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Subordination and Superordination Results of Non-Bazilevič Functions Involving Dziok-Srivastava Operator

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Abstract

The purpose of this paper is to derive subordination and superordination results involving Dziok-Srivastava operator for a family of analytic multivalent functions in the open unit disk. These results are applied to obtain sandwich results. Some results which are useful in geometric function theory are also obtained as special cases of the results presented in this paper.

Keywords: *Differential subordination, Differential superordination, Dominant, Subordinant.*

1 Introduction

Let $\mathcal{H}$ denote the class of functions analytic in the open unit disk $\Delta := \{z \in \mathcal{C} : \ |z| < 1\}$, and let $\mathcal{H}[a, p]$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + ..., \ \ p \in \mathbf{N} = \{1, 2, 3, ...\}\,. \tag{1.1}$$

Let $\mathcal{A}(p)$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \ \ p \in \mathbf{N}. \tag{1.2}$$

If f and g are analytic in Δ and there exists a *Schwarz function* $w(z)$, analytic in Δ with

$$w(0)=0, \ \ |w(z)|<1, \qquad z \in \Delta, \tag{1.3}$$

such that $f(z)=g(w(z))$, then the function f is called *subordinate* to g and is denoted by

$$f \prec g \ or \ f(z) \prec g(z), \qquad z \in \Delta. \tag{1.4}$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0)=g(0), \quad f(\Delta) \subset g(\Delta). \tag{1.5}$$

Suppose h and k are analytic functions in Δ and $\phi\,(r,s,t;z): \mathcal{C}^3 \times \Delta \to \mathcal{C}$. If h and $\phi(h(z), zh'(z), z^2h''(z); z)$ are univalent and if h satisfies the second-order *differential superordination*

$$k(z) \prec \phi(h(z), zh'(z), z^2h''(z); z), \tag{1.6}$$

then h is a solution of the differential superordination (1.6). Note that if f is subordinate to g, then g is superordinate to f. An analytic function q is called a *subordinant* if $q \prec h$ for all h satisfying (1.6). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.6) is said to be the *best subordinant*. Miller and Mocanu [8] obtained conditions on k, q and ϕ for which the following implication holds:

$$k(z) \prec \phi(h(z), zh'(z), z^2h''(z); z) \ \Rightarrow \ q(z) \prec h(z). \tag{1.7}$$

Ali et al. [1] have obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z), \tag{1.8}$$

where q_1 and q_2 are given univalent functions in Δ with $q_1(0)=1$ and $q_2(0)=1$.

Shanmugam et al. [11,12] and Goyal et al. [6] have obtained sandwich results for certain classes of analytic functions. Further subordination results can be found in [14-16].

Obradović [9] introduced a class of functions $f \in \mathcal{A}(1)=\mathcal{A}$, such that, for $0<\mu<1$,

$$Re\left\{f'(z)\left(\frac{z}{f(z)}\right)^{\mu}\right\}>0, \quad z \in \Delta. \tag{1.9}$$

He called this class of functions as "non-Bazilevič" type.

We consider a class $\mathcal{N}(\mu,p,\lambda;A,B)$ defined as

$$\mathcal{N}(\mu,p,\lambda;A,B)=\left\{f \in \mathcal{A}(p):(1+\lambda)\left(\frac{z^p}{f(z)}\right)^{\mu}-\lambda\frac{zf'(z)}{pf(z)}\left(\frac{z^p}{f(z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz}\right\}, \tag{1.10}$$

where $\lambda \in \mathcal{C},\ -1 \leq B < A \leq 1,\ 0 < \mu < 1,\ p \in N$.

Wang et al. [16] studied many subordination results using the above class for $p = 1$.

