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A Note On a Subclass of Analytic Functions Defined By Ruscheweyh Derivative and Multiplier Transformations

Alina Alb Lupaş

Department of Mathematics and Computer Science
University of Oradea
1 Universitatii Street, 410087 Oradea, Romania.
e-mail: dalb@uoradea.ro

Abstract

Let $\mathcal{A}(p,n) = \{f \in \mathcal{H}(U) : \ f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, \ \ z \in U\}$, ***with*** $\mathcal{A}(1,n) = \mathcal{A}_n$, $n \in N$. ***We consider in this paper the operator*** $RI^{\gamma}(m,\lambda,l) : \mathcal{A}_n \to \mathcal{A}_n$, ***defined by***
$RI^{\gamma}(m,\lambda,l)f(z) := (1-\gamma)\, R^m f(z) + \gamma I(m,\lambda,l)f(z)$ ***where***
$I(m,\lambda,l)f(z) = z + \sum_{j=n+1}^{\infty} \left[\frac{1+\lambda(j-1)+l}{l+1}\right]^m a_j z^j$ ***and***
$(m+1)R^{m+1}f(z) = z(R^m f(z))' + mR^m f(z)$, $\ m \in N_0$, $N_0 = N \cup \{0\}, \lambda \in R$, $\lambda \geq 0$, $l \geq 0$ ***is the Ruscheweyh operator. By making use of the above mentioned differential operator, a new subclass of univalent functions in the open unit disc is introduced. The new subclass is denoted by*** $\mathcal{RI}^{\gamma}(m,n,\mu,\alpha,\lambda,l)$. ***Parallel results, for some related classes including the class of starlike and convex functions respectively, are also obtained.***

Keywords: *Analytic function, starlike function, convex function, Ruscheweyh derivative, multiplier transformations.*

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1 Introduction and definitions

Denote by U the unit disc of the complex plane, $U = \{z \in C : \ |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let

$$\mathcal{A}(p,n) = \{f \in \mathcal{H}(U) : \ f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, \quad z \in U\},$$

with $\mathcal{A}(1,n) = \mathcal{A}_n$ and

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, \ z \in U\},$$

where $p, n \in N$, $a \in C$.

Let $\mathcal{S}$ denote the subclass of functions that are univalent in U.

By $\mathcal{S}^*(\alpha)$ we denote a subclass of $\mathcal{A}_n$ consisting of starlike univalent functions of order α, $0 \leq \alpha < 1$ which satisfies

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U. \tag{1.1}$$

Further, a function f belonging to $\mathcal{S}$ is said to be convex of order α in U, if and only if

$$Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha, \quad z \in U \tag{1.2}$$

for some α, $(0 \leq \alpha < 1)$. We denote by $\mathcal{K}(\alpha)$ the class of functions in $\mathcal{S}$ which are convex of order α in U and denote by $\mathcal{R}(\alpha)$ the class of functions in $\mathcal{A}_n$ which satisfy

$$Re f'(z) > \alpha, \quad z \in U. \tag{1.3}$$

It is well known that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}$.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Definition 1.1 *[6] For $f \in \mathcal{A}(p,n)$, $p, n \in N$, $m \in N \cup \{0\}$, $\lambda, l \geq 0$, the operator $I_p(m,\lambda,l) f(z)$ is defined by the following infinite series*

$$I_p(m,\lambda,l) f(z) := z^p + \sum_{j=p+n}^{\infty} \left(\frac{p + \lambda(j-1) + l}{p+l}\right)^m a_j z^j.$$

Remark 1.2 *It follows from the above definition that*

$$I_p(0,\lambda,l) f(z) = f(z),$$

$$(p+l) I_p(m+1,\lambda,l) f(z) = [p(1-\lambda)+l] I_p(m,\lambda,l) f(z) + \lambda z (I_p(m,\lambda,l) f(z))',$$

for $z \in U$.

Remark 1.3 *If* $p = 1$ *we have* $I_1(m, \lambda, l) f(z) = I(m, \lambda, l)$ *and*

$$(l+1) I(m+1, \lambda, l) f(z) = [l+1-\lambda] I(m, \lambda, l) f(z) + \lambda z (I(m, \lambda, l) f(z))',$$

for $z \in U$.

Remark 1.4 *If* $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, *then*

$$I(m, \lambda, l) f(z) = z + \sum_{j=n+1}^{\infty} \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^m a_j z^j,$$

for $z \in U$.

Remark 1.5 *For* $l = 0$, $\lambda \geq 0$, *the operator* $D_\lambda^m = I(m, \lambda, 0)$ *was introduced and studied by Al-Oboudi* [5], *which reduced to the Sălăgean differential operator* $S^m = I(m, 1, 0)$ [14] *for* $\lambda = 1$. *The operator* $I_l^m = I(m, 1, l)$ *was studied recently by Cho and Srivastava* [9] *and Cho and Kim* [10]. *The operator* $I_m = I(m, 1, 1)$ *was studied by Uralegaddi and Somanatha* [15], *the operator* $D_\lambda^\delta = I(\delta, \lambda, 0)$, *with* $\delta \in R$, $\delta \geq 0$, *was introduced by Acu and Owa* [1].

