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A Unified Class of Uniformly Convex Functions Involving Cho-Kim Operator

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Abstract

The main purpose of this paper is to introduce a new class $\mathcal{UH}(\alpha, \beta, \gamma, \lambda, k)$, ***of functions which are analytic in the open disc*** $\Delta = \{z \in \mathcal{C} : |z| < 1\}$. ***We obtain various results including characterization, coefficients estimates, distortion and covering theorems, radii of close-to-convexity, starlikeness and convexity for functions belonging to the class*** $\mathcal{UH}(\alpha, \beta, \gamma, \lambda, k)$.

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1 Introduction and Motivations

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

that are analytic in the open unit disc $\Delta := \{z \in \mathcal{C} : |z| < 1\}$. Let $\mathcal{S}$ be a subclass of $\mathcal{A}$ consisting of univalent functions in Δ. By $\mathcal{K}(\beta)$, and $\mathcal{S}^*(\beta)$ respectively, we mean the classes of analytic functions that satisfy the analytic conditions

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta \qquad \text{and} \qquad \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \qquad z \in \Delta$$

for $0 \le \beta < 1$. In particular, $\mathcal{K} = \mathcal{K}(0)$ and $\mathcal{S}^* = \mathcal{S}^*(0)$ respectively, are the well-known standard class of convex and starlike functions.

Let $\mathcal{T}$ be the subclass of $\mathcal{S}$ of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \ge 0, \tag{1}$$

that are analytic in the open unit disk Δ. This class was introduced and studied by Silverman [7]. Analogous to the subclasses $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$ of $\mathcal{S}$ respectively, the subclasses of $\mathcal{T}$ denoted by $\mathcal{T}^*(\beta)$ and $\mathcal{C}(\beta)$, $0 \le \beta < 1$, were also investigated by Silverman in [7].

The main class which we investigate in this present paper uses the operator known as the Cho-Kim operator. In fact, One important concept that is useful in discussing this operator is the convolution or Hadamard product. Here by convolution we mean the following: For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$, the Hadamard product(or convolution product) is given by $(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n$.

For functions $f \in \mathcal{A}$, we recall the multiplier transformation $I(\lambda, k)$ introduced by Cho and Kim [3] defined as

$$I(\lambda, k)f(z) = z + \sum_{n=2}^{\infty} \Psi_n a_n z^n \quad (\ \lambda \ge 0;\ k \in \mathcal{Z}) \tag{2}$$

where

$$\Psi_n := n\left(\frac{1+\lambda}{n+\lambda}\right)^k \tag{3}$$

so that, obviously,

$$I(\lambda, k)\,(I(\lambda, m)\ f(z)) = I(\lambda, k+m)\ f(z) \quad (k, m \in \mathcal{Z})\,. \tag{4}$$

The operators $I(\lambda, k)$ are closely related to the Komatu integral operators [4] and also to the differential and integral operators investigated by Sălăgean [5]. We also note that $I(0,0)f(z) = zf'(z)$ and $I(0,1)f(z) = f(z)$. Now we define an unified class of analytic function based on this operator.

Definition 1.1 *For* $0 \leq \gamma \leq 1,\ 0 \leq \beta < 1,\ \alpha \geq 0,$ *and for all* $z \in \Delta$, *we let the class* $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \lambda,\ k)$, *consists of functions* $f \in \mathcal{T}$ *is said to be in the class satisfying the condition*

$$\mathrm{Re}\left\{\frac{zF'(z)}{F(z)}\right\} > \alpha\left|\frac{zF'(z)}{F(z)} - 1\right| + \beta, \tag{5}$$

with,

$$F(z) := \gamma(1+\lambda)I(\lambda, k)f(z) + (1 - \gamma(1+\lambda))I(\lambda, k+1)f(z), \tag{6}$$

where $I(\lambda, k)f(z)$ *is the Cho-Kim operator as defined by* (2)

The family $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \lambda,\ k)$, unifies various well known classes of analytic univalent functions. We list a few of them. The class $\mathcal{UH}(0,\ \beta, \gamma,\ 0,\ 0)$ studied in [1]. Many classes including $\mathcal{UH}(0,\ \beta, 0,\ 0,\ 0)$ and $\mathcal{UH}(0,\ \beta, 1,\ 0,\ 0)$ given in [9], are particular cases of this class. Further that, the class $\mathcal{UH}(\alpha,\ \beta, 1,\ 0,\ 0)$ is the class of $\alpha-$uniformly convex of order β, was introduced and studied in [8] (also see [2]).

