$$\gamma_1 + \gamma_2 \left(\frac{\alpha f(z) + \beta z f'(z)}{z}\right)^{\mu} + \gamma_3 \mu \left(\frac{\alpha(zf'(z) - f(z)) + \beta z^2 f''(z)}{\alpha f(z) + \beta z f'(z)}\right)$$
$$\prec \gamma_1 + \gamma_2 q(z) + \gamma_3 \frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{\alpha f(z) + \beta z f'(z)}{z}\right)^{\mu} \prec q(z)$$

and q is the best dominant.

Putting $\alpha = 1$ and $\beta = 1$ in Corollary 3.2, we obtain the following corollary.

Corollary 3.3 *Let* $\gamma_i \in \mathcal{C}$ $(i = 1, 2, 3)(\gamma_3 \neq 0)$, $0 \neq \mu \in \mathcal{C}$, q *be convex univalent with* $q(0) = 1$, *and* (7) holds true. If $f \in \mathcal{A}$ satisfies

$$\gamma_1 + \gamma_2 \left(\frac{f(z) + zf'(z)}{z}\right)^{\mu} + \gamma_3 \mu \left(\frac{z^2 f''(z) + zf'(z) - f(z)}{f(z) + zf'(z)}\right)$$
$$\prec \gamma_1 + \gamma_2 q(z) + \gamma_3 \frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{f(z)+zf'(z)}{z}\right)^{\mu} \prec q(z)$$

and q is the best dominant.

Putting $\alpha = 1$ and $\beta = 0$ in Corollary 3.2, we obtain the following corollary.

Corollary 3.4 *Let* $\gamma_i \in \mathcal{C}$ $(i = 1, 2, 3)$ $(\gamma_3 \neq 0)$, $0 \neq \mu \in \mathcal{C}$, *q be convex univalent with* $q(0) = 1$, *and* (7) holds true. If $f \in \mathcal{A}$ satisfies

$$\gamma_1 + \gamma_2\left(\frac{f(z)}{z}\right)^{\mu} + \gamma_3\mu\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \gamma_1 + \gamma_2 q(z) + \gamma_3\frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec q(z)$$

and q is the best dominant.

Putting $\alpha = 0$ and $\beta = 1$ in Corollary 3.2, we obtain the following corollary.

Corollary 3.5 *Let* $\gamma_i \in \mathcal{C}$ $(i = 1, 2, 3)(\gamma_3 \neq 0)$, $0 \neq \mu \in \mathcal{C}$, *q be convex univalent with* $q(0) = 1$, *and* (7) holds true. If $f \in \mathcal{A}$ satisfies

$$\gamma_1 + \gamma_2\left(f'(z)\right)^{\mu} + \gamma_3\mu\frac{zf''(z)}{f'(z)} \prec \gamma_1 + \gamma_2 q(z) + \gamma_3\frac{zq'(z)}{q(z)},$$

then

$$(f'(z))^{\mu} \prec q(z)$$

and q is the best dominant.

By taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 3.1, we have the following corollary.

Corollary 3.6 *Let* $\Phi, \Psi \in \mathcal{A}$, $\gamma_i \in \mathcal{C}$ $(i = 1, 2, 3)(\gamma_3 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ *such that* $\mu \neq 0$ *and* $\alpha + \beta \neq 0$, *q be convex univalent with* $q(0) = 1$, *assume that* (7) holds. If $f \in \mathcal{A}$ and

$$\Delta^{(\gamma_i)_1^3}(f; \Phi, \Psi, \alpha, \beta) \prec \gamma_1 + \gamma_2\frac{1+Az}{1+Bz} + \gamma_3\frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$\left(\frac{\alpha(f*\Phi)(z) + \beta(f*\Psi)(z)}{z}\right)^{\mu} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Putting $\alpha = 1$ and $\beta = 0$ in Corollary 3.6, we obtain the following corollary.