Definition 1.6 [13] *Ruscheweyh has defined the operator* $R^m : \mathcal{A}_n \to \mathcal{A}_n$,

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z) \\ (m+1) R^{m+1} f(z) &= z [R^m f(z)]' + m R^m f(z), \quad z \in U. \end{aligned}$$

To prove our main theorem we shall need the following lemma.

Lemma 1.7 [12] *Let* u *be analytic in* U *with* $u(0) = 1$ *and suppose that*

$$Re \left(1 + \frac{z u'(z)}{u(z)} \right) > \frac{3\alpha - 1}{2\alpha}, \quad z \in U. \tag{1.4}$$

Then $Re u(z) > \alpha$ *for* $z \in U$ *and* $1/2 \leq \alpha < 1$.

2 Main results

Definition 2.1 *For a function* $f \in \mathcal{A}_n$ *we define the differential operator*

$$RI^\gamma(m, \lambda, l) f(z) = (1-\gamma) R^m f(z) + \gamma I(m, \lambda, l) f(z), \tag{2.5}$$

where $m, n \in N_0$, $N_0 = N \cup \{0\}$, $\lambda \in R$, $\lambda \geq 0$, $\gamma \geq 0$, $l \geq 0$.

Remark 2.2 *For* $l = 0$ *the above defined operator was introduced in* [4] *and for* $l = 0$ *and* $\lambda = 1$ *the operator was introduced in* [3].

Definition 2.3 *We say that a function $f \in \mathcal{A}_n$ is in the class $\mathcal{RI}^{\gamma}(m,n,\mu,\alpha,\lambda,l)$, $m,n \in N$, $\mu \geq 0$, $\alpha \in [0,p)$, $\gamma \geq 0$ if*

$$\left| \frac{RI^{\gamma}(m+1,\lambda,l)f(z)}{z} \left(\frac{z}{RI^{\gamma}(m,\lambda,l)f(z)} \right)^{\mu} - 1 \right| < 1-\alpha, \qquad z \in U. \quad (2.6)$$

Remark 2.4 *The family $\mathcal{RI}^{\gamma}(m,n,\mu,\alpha,\lambda,l)$ is a new comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, $\mathcal{RI}^{1}(m,n,\mu,\alpha,\lambda,l)$ was studied in [7], $\mathcal{RI}^{1}(0,1,1,\alpha,1,0) = \mathcal{S}^{*}(\alpha)$, $\mathcal{BI}^{1}(1,1,1,\alpha,1,0) = \mathcal{K}(\alpha)$ and $\mathcal{BI}^{1}(0,1,0,\alpha,1,0) = \mathcal{R}(\alpha)$. Another interesting subclass is the special case $\mathcal{RI}^{1}(0,1,2,\alpha,1,l)=\mathcal{B}(\alpha)$ which has been introduced by Frasin and Darus [11] and also the class $\mathcal{RI}^{1}(0,1,\mu,\alpha,1,0) = \mathcal{B}(\mu,\alpha)$ which has been introduced by Frasin and Jahangiri [12].*

In this note we provide a sufficient condition for functions to be in the class $\mathcal{RI}^{\gamma}(m,n,\mu,\alpha,\lambda,l)$. Consequently, as a special case, we show that convex functions of order $1/2$ are also members of the above defined family.

Theorem 2.5 *For the function $f \in \mathcal{A}_n$, $m,n \in N$, $\mu \geq 0$, $1/2 \leq \alpha < 1$ if*

$$\frac{(m+2)RI^{\gamma}(m+2,\lambda,l)f(z)}{RI^{\gamma}(m+1,\lambda,l)f(z)} - \mu(m+1)\frac{RI^{\gamma}(m+1,\lambda,l)f(z)}{RI^{\gamma}(m,\lambda,l)f(z)} + \quad (2.7)$$

$$\gamma\left(\frac{l+1}{\lambda} - m - 2\right)\frac{I(m+2,\lambda,l)f(z)}{RI^{\gamma}(m+1,\lambda,l)f(z)} +$$

$$\gamma\mu\left(\frac{l+1}{\lambda} - m - 1\right)\frac{I(m+1,\lambda,l)f(z)}{RI^{\gamma}(m,\lambda,l)f(z)} -$$

$$\gamma\left[\frac{l+1}{\lambda} - m - 2\right]\frac{I(m+1,\lambda,l)f(z)}{RI^{\gamma}(m+1,\lambda,l)f(z)} +$$

$$\gamma\mu\left[\frac{l+1}{\lambda} - m - 1\right]\frac{I(m,\lambda,l)f(z)}{RI^{\gamma}(m,\lambda,l)f(z)} + (m+1)(\mu-1) \prec 1+\beta z, \quad z \in U,$$

where

$$\beta = \frac{3\alpha - 1}{2\alpha},$$

then $f \in \mathcal{RI}^{\gamma}(m,n,\mu,\alpha,\lambda,l)$.