In this present paper, we obtain a characterization, coefficients estimates, distortion theorem and covering theorem, extreme points and radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \lambda,\ k)$,.

2 Characterization and Coefficient estimates

Theorem 2.1 *Let* $f \in \mathcal{T}$. *Then* $f \in \mathcal{UH}(\alpha,\ \beta, \gamma,\ \lambda,\ k)$, $\quad 0 \leq \gamma \leq 1$, $0 \leq \beta < 1$ *and* $\alpha \geq 0$,

$$\sum_{n=2}^{\infty} [n(\alpha+1) - (\alpha+\beta)]\,(\gamma(n-1)+1)\,\Psi_n|a_n| \leq 1 - \beta. \tag{7}$$

This result is sharp for the function

$$f(z) = z - \frac{1-\beta}{[n(\alpha+1) - (\alpha+\beta)][\gamma(n-1)+1]\Psi_n} z^n \ n \geq 2. \tag{8}$$

Proof: We employ the technique adopted by [2]. We have $f \in \mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k)$, if and only if the condition (5) is satisfied, which is equivalent to

$$\mathrm{Re}\left\{\frac{zF'(z)(1+ke^{i\theta}) - F(z)ke^{i\theta}}{F(z)}\right\} > \beta, \qquad -\pi \leq \theta < \pi. \tag{9}$$

Now, letting $G(z) = zF'(z)(1 + ke^{i\theta}) - F(z)ke^{i\theta}$, equation (9) is equivalent to

$$|G(z) + (1-\beta)F(z)| > |G(z) - (1+\beta)F(z)|, \ 0 \le \beta < 1.$$

where $F(z)$ is as defined in (6). Now a simple computation gives

$$\begin{aligned}
&|G(z) + (1-\beta)F(z)| \\
&\ge \ |(2-\beta)z| - \left|\sum_{n=2}^{\infty}\{n+1-\beta\}\{\gamma(n-1)+1\}\Psi_n a_n z^n\right| \\
&\qquad - \left|ke^{i\theta}\sum_{n=2}^{\infty}(n-1)\{\gamma(n-1)+1\}\Psi_n a_n z^n\right| \\
&\ge \ (2-\beta)|z| - \sum_{n=2}^{\infty}\{n+1-\beta\}\{\gamma(n-1)+1\}\Psi_n a_n |z|^n \\
&\qquad -k\sum_{n=2}^{\infty}(n-1)\{\gamma(n-1)+1\}\Psi_n a_n |z|^n \\
&\ge \ (2-\beta)|z| - \sum_{n=2}^{\infty}\Big(n(\alpha+1) - (\alpha+\beta) + 1\Big)\Big(\gamma(n-1)+1\Big)\Psi_n a_n |z|^n
\end{aligned}$$

and similarly,

$$\begin{aligned}
&|G(z) - (1+\beta)F(z)| \\
&\le \beta|z| + \sum_{n=2}^{\infty}\Big((n(\alpha+1) - (\alpha+\beta) - 1)\Big)\Big(\gamma(n-1)+1\Big)\Psi_n a_n |z|^n.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|G(z) + (1-\beta)F(z)| - |G(z) - (1+\beta)F(z)| \\
&\ge \ 2(1-\beta)|z| - 2\sum_{n=2}^{\infty}\Big((n(\alpha+1) - (\alpha+\beta))\Big)\Big(\gamma(n-1)+1\Big)\Psi_n a_n |z|^n \ge 0,
\end{aligned}$$

which is equivalent to the result (7).