Corollary 3.7 *Let* $\Phi \in \mathcal{A}$, $\gamma_i \in \mathcal{C}$ $(i = 1, 2, 3)(\gamma_3 \neq 0)$, $0 \neq \mu \in \mathcal{C}$ *and* q *be convex univalent with* $q(0) = 1$, *assume that* (7) holds. If $f \in \mathcal{A}$ and

$$\gamma_1 + \gamma_2 \left(\frac{(f * \Phi)(z)}{z}\right)^{\mu} + \gamma_3 \mu \left(\frac{z(f * \Phi)'(z)}{(f * \Phi)(z)} - 1\right)$$
$$\prec \gamma_1 + \gamma_2 \frac{1 + Az}{1 + Bz} + \gamma_3 \frac{(A - B)z}{(1 + Az)(1 + Bz)},$$

then

$$\left(\frac{(f * \Phi)(z)}{z}\right)^{\mu} \prec \frac{1 + Az}{1 + Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Putting $\gamma_1 = 1$, $\gamma_2 = 0$ and $\gamma_3 = 1$ in Corollary 3.7, we obtain the following corollary.

Corollary 3.8 *Let* $\Phi, \in \mathcal{A}$, $0 \neq \mu \in \mathcal{C}$ *and* q *be convex univalent with* $q(0) = 1$, *assume that* (7) holds. If $f \in \mathcal{A}$ and

$$1 + \mu \left(\frac{z(f * \Phi)'(z)}{(f * \Phi)(z)} - 1\right) \prec 1 + \frac{(A - B)z}{(1 + Az)(1 + Bz)},$$

then

$$\left(\frac{(f * \Phi)(z)}{z}\right)^{\mu} \prec \frac{1 + Az}{1 + Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

By setting $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_3 = 1$, $\alpha = 1$, $\beta = 0$, $\Phi(z) = \frac{z}{1-z}$ and $q(z) = (1 + Bz)^{\mu \frac{(A-B)}{B}}$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.9 *Let* $0 \neq \mu \in \mathcal{C}$, $-1 \leq B < A \leq 1$ *and* q *be convex univalent with* $q(0) = 1$, *assume that* (7) holds. If $f \in \mathcal{A}$ and

$$1 + \mu \left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1 + Az}{1 + Bz},$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec (1 + Bz)^{\mu \frac{(A-B)}{B}}$$

and $(1 + Bz)^{\mu \frac{(A-B)}{B}}$ is the best dominant.

We note that $q(z) = (1+Bz)^{\mu\frac{(A-B)}{B}}$ is univalent if and only if $\left|\frac{\mu(A-B)}{B} - 1\right| \leq 1$ or $\left|\frac{\mu(A-B)}{B} + 1\right| \leq 1$.

By setting $\gamma_1 = 1$, $\gamma_2 = 0$, $\alpha = 1$, $\beta = 0$, $\Phi(z) = \frac{z}{1-z}$, $q(z) = \frac{1}{(1-z)^{2b}}$ ($b \in \mathcal{C} - \{0\}$), $\mu = 1$ and $\gamma_3 = \frac{1}{b}$ in Theorem 3.1, we obtain the following corollary as stated in [19].

Corollary 3.10 *Let* $0 \neq b \in \mathcal{C}$ *and* q *be convex univalent with* $q(0) = 1$, *assume that* (7) holds. If $f \in \mathcal{A}$ and

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1+z}{1-z},$$

then

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^{2b}}$$

and $\frac{1}{(1-z)^{2b}}$ is the best dominant.

By setting $\gamma_1 = 1$, $\gamma_2 = 0$, $\alpha = 1$, $\beta = 0$, $\Phi(z) = \frac{z}{(1-z)^2}$, $q(z) = \frac{1}{(1-z)^{2b}}$ ($b \in \mathcal{C} - \{0\}$), $\mu = 1$ and $\gamma_3 = \frac{1}{b}$ in Theorem 3.1, we obtain the following corollary as stated in [19].

Corollary 3.11 *Let* $0 \neq b \in \mathcal{C}$ *and* q *be convex univalent with* $q(0) = 1$, *assume that* (7) holds. If $f \in \mathcal{A}$ and

$$1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z},$$

then

$$f'(z) \prec \frac{1}{(1-z)^{2b}}$$

and $\frac{1}{(1-z)^{2b}}$ is the best dominant.

By setting $\gamma_1 = 1$, $\gamma_2 = 0$, $\alpha = 1$, $\beta = 0$, $\Phi(z) = \frac{z}{1-z}$, $q(z) = \frac{1}{(1-z)^{2ab}}$ ($b \in \mathcal{C} - \{0\}$), $\mu = a$ and $\gamma_3 = \frac{1}{b}$ in Theorem 3.1, we obtain the following corollary as stated in [14].

