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Some Properties For An Integral Operator On The $\mathcal{CVH}(\beta)$-Class

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Abstract

In this paper we prove some properties for a general integral operator on the $\mathcal{CVH}(\beta)$-class of convex functions associated with some hyperbola.

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1 Introduction

Consider $\mathcal{H}(\mathcal{U})$ be the set of functions which are regular in the unit disc

$$\mathcal{U} = \{z \in C\,,\ |z| < 1\}\,,$$

$\mathcal{A}$ denotes the class of the functions $f(z)$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots\,,\ z \in \mathcal{U}$$

with the property $f(0) = f'(0) - 1 = 0$ and $\mathcal{S} = \{f \in \mathcal{A} : f\ is\ univalent\ in\ \mathcal{U}\}$.

A function $f \in \mathcal{S}$ is convex function of order $\alpha, 0 \leq \alpha < 1$ and denote this class by $\mathcal{K}(\alpha)$ if f satisfies the inequality

$$\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}+1\right\}>\alpha, z\in\mathcal{U}.$$

We recall here the definition of the class $\mathcal{CVH}(\beta)$ introduced by Acu and Owa in [1].

The function $f\in\mathcal{A}$ is in the class $\mathcal{CVH}(\beta)$, $\beta>0$, if

$$\left|\frac{zf''(z)}{f'(z)}-2\beta\left(\sqrt{2}-1\right)+1\right|<\mathbf{Re}\left\{\sqrt{2}\frac{zf''(z)}{f'(z)}\right\}+2\beta\left(\sqrt{2}-1\right)+\sqrt{2}\,,\ z\in\mathcal{U}\,. \tag{1.1}$$

Geometric interpretation: $f\in\mathcal{CVH}(\beta)$ if and only if $\frac{zf''(z)}{f'(z)}+1$ take all values in the convex domain $\Omega(\beta)=\{w=u+i\cdot v\,:\,v^2<4\beta u+u^2\,,\,u>0\}$. Note that $\Omega(\beta)$ is the interior of a hyperbola in the right half-plane which is symmetric about the real axis and has vertex at the origin.

Regarding the class $\mathcal{CVH}(\beta)$ we recall the coefficient estimations obtained in [1]:

Theorem 1.1. *If* $f(z)=z+\sum\limits_{j=2}^{\infty}a_jz^j$ *belongs to the class* $\mathcal{CVH}(\beta)$, $\beta>0$, *then*

$$|a_2|\le\frac{1+4\beta}{2(1+2\beta)}\ ,\ |a_3|\le\frac{(1+4\beta)(3+16\beta+24\beta^2)}{12(1+2\beta)^3}\,.$$

We consider the integral operator defined by

$$F_{\gamma_1,\dots,\gamma_n}(z)=\int_0^z\left[f_1'(t)\right]^{\gamma_1}\cdot\ldots\cdot\left[f_n'(t)\right]^{\gamma_n}dt \tag{1.2}$$

where $f_i\in\mathcal{A}$ and $\gamma_i>0$, for all $i\in\{1,\dots,n\}$. This operator was introduced by Breaz, Owa and Breaz in [2].

2 Main results

Theorem 2.1. *If* $f_i\in\mathcal{CVH}(\beta_i)$, $\beta_i>0$, $\gamma_i>0$, *for all* $i\in\{1,\dots,n\}$, *then the integral operator defined in (2) is in the class* $\mathcal{K}(\alpha)$, *where* $\alpha=1-\sum\limits_{i=1}^{n}\gamma_i-\left(2-\sqrt{2}\right)\sum\limits_{i=1}^{n}\gamma_i\beta_i$, $0\le\alpha<1$.

Proof. After some simple calculus, we have:

$$\frac{zF''_{\gamma_1,...,\gamma_n}(z)}{F'_{\gamma_1,...,\gamma_n}(z)} = \gamma_1 \frac{zf''_1(z)}{f'_1(z)} + ... + \gamma_n \frac{zf''_n(z)}{f'_n(z)} = \sum_{i=1}^{n} \gamma_i \frac{zf''_i(z)}{f'_i(z)}.$$

Since $f_i \in \mathcal{CVH}(\beta_i)$, for all $i \in \{1, ..., n\}$ we have that every function f_i satisfies the inequality (1).