On the other hand, for all $-\pi \le \theta < \pi$, we must have

$$\mathrm{Re}\left\{\frac{zF'(z)}{F(z)}(1+ke^{i\theta}) - ke^{i\theta}\right\} > \beta.$$

Now, choosing the values of z on the positive real axis, where $0 \le |z| = r < 1$, and using $\mathrm{Re}\{-e^{i\theta}\} \ge -|e^{i\theta}| = -1$, the above inequality can be written as

$$\mathrm{Re}\left\{\frac{(1-\beta) - \sum_{n=2}^{\infty}\Big(n(\alpha+1) - (\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\Psi_n a_n r^{n-1}}{1 - \sum_{n=2}^{\infty}\Big(\gamma(n-1)+1\Big)\Psi_n a_n r^{n-1}}\right\} \ge 0.$$

Setting $r \to 1^-$, we get the desired result.

Many known results can be obtained as particular cases of Theorem 2.1. For details, we refer to the work of Shanmugam *et al.* [6].

By taking $\alpha = 0, \gamma = 0$, $\lambda = 0$ and $k = 0$ in Theorem 2.1, we get the following interesting result given in [7].

Corollary 2.2 *[7] If $f \in \mathcal{T}$, then $f \in \mathcal{C}(\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n(n-\beta)a_n \leq 1-\beta.$$

Indeed, since $f \in \mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k)$, (7), we have

$$\sum_{n=2}^{\infty} \Big(n(\alpha+1)-(\alpha+\beta)\Big) \Big(\gamma(n-1)+1\Big) \Psi_n a_n \leq 1-\beta.$$

Hence for all $n \geq 2$, we have

$$a_n \leq \frac{1-\beta}{\Big(n(\alpha+1)-(\alpha+\beta)\Big) \Big(\gamma(n-1)+1\Big) \Psi_n},$$

whenever $0 \leq \gamma \leq 1,\ \ 0 \leq \beta < 1$ and $\alpha \geq 0$. Hence we state this important observation as a separate theorem.

Theorem 2.3 *If $f \in \mathcal{UH}(q, s, \lambda, \beta, k)$, then*

$$a_n \leq \frac{1-\beta}{\Big(n(\alpha+1)-(\alpha+\beta)\Big) \Big(\gamma(n-1)+1\Big) \Psi_n},\ \ n \geq 2, \tag{10}$$

where $0 \leq \gamma \leq 1,\ \ 0 \leq \beta < 1$ and $\alpha \geq 0$. Equality in (10) holds for the function

$$f(z) = z - \frac{1-\beta}{\Big(n(\alpha+1)-(\alpha+\beta)\Big) \Big(\gamma(n-1)+1\Big) \Psi_n}. \tag{11}$$

This theorem also contains many known results for the special values of the parameters. For example, see the works of Shanmugam *et al.* [6].

3 Distortion and Covering Theorems

Theorem 3.1 *If $f \in \mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k)$, then $f \in \mathcal{T}^*(\delta)$, where*

$$\delta = 1 - \frac{1-\beta}{\Big(2(\alpha+1)-(\alpha+\beta)\Big) \Big(\gamma+1\Big) \Psi_2 - (1-\beta)}.$$

This result is sharp with the extremal function being

$$f(z) = z - \frac{1-\beta}{\Big(2(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma+1\Big)\Psi_2} z^2.$$

Proof: It is sufficient to show that (7) implies $\sum_{n=2}^{\infty}(n-\delta)a_n \le 1-\delta$ [7], that is,

$$\frac{n-\delta}{1-\delta} \le \frac{\Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\Psi_n}{1-\beta}, \; n \ge 2. \tag{12}$$

Since, for $n \ge 2$, (12) is equivalent to

$$\delta \;\le\; 1 - \frac{(n-1)(1-\beta)}{\Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\Psi_n - (1-\beta)} = \Phi(n),$$

and $\Phi(n) \le \Phi(2)$, (12) holds true for any $0 \le \gamma \le 1,\; 0 \le \beta < 1$ and $\alpha \ge 0$. This completes the proof of the Theorem 3.1.

As in the previous cases we note this result has many special cases. If we take By taking $\alpha = 0, \gamma = 0$, $\lambda = 0$ and $k = 0$ in Theorem 3.1, then we have the following result of Silverman [7].