Corollary 3.12 *Let* $0 \neq b \in \mathcal{C}$ *and* q *be convex univalent with* $q(0) = 1$, *assume that* (7) holds. If $f \in \mathcal{A}$ and

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1+z}{1-z},$$

then

$$\left(\frac{f(z)}{z}\right)^a \prec \frac{1}{(1-z)^{2ab}}$$

and $\frac{1}{(1-z)^{2ab}}$ is the best dominant.

By taking $q(z) = \left(\frac{1+z}{1-z}\right)^\delta$ $(0 < \delta \le 1)$, in Theorem 3.1, we have the following corollary.

Corollary 3.13 *Let* $\Phi, \Psi \in \mathcal{A}$, $\gamma_i \in \mathcal{C}$ $(i = 1, 2, 3)(\gamma_3 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ *such that* $\mu \neq 0$ *and* $\alpha + \beta \neq 0$, q *be convex univalent with* $q(0) = 1$, *assume that* (7) holds. If $f \in \mathcal{A}$ and

$$\Delta^{(\gamma_i)_1^3}(f; \Phi, \Psi, \alpha, \beta) \prec \gamma_1 + \gamma_2 \left(\frac{1+z}{1-z}\right)^\delta + \gamma_3 \frac{2\delta z}{(1-z^2)},$$

then

$$\left(\frac{\alpha(f * \Phi)(z) + \beta(f * \Psi)(z)}{z}\right)^\mu \prec \left(\frac{1+z}{1-z}\right)^\delta$$

and $\left(\frac{1+z}{1-z}\right)^\delta$ is the best dominant.

Putting $q(z) = e^{\mu A z}$ $(|\mu A| \le \pi)$, in Theorem 3.1, we have the following corollary.

Corollary 3.14 *Let* $\Phi, \Psi \in \mathcal{A}$, $\gamma_i \in \mathcal{C}$ $(i = 1, 2, 3)(\gamma_3 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ *such that* $\mu \neq 0$ *and* $\alpha + \beta \neq 0$, q *be convex univalent with* $q(0) = 1$, *assume that* (7) holds. If $f \in \mathcal{A}$ and

$$\Delta^{(\gamma_i)_1^3}(f; \Phi, \Psi, \alpha, \beta) \prec \gamma_1 + \gamma_2 e^{\mu A z} + \gamma_3 \mu A z,$$

then

$$\left(\frac{\alpha(f * \Phi)(z) + \beta(f * \Psi)(z)}{z}\right)^\mu \prec e^{\mu A z}$$

and $e^{\mu A z}$ is the best dominant.

3.1 Superordination results

Now, by applying Lemma 2.4, we prove the following theorem.

Theorem 3.15 *Let* $\Phi, \Psi \in \mathcal{A}$, $\gamma_i \in \mathcal{C}$ $(i = 1, 2, 3)(\gamma_3 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ *such that* $\mu \neq 0$ *and* $\alpha + \beta \neq 0$, q *be convex univalent with* $q(0) = 1$, *and assume that*

$$Re \left\{\frac{\gamma_2}{\gamma_3} q(z)\right\} \ge 0. \tag{13}$$

If $f \in \mathcal{A}$, $\left(\frac{\alpha(f*\Phi)(z)+\beta(f*\Psi)(z)}{z}\right)^\mu \in \mathcal{H}[q(0), 1] \cap Q$. *Let* $\Delta^{(\gamma_i)_1^3}(f; \Phi, \Psi, \alpha, \beta)$ *be univalent in* $\mathcal{U}$ *and*

$$\gamma_1 + \gamma_2 q(z) + \gamma_3 \frac{zq'(z)}{q(z)} \prec \Delta^{(\gamma_i)_1^3}(f; \Phi, \Psi, \alpha, \beta), \tag{14}$$

where $\Delta^{(\gamma_i)_1^3}(f;\Phi,\Psi,\alpha,\beta)$ *is given by (9), then*

$$q(z) \prec \left(\frac{\alpha(f*\Phi)(z)+\beta(f*\Psi)(z)}{z}\right)^{\mu}$$

and q *is the best subordinant.*

Proof Define the function p by

$$p(z) := \left(\frac{\alpha(f*\Phi)(z)+\beta(f*\Psi)(z)}{z}\right)^{\mu}. \tag{15}$$