2 Definition and preliminaries

Definition 2.1. Corresponding to the function

$$h_p(\alpha_1, \alpha_2, ..., \alpha_l; \beta_1, \beta_2, ..., \beta_m; z) = z^p\ {}_l\mathcal{F}_m(\alpha_1, \alpha_2, ..., \alpha_l; \beta_1, \beta_2, ..., \beta_m; z) \quad (2.1)$$

$$(l \leq m+1;\ l, m \in N_0 = N \cup \{0\}, \alpha_i \in \mathcal{C}, \beta_j \in \mathcal{C} \backslash Z_0^- = \{0, -1, -2, ...\} \\ ; 1 \leq i \leq l, 1 \leq j \leq m)$$

(where ${}_l\mathcal{F}_m$ is the well-known generalized hypergeometric function), and $f \in \mathcal{A}(p)$, Dziok and Srivastava [4] studied a linear operator $(\mathcal{H}_p^{(l,m)}[\alpha_1]f)(z)$ defined in terms of the Hadamard product as

$$\begin{aligned}(\mathcal{H}_p^{(l,m)}[\alpha_1]f)(z) &= h_p(\alpha_1, \alpha_2, ..., \alpha_l; \beta_1, \beta_2, ..., \beta_m; z) * f(z) \\ &= z^p + \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p}(\alpha_2)_{k-p}...(\alpha_l)_{k-p}}{(\beta_1)_{k-p}(\beta_2)_{k-p}...(\beta_m)_{k-p}} \frac{a_k}{(k-p)!} z^k, \quad z \in \Delta.\end{aligned} \quad (2.2)$$

under the conditions mentioned with (2.1).
We observe that

$$z[(\mathcal{H}_p^{(l,m)}[\alpha_1]f)(z)]' = \alpha_1(\mathcal{H}_p^{(l,m)}[\alpha_1+1]f)(z) - (\alpha_1 - p)(\mathcal{H}_p^{(l,m)}[\alpha_1]f)(z). \quad (2.3)$$

Differentiating (2.3), $(j-1)$ times, we get

$$z[(\mathcal{H}_p^{(l,m)}[\alpha_1]f)(z)]^{(j)} = \alpha_1[(\mathcal{H}_p^{(l,m)}[\alpha_1+1]f)(z)]^{(j-1)} - (\alpha_1 - p + j - 1)[(\mathcal{H}_p^{(l,m)}[\alpha_1]f)(z)]^{(j-1)}. \quad (2.4)$$

The Dziok-Srivastava linear operator [4], is a generalization of a number of well-known operators such as the Hohlov linear operator, Saitoh's generalized linear operator, the Carlson-Shaffer linear operator, the Ruscheweyh derivative operator as well as its generalized versions, the Bernardi-Libera-Livingston operator and Srivastava-Owa fractional derivative operator. For details, one may refer to the papers cited above. The Dziok-Srivastava linear operator defined by (2.2) was further extended by Dziok and Raina [5] and also studied by Darus et al. [3].

Definition 2.2 *(see [8]). Denote by* $\boldsymbol{Q}$*, the set of all functions* $f(z)$ *that are analytic and injective on* $\overline{\Delta} - E(f)$*, where*

$$E(f) = \left\{ \xi \in \partial\Delta : \quad \lim_{z \to \xi} f(z) = \infty \right\}, \quad (2.5)$$

and are such that $f'(\xi) \neq 0$ *for* $\xi \in \partial\Delta - E(f)$.

We will require certain results due to Miller and Mocanu [7,8], Bulboacă [2], and Shanmugum et al. [11] contained in the following lemmas:

Lemma 2.3 *(see [7]). Let* $q(z)$ *be univalent in the unit disk* Δ, *and let* ϑ *and* ϕ *be analytic in the domain* D *containing* $q(\Delta)$ *with* $\phi(w) \neq 0$ *when* $w \in q(\Delta)$. *Set* $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \vartheta(q(z)) + Q(z)$. *Suppose that*

(i) $Q(z)$ *is starlike univalent in* Δ.

(ii) $Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ *for* $z \in \Delta$.