Corollary 3.2 *[7] If $f \in \mathcal{C}(\beta)$, then $f \in \mathcal{T}^*\left(\frac{2}{3-\beta}\right)$. The result is sharp for the extremal function*

$$f(z) = z - \frac{1-\beta}{2(2-\beta)} z^2.$$

Remark. Since distortion theorem and covering theorem are available for the class $\mathcal{T}^*(\beta)$ [7], we can also obtain the corresponding results for the class $\mathcal{UH}(\alpha, \beta, \gamma, \lambda, k)$, from the respective results of $\mathcal{T}^*(\beta)$ by using Theorem 3.1, and we state them without proof.

Theorem 3.3 *Let Ψ_n be defined as in* (3)*. Then, for $f \in \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k)$, with $z = re^{i\theta} \in \Delta$, we have*

$$r - B(\alpha, \beta, \gamma, \lambda)r^2 \le |f(z)| \le r + B(\alpha, \beta, \gamma, \lambda)r^2, \tag{13}$$

where,

$$B(\alpha, \beta, \gamma, \lambda) := \frac{1-\beta}{\Big(2(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma+1\Big)\Psi_2}.$$

Theorem 3.4 *If $f \in \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k)$, then for $|z| = r < 1$*

$$1 - B(\alpha, \beta, \gamma, \lambda)r \le |f'(z)| \le 1 + B(\alpha, \beta, \gamma, \lambda)r\,, \tag{14}$$

where $B(\alpha, \beta, \gamma, \lambda)$ as in Theorem 3.3.

Note that in Theorem 3.3 and Theorem 3.4 equality holds for the function

$$f(z) = z - \frac{1-\beta}{\Big(2(\alpha+1) - (\alpha+\beta)\Big)\Big(\gamma+1\Big)\Psi_2} z^2.$$

4 Extreme points of the class $\mathcal{UH}(\alpha, \beta, \gamma, \lambda, k)$,

Theorem 4.1 *Let $f_1(z) = z$ and*

$$f_n(z) = z - \frac{1-\beta}{\Big(n(\alpha+1) - (\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\Psi_n} z^n, \qquad n \ge 2$$

and Ψ_n be as defined in (3)*. Then $f \in \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k)$, if and only if it can be represented in the form*

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \qquad \mu_n \ge 0, \qquad \sum_{n=1}^{\infty} \mu_n = 1. \tag{15}$$

Proof: Suppose $f(z)$ can be written as in (15). Then

$$f(z) = z - \sum_{n=2}^{\infty} \mu_n \left\{ \frac{1-\beta}{\Big(n(\alpha+1) - (\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\Psi_n} \right\} z^n.$$

Now,

$$\sum_{n=2}^{\infty} \mu_n \frac{\Big(n(\alpha+1) - (\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\Psi_n(1-\beta)}{(1-\beta)\Big(n(\alpha+1) - (\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\Psi_n} = \sum_{n=2}^{\infty} \mu_n = 1 - \mu_1 \le 1.$$

Thus $f \in \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k)$. Conversely, let us have $f \in \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k)$. Then by using (10), we may write

$$\mu_n = \frac{\Big(n(\alpha+1) - (\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\Psi_n}{1-\beta} a_n, \qquad n \ge 2,$$

and $\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n$. Then $f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z)$, with $f_n(z)$ is as in the Theorem.

Corollary 4.2 *The extreme points of* $f \in \mathcal{UH}(\alpha,\, \beta,\, \gamma,\, \lambda,\, k)$, *are the functions* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{1-\beta}{\Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\,\Psi_n} z^n, \quad n \geq 2.$$

Remark. As in earlier theorems, we can deduce known results for various other classes and we omit details.

Theorem 4.3 *The class* $\mathcal{UH}(\alpha,\, \beta,\, \gamma,\, \lambda,\, k)$ *is a convex set.*