Simple computation from (15), we get,

$$\Delta^{(\gamma_i)_1^3}(f;\Phi,\Psi,\alpha,\beta) = \gamma_1 + \gamma_2 p(z) + \gamma_3 \frac{zp'(z)}{p(z)},$$

then

$$\gamma_1 + \gamma_2 q(z) + \gamma_3 \frac{zq'(z)}{q(z)} \prec \gamma_1 + \gamma_2 p(z) + \gamma_3 \frac{zp'(z)}{p(z)}.$$

By setting $\vartheta(w) = \gamma_1 + \gamma_2 w$ and $\phi(w) = \frac{\gamma_3}{w}$, it is easily observed that $\vartheta(w)$ is analytic in $\mathcal{C}$. Also, $\phi(w)$ is analytic in $\mathcal{C} - \{0\}$ and $\phi(w) \neq 0$.

If we let

$$\begin{aligned} L(z,t) &= \vartheta(q(z)) + \phi(q(z))tzq'(z) = \gamma_1 + \gamma_2 q(z) + \gamma_3 t \frac{zq'(z)}{q(z)} \\ &= a_1(t)z + \dots \end{aligned} \tag{16}$$

Differentiating (16) with respect to z and t, we have

$$\begin{aligned} \frac{\partial L(z,t)}{\partial z} &= \gamma_2 q'(z) + t\gamma_3 \left[\frac{zq''(z)}{q(z)} + \frac{q'(z)}{q(z)} - \frac{z(q'(z))^2}{q^2(z)}\right] \\ &= a_1(t) + \dots \end{aligned}$$

and

$$\frac{\partial L(z,t)}{\partial t} = \gamma_3 \frac{zq'(z)}{q(z)}.$$

Also,

$$\frac{\partial L(0,t)}{\partial z} = \gamma_3 q'(0) \left[\frac{\gamma_2}{\gamma_3} + t\frac{1}{q(0)}\right]$$

From the univalence of q we have $q'(0) \neq 0$ and $q(0) = 1$, it follows that $a_1(t) \neq 0$ for $t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = +\infty$.

A simple computation yields,

$$\operatorname{Re}\left\{ z\frac{\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} \right\} = \operatorname{Re}\left\{ \frac{\gamma_2}{\gamma_3} q(z) + t\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) \right\}.$$

Using the fact that q is convex univalent function in $\mathcal{U}$ and $\gamma_4 \neq 0$, we have,

$$\operatorname{Re}\left\{ z\frac{\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} \right\} > 0 \text{ if } \Re\left\{ \frac{\gamma_2}{\gamma_3} q(z) \right\} > 0, \quad z \in \mathcal{U}, \ t \geq 0.$$

Now Theorem 3.15 follows by applying Lemma 2.4.

By fixing $\Phi(z) = \frac{z}{1-z}$ and $\Psi(z) = \frac{z}{(1-z)^2}$ in Theorem 3.15, we obtain the following corollary.

Corollary 3.16 *Let* $\gamma_i \in \mathcal{C}$ $(i = 1, 2, 3)(\gamma_3 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ *such that* $\mu \neq 0$ *and* $\alpha + \beta \neq 0$, q *be convex univalent with* $q(0) = 1$, *and (13) hold true. If* $f \in \mathcal{A}$, $\left(\frac{\alpha f(z)+\beta z f'(z)}{z}\right)^{\mu} \in \mathcal{H}[q(0), 1] \cap Q$. *Let* $\gamma_1 + \gamma_2 \left(\frac{\alpha f(z)+\beta z f'(z)}{z}\right)^{\mu} + \gamma_3\mu\left(\frac{\alpha(zf'(z)-f(z))+\beta z^2 f''(z)}{\alpha f(z)+\beta z f'(z)}\right)$, *be univalent in* $\mathcal{U}$ *and*

$$\gamma_1 + \gamma_2 q(z) + \gamma_3 \frac{zq'(z)}{q(z)} \prec \gamma_1 + \gamma_2 \left(\frac{\alpha f(z) + \beta z f'(z)}{z}\right)^{\mu} + \gamma_3\mu\left(\frac{\alpha(zf'(z) - f(z)) + \beta z^2 f''(z)}{\alpha f(z) + \beta z f'(z)}\right),$$

then

$$q(z) \prec \left(\frac{\alpha f(z) + \beta z f'(z)}{z}\right)^{\mu}$$

and q *is the best subordinant.*

Putting $\alpha = 1$ and $\beta = 1$ in Corollary 3.16, we obtain the following corollary.