If $r(z)$ *is analytic in* Δ, *with* $r(0) = q(0), r(\Delta) \subset D$ *and*

$$\vartheta(r(z)) + zr'(z)\phi(r(z)) \prec \vartheta(q(z)) + zq'(z)\phi(q(z)), \tag{2.6}$$

then $r(z) \prec q(z)$ *and* $q(z)$ *is the best dominant.*

Lemma 2.4 *(see [11]). Let* $q(z)$ *be a convex univalent function in* Δ *and* $\psi, \gamma \in \mathcal{C}$ *with*
$Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > max\left\{0, -Re\left(\frac{\psi}{\gamma}\right)\right\}$. *If* $r(z)$ *is analytic in* Δ *and*

$$\psi r(z) + \gamma zr'(z) \prec \psi q(z) + \gamma zq'(z), \tag{2.7}$$

then $r(z) \prec q(z)$ *and* $q(z)$ *is the best dominant.*

Lemma 2.5 *(see [8]). Let* $q(z)$ *be convex univalent function in the unit disk* Δ *and* $\gamma \in \mathcal{C}$. *Further assume that* $Re\,(\gamma) > 0$. *If* $r(z) \in \mathcal{H}[q(0), 1] \cap Q$ *and* $r(z) + \gamma r'(z)$ *is univalent in* Δ, *then*

$$q(z) + \gamma zq'(z) \prec r(z) + \gamma zr'(z), \tag{2.8}$$

which implies that $q(z) \prec r(z)$ *and* $q(z)$ *is the best subordinant.*

Lemma 2.6 *(see [2]). Let* $q(z)$ *be convex univalent in the disk* Δ, *and let* ϑ *and* ϕ *be analytic in a domain* D *containing* $q(\Delta)$. *Suppose that*

(i) $Re\left\{\frac{\vartheta'(q(z))}{\phi(q(z))}\right\} > 0$ *for* $z \in \Delta$;

(ii) $zq'(z)\phi(q(z))$ *is starlike univalent in* $z \in \Delta$.

If $r(z) \in \mathcal{H}[q(0), 1] \cap Q$, *with* $r(\Delta) \subseteq D$, *and if* $\vartheta(r(z)) + zr'(z)\phi(r(z))$ *is univalent in* Δ *and*

$$\vartheta(q(z)) + zq'(z)\phi(q(z)) \prec \vartheta(r(z)) + zr'(z)\phi(r(z)), \tag{2.9}$$

then $q(z) \prec r(z)$ *and* $q(z)$ *is the best subordinant.*

The main object of this paper is to apply a method based on the differential subordination in order to derive several subordination results.

3 Subordination for analytic functions

Theorem 3.1. *Let* $q(z)$ *be univalent in the unit disk* Δ *and let* $l \leq m+1$, $l, m \in N_0$, $\alpha_i, \beta_k \in \mathcal{C}\backslash Z_0^-; 1 \leq i \leq l, 1 \leq k \leq m$, $\lambda \in \mathcal{C}^* = \mathcal{C}\backslash\{0\}$ *and* $0 < \mu < 1$.

Suppose that $f \in \mathcal{A}(p)$, q *satisfies the inequality*

$$Re\left(1+\frac{zq''(z)}{q'(z)}\right) > max\left\{0, -Re\left(\frac{\mu(p-j+1)}{\lambda}\right)\right\}. \tag{3.1}$$

and

$$\Phi_1(z) = \left[\frac{p!}{(p-j+1)!}\ \frac{z^{p-j+1}}{\left[(H_p^{(l,m)}[\alpha_1]f)(z)\right]^{(j-1)}}\right]^{\mu}\left\{\left(1+\frac{\alpha_1\lambda}{p-j+1}\right)\right.$$

$$\left. -\frac{\alpha_1\lambda}{(p-j+1)}\ \frac{\left[(H_p^{(l,m)}[\alpha_1+1]f)(z)\right]^{(j-1)}}{\left[(H_p^{(l,m)}[\alpha_1]f)(z)\right]^{(j-1)}}\right\}, \tag{3.2}$$

where $(\mathcal{H}_p^{(l,m)}[\alpha_1]f)(z)$ *is defined by (2.2). If*

$$\Phi_1(z) \prec q(z) + \frac{\lambda}{\mu(p-j+1)}zq'(z) \qquad (1 \leq j \leq p), \tag{3.3}$$