Proof: Let the function

$$f_j(z) = \sum_{n=2}^{\infty} a_{n,\,j} z^n, \qquad a_{n,\,j} \geq 0, \qquad j = 1, 2\,, \tag{16}$$

be the class $\mathcal{UH}(\alpha,\, \beta,\, \gamma,\, \lambda,\, k)$. It sufficient to show that the function $g(z)$ defined by

$$g(z) = \mu f_1(z) + (1-\mu) f_2(z), \qquad 0 \leq \mu \leq 1,$$

is in the class $\mathcal{UH}(\alpha,\, \beta,\, \gamma,\, \lambda,\, k)$. Since

$$g(z) = z - \sum_{n=2}^{\infty} [\mu a_{n,1} + (1-\mu) a_{n,2}] z^n,$$

an easy computation with the aid of Theorem 2.1 gives,

$$\sum_{n=2}^{\infty} \Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\,\Psi_n [\mu a_{n,1} + (1-\mu) a_{n,2}]$$

$$+(1-\mu) \sum_{n=2}^{\infty} (\, n(\alpha+1)-(\alpha+\beta))(\,\gamma(n-1)+1)\Psi_n$$

$$\begin{aligned} &\leq \mu(1-\beta) + (1-\mu)(1-\beta) \\ &\leq 1-\beta, \end{aligned}$$

which implies that $g \in \mathcal{UH}(\alpha,\, \beta,\, \gamma,\, \lambda,\, k)$. Hence $\mathcal{UH}(\alpha,\, \beta,\, \gamma,\, \lambda,\, k)$ is convex.

5 Modified Hadamard products

For functions of the form (16), we define the modified Hadamard product as

$$(f_1 {*} f_2)(z) = z - \sum_{n=2}^{\infty} a_{n,\,1}\, a_{n,\,2} z^n. \tag{17}$$

Theorem 5.1 *If* $f_j(z) \in \mathcal{UH}(q,s,\lambda,\beta,k)$, $j = 1,\ 2$, *then*

$$(f_1 * f_2)(z) \in \mathcal{UH}(q,s,\lambda,\beta,k,\ \xi),$$

where

$$\xi = \frac{(2-\beta)\left(2(\alpha+1)-(\alpha+\beta)\right)\left(\gamma+1\right)\Psi_2 - 2(1-\beta)^2}{(2-\beta)\left(2(\alpha+1)-(\alpha+\beta)\right)\left(\gamma+1\right)\Psi_2 - (1-\beta)^2},$$

with Ψ_n *be defined as in* (3).

Proof: Since $f_j(z) \in \mathcal{UH}(q,s,\lambda,\beta,k)$, $j = 1,\ 2$, we have

$$\sum_{n=2}^{\infty}\left(n(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(n-1)+1\right)\Psi_n a_{n,\,j} \leq 1-\beta, \qquad j = 1,\ 2. \tag{18}$$

The Cauchy-Schwartz inequality leads to

$$\sum_{n=2}^{\infty}\frac{\left(n(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(n-1)+1\right)\Psi_n a_{n,\,j}}{1-\beta}\sqrt{a_{n,\,1}\, a_{n,\,2}} \leq 1. \tag{19}$$

Note that we need to find the largest ξ such that

$$\sum_{n=2}^{\infty}\frac{\left(n(k+1)-(k+\xi)\right)\left(\gamma(n-1)+1\right)\Psi_n a_{n,\,j}}{1-\xi}a_{n,\,1}\, a_{n,\,2} \leq 1. \tag{20}$$

Therefore, in view of (19) and (20), whenever

$$\frac{n-\xi}{1-\xi}\sqrt{a_{n,\,1}\, a_{n,\,2}} \leq \frac{n-\beta}{1-\beta},\ n \geq 2$$

holds, then (20) is satisfied. We have, from (19),

$$\sqrt{a_{n,\,1}\, a_{n,\,2}} \leq \frac{1-\beta}{\left(n(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(n-1)+1\right)\Psi_n},\ n \geq 2. \tag{21}$$

Thus, if

$$\left(\frac{n-\xi}{1-\xi}\right)\left[\frac{1-\beta}{\left(n(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(n-1)+1\right)\Psi_n}\right] \leq \frac{n-\beta}{1-\beta},\ n \geq 2,$$

or, if

$$\xi \leq \frac{(n-\beta)\left(n(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(n-1)+1\right)\Psi_n - n(1-\beta)^2}{(n-\beta)\left(n(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(n-1)+1\right)\Psi_n - (1-\beta)^2},\ n \geq 2,$$

then (19) is satisfied. Note that the right hand side of the above expression is an increasing function on n. Hence, setting $n = 2$ in the above inequality gives the required result. Finally, by taking the function

$$f(z) = z - \frac{1-\beta}{(2-\beta)\Big(2(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma+1\Big)\Psi_2} z^2,$$

we see that the result is sharp.