Corollary 3.17 *Let* $\gamma_i \in \mathcal{C}$ $(i = 1, 2, 3)(\gamma_3 \neq 0)$, $0 \neq \mu \in \mathcal{C}$, q *be convex univalent with* $q(0) = 1$, *and (13) hold true. If* $f \in \mathcal{A}$, $\left(\frac{f(z)+zf'(z)}{z}\right)^{\mu} \in \mathcal{H}[q(0), 1] \cap Q$. *Let* $\gamma_1 + \gamma_2\left(\frac{f(z)+zf'(z)}{z}\right)^{\mu} + \gamma_3\mu\left(\frac{z^2 f''(z)+zf'(z)-f(z)}{f(z)+zf'(z)}\right)$, *be univalent in* $\mathcal{U}$ *and*

$$\gamma_1 + \gamma_2 q(z) + \gamma_3 \frac{zq'(z)}{q(z)} \prec \gamma_1 + \gamma_2\left(\frac{f(z) + zf'(z)}{z}\right)^{\mu} + \gamma_3\mu\left(\frac{z^2 f''(z) + zf'(z) - f(z)}{f(z) + zf'(z)}\right),$$

then

$$q(z) \prec \left(\frac{f(z) + zf'(z)}{z}\right)^{\mu}$$

and q *is the best subordinant.*

By taking $q(z) = (1 + Az)/(1 + Bz)$ $(-1 \le B < A \le 1)$ in Theorem 3.15, we obtain the following corollary.

Corollary 3.18 *Let* $\gamma_i \in \mathcal{C}$ $(i = 1, 2, 3)(\gamma_3 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ *such that* $\mu \neq 0$ *and* $\alpha + \beta \neq 0$, q *be convex univalent with* $q(0) = 1$, *and (13) hold true. If* $f \in \mathcal{A}$, $\left(\frac{\alpha(f*\Phi)(z)+\beta(f*\Psi)(z)}{z}\right)^{\mu} \in \mathcal{H}[q(0), 1] \cap Q$. *Let* $\Delta^{(\gamma_i)_1^3}(f; \Phi, \Psi, \alpha, \beta)$ *be univalent in* $\mathcal{U}$ *and*

$$\gamma_1 + \gamma_2 \frac{1 + Az}{1 + Bz} + \gamma_3 \frac{(A - B)z}{(1 + Az)(1 + Bz)} \prec \Delta^{(\gamma_i)_1^3}(f; \Phi, \Psi, \alpha, \beta),$$

then

$$\frac{1 + Az}{1 + Bz} \prec \left(\frac{\alpha(f * \Phi)(z) + \beta(f * \Psi)(z)}{z}\right)^{\mu}$$

and $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

4 Sandwich results

There is a complete analog of Theorem 3.1 for differential subordination and Theorem 3.15 for differential superordination. We can combine the results of Theorem 3.1 with Theorem 3.15 and obtain the following sandwich theorem.

Theorem 4.1 *Let* q_1 *and* q_2 *be convex univalent in* $\mathcal{U}$, $\gamma_i \in \mathcal{C}$ $(i = 1, 2, 3)(\gamma_3 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ *such that* $\mu \neq 0$ *and* $\alpha + \beta \neq 0$, *and let* q_2 *satisfy (7) and* q_1 *satisfy (13). For* $f, \Phi, \Psi \in \mathcal{A}$, *let* $\left(\frac{\alpha(f*\Phi)(z)+\beta(f*\Psi)(z)}{z}\right)^{\mu} \in \mathcal{H}[1, 1] \cap Q$ *and* $\Delta^{(\gamma_i)_1^3}(f; \Phi, \Psi, \alpha, \beta)$ *defined by (9) be univalent in* $\mathcal{U}$ *satisfying*

$$\gamma_1 + \gamma_2 q_1(z) + \gamma_3 \frac{zq_1'(z)}{q_1(z)} \prec \Delta^{(\gamma_i)_1^3}(f; \Phi, \Psi, \alpha, \beta) \prec \gamma_1 + \gamma_2 q_2(z) + \gamma_3 \frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{\alpha(f * \Phi)(z) + \beta(f * \Psi)(z)}{z}\right)^{\mu} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and best dominant.*

By taking $q_1(z) = \frac{1+A_1z}{1+B_1z}$ $(-1 \le B_1 < A_1 \le 1)$ and $q_2(z) = \frac{1+A_2z}{1+B_2z}$ $(-1 \le B_2 < A_2 \le 1)$ in Theorem 4.1 we obtain the following result.