Proof If we consider

$$u(z) = \frac{RI^{\gamma}(m+1,\lambda,l)\,f(z)}{z}\left(\frac{z}{RI^{\gamma}(m,\lambda,l)\,f(z)}\right)^{\mu}, \quad (2.8)$$

then $u(z)$ is analytic in U with $u(0)=1$. Taking into account the relation

$$(l+1)I(m+1,\lambda,l)f(z)=(1-\lambda+l)\,I(m,\lambda,l)f(z)+\lambda z\left(I(m,\lambda,l)f(z)\right)',$$

a simple differentiation yields

$$\frac{zu'(z)}{u(z)}=\frac{(m+2)RI^{\gamma}(m+2,\lambda,l)f(z)}{RI^{\gamma}(m+1,\lambda,l)f(z)}-\mu(m+1)\frac{RI^{\gamma}(m+1,\lambda,l)f(z)}{RI^{\gamma}(m,\lambda,l)f(z)}+ \tag{2.9}$$

$$\gamma\left(\frac{l+1}{\lambda}-m-2\right)\frac{I(m+2,\lambda,l)f(z)}{RI^{\gamma}(m+1,\lambda,l)f(z)}+$$

$$\gamma\mu\left(\frac{l+1}{\lambda}-m-1\right)\frac{I(m+1,\lambda,l)f(z)}{RI^{\gamma}(m,\lambda,l)f(z)}-$$

$$\gamma\left[\frac{l+1}{\lambda}-m-2\right]\frac{I(m+1,\lambda,l)f(z)}{RI^{\gamma}(m+1,\lambda,l)f(z)}+$$

$$\gamma\mu\left[\frac{l+1}{\lambda}-m-1\right]\frac{I(m,\lambda,l)f(z)}{RI^{\gamma}(m,\lambda,l)f(z)}+(m+1)(\mu-1)-1.$$

Using (2.7) we get

$$Re\left(1+\frac{zu'(z)}{u(z)}\right)>\frac{3\alpha-1}{2\alpha}.$$

Thus, from Lemma 1.7 we deduce that

$$Re\left\{\frac{RI^{\gamma}(m+1,\lambda,l)f(z)}{z}\left(\frac{z}{RI^{\gamma}(m,\lambda,l)f(z)}\right)^{\mu}\right\}>\alpha.$$

Therefore, $f\in\mathcal{RI}^{\gamma}(m,n,\mu,\alpha,\lambda,l)$, by Definition 2.3.

As a consequence of the above theorem we have the following interesting corollaries [2].

Corollary 2.6 *If* $f\in\mathcal{A}_1$ *and*

$$Re\left\{\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right\}>-\frac{1}{2},\quad z\in U, \tag{2.10}$$

then

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\frac{1}{2},\quad z\in U. \tag{2.11}$$

That is, f *is convex of order* $\frac{1}{2}$, *or* $f\in\mathcal{RI}^{1}\left(1,1,1,\frac{1}{2},1,0\right)$.

Corollary 2.7 *If $f \in \mathcal{A}_1$ and*

$$Re\left\{\frac{2zf'(z) + z^2 f''(z)}{f'(z) + zf''(z)}\right\} > -\frac{1}{2}, \quad z \in U, \tag{2.12}$$

then $f \in \mathcal{RI}^1\left(1,1,0,\frac{1}{2},1,0\right)$, that is

$$Re\{f'(z) + zf''(z)\} > \frac{1}{2}, \quad z \in U. \tag{2.13}$$

Corollary 2.8 *If $f \in \mathcal{A}_1$ and*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2}, \quad z \in U, \tag{2.14}$$

then

$$Ref'(z) > \frac{1}{2}, \quad z \in U. \tag{2.15}$$

In another words, if the function f is convex of order $\frac{1}{2}$, then $f \in \mathcal{RI}^1(0,1,0,\frac{1}{2}, 1,0) \equiv \mathcal{R}\left(\frac{1}{2}\right)$.

Corollary 2.9 *If $f \in \mathcal{A}_1$ and*

$$Re\left\{\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} > -\frac{3}{2}, \quad z \in U, \tag{2.16}$$

then f is starlike of order $\frac{1}{2}$, hence $f \in \mathcal{RI}^1(0,1,1,\frac{1}{2},1,0)$.

3 Open Problem

The open problem is to define a generic class of analytic functions such that the class $\mathcal{RI}^\gamma(m,n,\mu,\alpha,\lambda,l)$, $m,n \in N$, $\mu \geq 0$, $\alpha \in [0,p)$, $\gamma \geq 0$ is contained inside and is possible to obtain. Compare the new results with the results given by [12] and [8].

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