6 Radii of close-to-convexity, starlikeness and convexity

Theorem 6.1 *Let the function $f \in \mathcal{T}$ be in the class $\mathcal{UH}(q, s, \lambda, \beta, k)$. Then $f(z)$ is close-to-convex of order ρ, $0 \le \rho < 1$ in $|z| < r_1(\alpha, \beta, \gamma, \rho)$, where $r_1(\alpha, \beta, \gamma, \rho)$*

$$= \inf_n \left[\frac{(1-\rho)\Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\Psi_n}{n(1-\beta)} \right]^{\frac{1}{n-1}}, \qquad n \ge 2,$$

with Ψ_n be defined as in (3). This result is sharp for the function $f(z)$ given by (8).

Proof: It is sufficient to show that $|f'(z) - 1| \le 1 - \rho$, $0 \le \rho < 1$, for $|z| < r_1(\alpha, \beta, \gamma, \rho)$, or equivalently

$$\sum_{n=2}^{\infty} \left(\frac{n}{1-\rho} \right) a_n |z|^{n-1} \le 1. \tag{22}$$

By Theorem 2.1, (22) will be true if

$$\left(\frac{n}{1-\rho} \right) |z|^{n-1} \le \frac{\Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\Psi_n}{1-\beta}$$

or, if

$$|z| \le \left[\frac{(1-\rho)\Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\Psi_n}{n(1-\beta)} \right]^{\frac{1}{n-1}}, \qquad n \ge 2. \tag{23}$$

The theorem follows easily from (23).

Theorem 6.2 *Let the function $f(z)$ defined by* (1) *be in the class* $\mathcal{UH}(\alpha,\ \beta,\gamma,\ \lambda,\ k)$. *Then $f(z)$ is starlike of order ρ, $0\le\rho<1$ in $|z|<r_2(\alpha,\ \beta,\gamma,\rho)$, where* $r_2(\alpha,\ \beta,\gamma,\rho)$

$$= \inf_n \left[\frac{(1-\rho)\left(n(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(n-1)+1\right)\Psi_n}{(n-\rho)(1-\beta)}\right]^{\frac{1}{n-1}}, \qquad n\ge 2,$$

with Ψ_n be defined as in (3). *This result is sharp for the function $f(z)$ given by* (8).

Proof: It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right|\le 1-\rho, \text{or equivalently} \qquad \sum_{n=2}^{\infty}\left(\frac{n-\rho}{1-\rho}\right)a_n|z|^{n-1}\le 1, \tag{24}$$

for $0\le\rho<1$, and $|z|<r_2(\alpha,\ \beta,\gamma,\rho)$. Proceeding as in Theorem 6.1, with the use of Theorem 2.1, we get the required result. Hence, by Theorem 2.1, (24) will be true if

$$\left(\frac{n-\rho}{1-\rho}\right)|z|^{n-1}\le\frac{\{n(\alpha+1)-(\alpha+\beta)\}\{\gamma(n-1)+1\}\Psi_n}{1-\beta}$$

or, if

$$|z|\le\left[\frac{\{n(\alpha+1)-(\alpha+\beta)\}\{\gamma(n-1)+1\}\Psi_n}{(n-\rho)(1-\beta)}\right]^{\frac{1}{n-1}},\ n\ge 2. \tag{25}$$

The theorem follows easily from (25).

Theorem 6.3 *Let the function $f(z)$ defined by* (1) *be in the class* $\mathcal{UH}(\alpha,\ \beta,\gamma,\lambda,k)$. *Then $f(z)$ is convex of order ρ, $0\le\rho<1$ in $|z|<r_3(\alpha,\ \beta,\gamma,\rho)$, where*

$$r_3(\alpha,\ \beta,\gamma,\rho)=\inf_n\left[\frac{(1-\rho)\left(n(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(n-1)+1\right)\Psi_n}{n(n-\rho)(1-\beta)}\right]^{\frac{1}{n-1}},$$

$n\ge 2$, with Ψ_n be defined as in (3). *This result is sharp for the function $f(z)$ given by* (8).