Corollary 4.2 *For* $f, \Phi, \Psi \in \mathcal{A}$, *let* $\left(\frac{\alpha(f*\Phi)(z)+\beta(f*\Psi)(z)}{z}\right)^{\mu} \in \mathcal{H}[1, 1] \cap Q$ *and* $\Delta^{(\gamma_i)_1^3}(f; \Phi, \Psi, \alpha, \beta)$ *defined by (9) be univalent in* $\mathcal{U}$ *satisfying*

$$\begin{aligned} \gamma_1 + \gamma_2 \frac{1 + A_1z}{1 + B_1z} &+ \gamma_3 \frac{(A_1 - B_1)z}{(1 + A_1z)(1 + B_1z)} \\ &\prec \Delta^{(\gamma_i)_1^3}(f; \Phi, \Psi, \alpha, \beta) \\ &\qquad \prec \gamma_1 + \gamma_2 \frac{1 + A_2z}{1 + B_2z} + \gamma_3 \frac{(A_2 - B_2)z}{(1 + A_2z)(1 + B_2z)} \end{aligned}$$

then

$$\frac{1+A_1z}{1+B_1z} \prec \left(\frac{\alpha(f*\Phi)(z)+\beta(f*\Psi)(z)}{z}\right)^{\mu} \prec \frac{1+A_2z}{1+B_2z}$$

and $\frac{1+A_1z}{1+B_1z}$, $\frac{1+A_2z}{1+B_2z}$ *are respectively the best subordinant and best dominant.*

5 Conclusion and Open Problem

We conclude this paper by remarking that in view of the function class defined by the subordination relation (4) and expressed in terms of the convolution (3) involving arbitrary coefficients, the main results would lead to additional new results. In fact, by appropriately selecting the arbitrary sequences ($\Phi(z)$ and $\Psi(z)$), the results presented in this paper would find further applications for the classes which incorparate generalized forms of linear operators illustrated below

1. $$\Phi(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}\dots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_m)_{n-1}} \frac{z^n}{(n-1)!}$$

   and

   $$\Psi(z) = z + \sum_{n=2}^{\infty} n\frac{(\alpha_1)_{n-1}\dots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_m)_{n-1}} \frac{z^n}{(n-1)!}$$

   where $\alpha_1,\dots,\alpha_l$ and $\beta_1,\dots,\beta_m$, $(l,m \in \mathcal{N} = 1,2,3,...)$ are complex parameters $\beta_j \notin \{0,-1,-2,\dots\}$ for $j=1,2,\dots m$, $l \le m+1$ (results for Dziok-Srivastava operator [6])

2. $\Phi(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} \frac{z^n}{(n-1)!}$ and $\Psi(z) = z + \sum_{n=2}^{\infty} n\frac{(a)_{n-1}}{(c)_{n-1}} \frac{z^n}{(n-1)!}$, where $c \ne 0,-1,-2,\dots$, (results for Carlson and Shaffer operator [3])

3. $\Phi(z) = \frac{z}{(1-z)^{\lambda+1}}, \lambda \ge -1$, and $\Psi(z) = \frac{z}{(1-z)^{\lambda+2}}, \lambda \ge -1$, (results for Ruscheweyh derivative operator [16])

4. $\Phi(z) = z + \sum_{n=2}^{\infty} n^k z^n$, and $\Psi(z) = z + \sum_{n=2}^{\infty} n^{k+1} z^n, k \ge 0$, (results for Salagean operator [17])

5. $\Phi(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^k z^n$, and $\Psi(z) = z + \sum_{n=2}^{\infty} n\left(\frac{n+\lambda}{1+\lambda}\right)^k z^n$, where $\lambda \ge 0$ $k \in \mathcal{Z}$,(results for Multiplier transformation [4, 5])

and specializing the parameters α, β, μ, $\gamma_1, \gamma_2, \gamma_3$ and γ_4 and the function $q(z)$ Theorem 3.1, Theorem 3.15 and Theorem 4.1 would eventually lead further new results for the classes of functions defined analogously by associating in the process of $\Phi(z)$ and $\Psi(z)$. These considerations can fruitfully be worked out and we skip the details in this regard.

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