then

$$\left[\frac{p!}{(p-j+1)!}\ \frac{z^{p-j+1}}{\left[(H_p^{(l,m)}[\alpha_1]f)(z)\right]^{(j-1)}}\right]^{\mu} \prec q(z), \tag{3.4}$$

and q *is the best dominant.*

Proof. Consider

$$r(z) := \left[\frac{p!}{(p-j+1)!}\ \frac{z^{p-j+1}}{\left[(H_p^{(l,m)}[\alpha_1]f)(z)\right]^{(j-1)}}\right]^{\mu}, \tag{3.5}$$

Differentiating (3.5) logarithmically with respect to z and using (2.4), we get

$$\frac{zr'(z)}{r(z)} = \alpha_1\ \mu\left(1-\frac{\left[(H_p^{(l,m)}[\alpha_1+1]f)(z)\right]^{(j-1)}}{\left[(H_p^{(l,m)}[\alpha_1]f)(z)\right]^{(j-1)}}\right), \tag{3.6}$$

which, in light of the hypothesis (3.3) of Theorem 3.1, yields the following subordination

$$r(z) + \frac{\lambda}{\mu(p-j+1)}zr'(z) \prec q(z) + \frac{\lambda}{\mu(p-j+1)}zq'(z). \tag{3.7}$$

An application of Lemma 2.4, with $\psi = 1$, $\gamma = \frac{\lambda}{\mu(p-j+1)}$, leads to (3.4).

Taking $l = 2, m = 1, \alpha_2 = 1$ in Theorem 3.1, we get the following corollary

Corollary 3.2. *Let* $q(z)$ *be univalent in the unit disk* Δ *and let* $\alpha_1, \beta_1 \in \mathcal{C}\backslash Z_0^-$, $\lambda \in \mathcal{C}^*$ *and* $0 < \mu < 1$. *Suppose that* $f \in \mathcal{A}(p)$, q *satisfies (3.1) and*

$$\Psi_1(z) = \left[\frac{p!}{(p-j+1)!}\ \frac{z^{p-j+1}}{\left[(L_p[\alpha_1,\beta_1]f)(z)\right]^{(j-1)}}\right]^{\mu}\left\{\left(1+\frac{\alpha_1\lambda}{p-j+1}\right)\right.$$

$$-\frac{\alpha_1\lambda}{(p-j+1)}\frac{[(L_p[\alpha_1+1,\beta_1]f)(z)]^{(j-1)}}{[(L_p[\alpha_1,\beta_1]f)(z)]^{(j-1)}}\Bigg\}, \tag{3.8}$$

where $(L_p(\alpha_1,\beta_1)f)(z)$ *is the Saitoh linear operator [10]. If*

$$\Psi_1(z) \prec q(z) + \frac{\lambda}{\mu(p-j+1)}zq'(z) \qquad (1 \le j \le p), \tag{3.9}$$

then

$$\left[\frac{p!}{(p-j+1)!}\ \frac{z^{p-j+1}}{[(L_p(\alpha_1,\beta_1)f)(z)]^{(j-1)}}\right]^{\mu} \prec q(z), \tag{3.10}$$

and q *is the best dominant.*

Taking $l = 1, m = 0, \alpha_1 = 1$ and $j = 1$ in Theorem 3.1, we arrive at the following

Corollary 3.3. *Let* $q(z)$ *be univalent in* Δ, $\lambda \in \mathcal{C}^*$ *and* $0 < \mu < 1$. *Suppose* q *satisfies (3.1). If* $f \in \mathcal{A}(p)$, *and*

$$\left(\frac{z^p}{f(z)}\right)^{\mu}\left[1+\lambda\left(1-\frac{zf'(z)}{pf(z)}\right)\right] \prec q(z) + \frac{\lambda}{p\mu}zq'(z), \tag{3.11}$$

then

$$\left(\frac{z^p}{f(z)}\right)^{\mu} \prec q(z), \tag{3.12}$$

and $q(z)$ *is the best dominant.*

Putting $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 3.1, we get the following corollary