Proof: It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}\right|\le 1-\rho \qquad \text{or equivalently} \qquad \sum_{n=2}^{\infty}\left(\frac{n(n-\rho)}{1-\rho}\right)a_n|z|^{n-1}\le 1, \tag{26}$$

for $0\le\rho<1$ and $|z|<r_3(\alpha,\ \beta,\gamma,\rho)$. Proceeding as in Theorem 6.1, we get the required result.

7 Appendix

We use $r_1(\alpha,\beta,\gamma,\rho)$ in Theorem 6.1 and $r_2(\alpha,\beta,\gamma,\rho)$ in Theorem 6.2. If we consider $k=0,\alpha=0,\beta=0$, and $\rho=0$ for $r_1(\alpha,\beta,\gamma,\rho)$ and $r_2(\alpha,\beta,\gamma,\rho)$, then $\Psi_n=n$ and

$$r_1=r_2=(n(\gamma(n-1)+1))^{\frac{1}{n-1}} \quad (n\geq 2).$$

Let

$$h(n)=(n(\gamma(n-1)+1))^{\frac{1}{n-1}} \quad (n\geq 2).$$

Then we have that

$$h(2)=2(1+\gamma),\ h(3)=(3(1+2\gamma))^{\frac{1}{2}},\ h(4)=(4(1+3\gamma))^{\frac{1}{3}},$$

and

$$h(5)=(5(1+4\gamma))^{\frac{1}{4}}.$$

This gives us that

$$h(2)^2-h(3)^2=1+2\gamma+4\gamma^2>0,$$

$$h(3)^6-h(4)^6=11+66\gamma+180\gamma^2+216\gamma^3>0,$$

and

$$h(4)^{12}-h(5)^{12}=131+1572\gamma+7824\gamma^2+19648\gamma^3+20736\gamma^4>0.$$

Thus, we see that

$$h(2)>h(3)>h(4)>h(5)>\cdots.$$

Therefore, if

$$f(z)=z+\sum_{n=2}^{p}a_nz^n,$$

then we expect that

$$r_1(\alpha,\,\beta,\gamma,\rho)=\left[\frac{(1-\rho)\left(p(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(p-1)+1\right)\Psi_p}{p(1-\beta)}\right]^{\frac{1}{p-1}},$$

and

$$r_2(\alpha,\,\beta,\gamma,\rho)=\left[\frac{(1-\rho)\left(p(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(p-1)+1\right)\Psi_p}{(p-\rho)(1-\beta)}\right]^{\frac{1}{p-1}}.$$

Similarly, letting $k=0,\alpha=0,\beta=0$, and $\rho=0$ in Theorem 6.3, we have that

$$r_3(\alpha,\beta,\gamma,\rho)=(\gamma(n-1)+1)^{\frac{1}{n-1}} \quad (n\geq 2).$$

Thus, letting

$$g(n) = (\gamma(n-1)+1)^{\frac{1}{n-1}} \ \ (n \geq 2),$$

we see that

$$g(2) = 1+\gamma, \ \ g(3) = (1+2\gamma)^{\frac{1}{2}}, \ \ g(4) = (1+3\gamma)^{\frac{1}{3}},$$

and

$$g(5) = (1+4\gamma)^{\frac{1}{4}}.$$

It follows from the above that

$$g(2) > g(3) > g(4) > g(5) > \ldots.$$

Therefore, if

$$f(z) = z + \sum_{n=2}^{p} a_n z^n,$$

then we expect that

$$r_3(\alpha,\, \beta, \gamma, \rho) = \left[\frac{(1-\rho)\Big(p(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(p-1)+1\Big)\,\Psi_p}{p(p-\rho)(1-\beta)} \right]^{\frac{1}{p-1}}.$$

But, as we know in general, it is not so easy to consider the exact values for $r_1(\alpha,\beta,\gamma,\rho), r_2(\alpha,\beta,\gamma,\rho)$, and $r_3(\alpha,\beta,\gamma,\rho)$. Thus we leave this problem as our open question.

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