Thus, we obtain:

$$\sqrt{2}\frac{zF''_{\gamma_1,...,\gamma_n}(z)}{F'_{\gamma_1,...,\gamma_n}(z)} = \sqrt{2}\sum_{i=1}^{n} \gamma_i \frac{zf''_i(z)}{f'_i(z)},$$

$$\sqrt{2}\frac{zF''_{\gamma_1,...,\gamma_n}(z)}{F'_{\gamma_1,...,\gamma_n}(z)} =$$

$$= \sum_{i=1}^{n}\left(\sqrt{2}\gamma_i \frac{zf''_i(z)}{f'_i(z)} + 2\gamma_i\beta_i\left(\sqrt{2}-1\right) + \sqrt{2}\gamma_i - 2\gamma_i\beta_i\left(\sqrt{2}-1\right) - \sqrt{2}\gamma_i\right),$$

$$\sqrt{2}\mathbf{Re}\left(\frac{zF''_{\gamma_1,...,\gamma_n}(z)}{F'_{\gamma_1,...,\gamma_n}(z)} + 1\right) =$$

$$= \mathbf{Re}\sum_{i=1}^{n}\left(\sqrt{2}\gamma_i \frac{zf''_i(z)}{f'_i(z)} + 2\gamma_i\beta_i\left(\sqrt{2}-1\right) + \sqrt{2}\gamma_i - 2\gamma_i\beta_i\left(\sqrt{2}-1\right) - \sqrt{2}\gamma_i\right) + \sqrt{2},$$

$$\sqrt{2}\mathbf{Re}\left(\frac{zF''_{\gamma_1,...,\gamma_n}(z)}{F'_{\gamma_1,...,\gamma_n}(z)} + 1\right) =$$

$$= \sum_{i=1}^{n}\gamma_i\left[\mathbf{Re}\left\{\sqrt{2}\frac{zf''_i(z)}{f'_i(z)}\right\} + 2\beta_i\left(\sqrt{2}-1\right) + \sqrt{2}\right] -$$

$$-\sum_{i=1}^{n}\left(2\gamma_i\beta_i\left(\sqrt{2}-1\right) + \sqrt{2}\gamma_i\right) + \sqrt{2} >$$

$$\sum_{i=1}^{n}\gamma_i\left|\frac{zf''_i(z)}{f'_i(z)} - 2\beta_i\left(\sqrt{2}-1\right) + 1\right| - \sum_{i=1}^{n}\left(2\gamma_i\beta_i\left(\sqrt{2}-1\right) + \gamma_i\sqrt{2}\right) + \sqrt{2} >$$

$$> -\sqrt{2}\sum_{i=1}^{n}\gamma_i - 2\left(\sqrt{2}-1\right)\sum_{i=1}^{n}\gamma_i\beta_i + \sqrt{2}.$$

Thus, we obtain:

$$\mathbf{Re}\left(\frac{zF''_{\gamma_1,\dots,\gamma_n}(z)}{F'_{\gamma_1,\dots,\gamma_n}(z)}+1\right) > 1-\sum_{i=1}^{n}\gamma_i-\left(2-\sqrt{2}\right)\sum_{i=1}^{n}\gamma_i\beta_i$$

which implies that $F_{\gamma_1,\dots,\gamma_n}(z)\in\mathcal{K}(\alpha)$, where $\alpha=1-\sum\limits_{i=1}^{n}\gamma_i-\left(2-\sqrt{2}\right)\sum\limits_{i=1}^{n}\gamma_i\beta_i$.

If we consider $\beta_1=\beta_2=\cdots=\beta_n=\beta>0$ in the above theorem we obtain:

Corollary 2.1. *If $f_i\in\mathcal{CVH}(\beta)$, $\beta>0$ and $\gamma_i>0$, for all $i\in\{1,...,n\}$, then the integral operator defined in (2) is in the class $\mathcal{K}(\alpha)$, where $\alpha=1-\left[1+\beta(2-\sqrt{2})\right]\sum\limits_{i=1}^{n}\gamma_i$, $0\le\alpha<1$.*

If we consider $n=1$ in the Theorem 2.1 we obtain:

Corollary 2.2. *If $f\in\mathcal{CVH}(\beta)$, $\beta>0$ and $\gamma>0$, then the integral operator defined by $F_\gamma(z)=\int\limits_0^z[f'(t)]^\gamma\,dt$ is in the class $\mathcal{K}(\alpha)$, where $\alpha=1-\gamma-(2-\sqrt{2})\gamma\beta$, $0\le\alpha<1$.*

If in (2) we consider $\gamma_1=\gamma_2=\cdots=\gamma_n=1$ we obtain the integral operator

$$F_1(z)=\int_0^z f_1'(t)\cdot\ldots\cdot f_n'(t)dt\,. \tag{2.3}$$

For this integral operator we have:

Corollary 2.3. *If $f_i\in\mathcal{CVH}(\beta_i)$, $\beta_i>0$ for all $i\in\{1,...,n\}$, then the integral operator defined by F_1 is in the class $\mathcal{K}(\alpha)$, $0\le\alpha<1$, where $\alpha=1-n-\left(2-\sqrt{2}\sum\limits_{i=1}^{n}\beta_i\right)$.*

Also for the operator defined in (3) we have the following estimations for the coefficients:

Theorem 2.2. *Let $f_i\in\mathcal{CVH}(\beta_i)$, $\beta_i>0$, for all $i\in\{1,...,n\}$, $f_i(z)=z+\sum\limits_{j=2}^{\infty}a_{i,j}z^j$, $i=\overline{1,n}$. If we consider the integral operator defined by (3), with $F_1(z)=z+\sum\limits_{j=2}^{\infty}b_jz^j$, then:*

$$|b_2|\le\frac{1}{2}\sum_{i=1}^{n}\frac{1+4\beta_i}{(1+2\beta_i)}$$

and

$$|b_3| \le \sum_{i=1}^{n} \frac{(1+4\beta_i)(3+16\beta_i+24\beta_i^2)}{12(1+2\beta_i)^3} + \frac{1}{3}\sum_{k=1}^{n-1}\left(\frac{1+4\beta_k}{(1+2\beta_k)} \cdot \sum_{i=k+1}^{n} \frac{1+4\beta_i}{(1+2\beta_i)}\right).$$

Proof. From (3) we obtain

$$F_1'(z) = f_1'(z) \cdot f_2'(z) \cdots f_n'(z),$$

namely

$$1+\sum_{j=2}^{\infty} j b_j z^{j-1} = \left(1+\sum_{j=2}^{\infty} j a_{1,j} \cdot z^{j-1}\right)\cdot\left(1+\sum_{j=2}^{\infty} j a_{2,j} \cdot z^{j-1}\right)\cdots\left(1+\sum_{j=2}^{\infty} j a_{n,j} \cdot z^{j-1}\right).$$

Thus we have:

$$b_2 = \sum_{i=1}^{n} a_{i,2}$$

$$b_3 = \sum_{i=1}^{n} a_{i,3} + \frac{4}{3}\sum_{k=1}^{n-1}\left(a_{k,2}\sum_{i=k+1}^{n} a_{i,2}\right).$$

But, from Theorem 1.1, we have

$$|a_{i,2}| \le \frac{1+4\beta_i}{2(1+2\beta_i)}\ ,\ i=\overline{1,n}$$

$$|a_{i,3}| \le \frac{(1+4\beta_i)(3+16\beta_i+24\beta_i^2)}{12(1+2\beta_i)^3}\ ,\ i=\overline{1,n}$$

In these conditions we obtain

$$|b_2| \le \sum_{i=1}^{n} |a_{i,2}| \le \frac{1}{2}\sum_{i=1}^{n}\frac{1+4\beta_i}{1+2\beta_i}$$

$$|b_3| \le \sum_{i=1}^{n} |a_{i,3}| + \frac{4}{3}\sum_{k=1}^{n-1}\left(|a_{k,2}|\sum_{i=k+1}^{n} |a_{i,2}|\right)$$

$$\le \sum_{i=1}^{n} \frac{(1+4\beta_i)(3+16\beta_i+24\beta_i^2)}{12(1+2\beta_i)^3} + \frac{1}{3}\sum_{k=1}^{n-1}\left(\frac{1+4\beta_k}{1+2\beta_k}\sum_{i=k+1}^{n}\frac{1+4\beta_i}{1+2\beta_i}\right)$$

For $\beta_1 = \beta_2 = \cdots = \beta_n = \beta > 0$, we obtain

Corollary 2.4. *Let* $f_i \in \mathcal{CVH}(\beta)$, $\beta > 0$, $f_i(z) = z + \sum_{j=2}^{\infty} a_{i,j} z^j$, $i = \overline{1, n}$. *If we consider the integral operator defined by (3), with* $F_1(z) = z + \sum_{j=2}^{\infty} b_j z^j$, *then:*

$$|b_2| \leq \frac{n(1+4\beta)}{2(1+2\beta)}$$

$$|b_3| \leq \frac{n(1+4\beta) \cdot [2n+1+4(3n+1)\beta + 8(2n+1)\beta^2]}{12(1+2\beta)^3} .$$

An open problem. Similar results can be obtained using other integral operators.

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