Corollary 3.4. *Let* $-1 \le B < A \le 1$ *and (3.1) hold. Suppose that* $f \in \mathcal{A}(p)$, *and* $\Phi_1(z)$ *is given by (3.2). If*

$$\Phi_1(z) \prec \frac{\lambda(A-B)z}{(p-j+1)\mu(1+Bz)^2} + \frac{1+Az}{1+Bz} \qquad (1 \le j \le p), \tag{3.13}$$

then

$$\left[\frac{p!}{(p-j+1)!}\ \frac{z^{p-j+1}}{[(H_p^{(l,m)}[\alpha_1]f)(z)]^{(j-1)}}\right]^{\mu} \prec \frac{1+Az}{1+Bz}, \tag{3.14}$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Further taking $A = 1,\ B = -1$ in Corollary 3.4, we get

Corollary 3.5. *Let (3.1) hold. Further suppose that* $f \in \mathcal{A}(p)$, *and* $\Phi_1(z)$ *is defined in (3.2). If*

$$\Phi_1(z) \prec \frac{2\lambda z}{(p-j+1)\mu(1-z)^2} + \frac{1+z}{1-z} \qquad (1 \le j \le p), \tag{3.15}$$

then

$$\left[\frac{p!}{(p-j+1)!}\ \frac{z^{p-j+1}}{[(H_p^{(l,m)}[\alpha_1]f)(z)]^{(j-1)}}\right]^{\mu} \prec \frac{1+z}{1-z}, \tag{3.16}$$

and $\frac{1+z}{1-z}$ *is the best dominant.*

For $l = 2, m = 1, j = 1, \alpha_2 = 1$ and $p = 1$ the aforementioned result reduces at once to the result obtained recently by Shanmugum et al. [13].

Theorem 3.6. *Let $q(z)$ be univalent in Δ, and let $l \le m+1, l, m \in N_0,\ \alpha_i, \beta_k \in \mathcal{C}\backslash Z_0^-; 1 \le i \le l, 1 \le k \le m,\ 0 < \mu < 1, 0 \ne \gamma, \beta \in \mathcal{C}$ and $f \in \mathcal{A}(p)$. Suppose that q satisfies*

$$Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > max\left\{0, -Re\left(\frac{\beta}{\gamma}\right)\right\}. \tag{3.17}$$

$$\Phi_2(z) = \left[\frac{[(H_p^{(l,m)}[\alpha_1+1]f)(z)]^{(j-1)}}{[(H_p^{(l,m)}[\alpha_1]f)(z)]^{(j-1)}}\right]^{\mu}\left\{\mu\gamma\left[(\alpha_1+1)\frac{[(H_p^{(l,m)}[\alpha_1+2]f)(z)]^{(j-1)}}{[(H_p^{(l,m)}[\alpha_1+1]f)(z)]^{(j-1)}}\right.\right.$$
$$\left.\left.-\alpha_1\frac{[(H_p^{(l,m)}[\alpha_1+1]f)(z)]^{(j-1)}}{[(H_p^{(l,m)}[\alpha_1]f)(z)]^{(j-1)}} - 1\right] + \beta\right\},$$
$$(1 \le j \le p). \tag{3.18}$$

If

$$\Phi_2(z) \prec \gamma z q'(z) + \beta q(z), \tag{3.19}$$

then

$$\left[\frac{[(H_p^{(l,m)}[\alpha_1+1]f)(z)]^{(j-1)}}{[(H_p^{(l,m)}[\alpha_1]f)(z)]^{(j-1)}}\right]^{\mu} \prec q(z), \tag{3.20}$$

and $q(z)$ is the best dominant.

Proof. Define the function $r(z)$ by

$$r(z) := \left[\frac{[(H_p^{(l,m)}[\alpha_1+1]f)(z)]^{(j-1)}}{[(H_p^{(l,m)}[\alpha_1]f)(z)]^{(j-1)}}\right]^{\mu}. \tag{3.21}$$

Then, a computation shows that

$$\frac{zr'(z)}{r(z)} = \mu\left[(\alpha_1+1)\frac{[(H_p^{(l,m)}[\alpha_1+2]f)(z)]^{(j-1)}}{[(H_p^{(l,m)}[\alpha_1+1]f)(z)]^{(j-1)}} - \alpha_1\frac{[(H_p^{(l,m)}[\alpha_1+1]f)(z)]^{(j-1)}}{[(H_p^{(l,m)}[\alpha_1]f)(z)]^{(j-1)}} - 1\right],$$

and hence

$$zr'(z) = \mu r(z)\left[(\alpha_1+1)\frac{[(H_p^{(l,m)}[\alpha_1+2]f)(z)]^{(j-1)}}{[(H_p^{(l,m)}[\alpha_1+1]f)(z)]^{(j-1)}} - \alpha_1\frac{[(H_p^{(l,m)}[\alpha_1+1]f)(z)]^{(j-1)}}{[(H_p^{(l,m)}[\alpha_1]f)(z)]^{(j-1)}} - 1\right]. \tag{3.22}$$

Set

$$\vartheta(w) = \beta w, \qquad \phi(w) = \gamma, \tag{3.23}$$

and let

$$Q(z) = zq'(z)\phi(q(z)) = \gamma zq'(z),$$

$$h(z) = \vartheta(q(z)) + Q(z) = \beta q(z) + \gamma zq'(z). \tag{3.24}$$

From (3.17), we see that $Q(z)$ is starlike in Δ and that

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{\frac{\beta}{\gamma} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0. \tag{3.25}$$

Thus applying Lemma 2.3, the proof of Theorem 3.6 is completed.

Taking $l = 2, m = 1, \alpha_2 = 1$ in Theorem 3.6, we get the following corollary.

Corollary 3.7. *Let $q(z)$ be univalent in Δ, and let $\alpha_1, \beta_1 \in \mathcal{C}\backslash Z_0^-$, $0 \neq \gamma, \beta \in \mathcal{C}, 0 < \mu < 1$, and $f \in \mathcal{A}(p)$. Suppose that q satisfies (3.17) and let*

$$\phi_2(z) = \left[\frac{[(L_p[\alpha_1+1,\beta_1]f)(z)]^{(j-1)}}{[(L_p[\alpha_1,\beta_1]f)(z)]^{(j-1)}}\right]^{\mu}\left\{\mu\gamma\left[(\alpha_1+1)\frac{[(L_p[\alpha_1+2,\beta_1]f)(z)]^{(j-1)}}{[(L_p[\alpha_1+1,\beta_1]f)(z)]^{(j-1)}}\right.\right.$$
$$\left.\left.-\alpha_1\frac{[(L_p[\alpha_1+1,\beta_1]f)(z)]^{(j-1)}}{[(L_p[\alpha_1,\beta_1]f)(z)]^{(j-1)}} - 1\right] + \beta\right\},$$
$$(1 \leq j \leq p). \tag{3.26}$$

If

$$\phi_2(z) \prec \gamma zq'(z) + \beta q(z), \tag{3.27}$$

then

$$\left[\frac{[(L_p(\alpha_1+1,\beta_1)f)(z)]^{(j-1)}}{[(L_p(\alpha_1,\beta_1)f)(z)]^{(j-1)}}\right]^{\mu} \prec q(z), \tag{3.28}$$

and $q(z)$ is the best dominant.

Note that for $l = 1, m = 0, \alpha_1 = 1$ and $f \in \mathcal{A}(1)$, we have

$$(\mathcal{H}_p^{(l,m)}[\alpha_1]f)(z) = f(z), \tag{3.29}$$

$$(\mathcal{H}_p^{(l,m)}[\alpha_1+1]f)(z) = zf'(z), \tag{3.30}$$

$$(\mathcal{H}_p^{(l,m)}[\alpha_1+2]f)(z) = zf'(z) + \frac{z^2f''(z)}{2}. \tag{3.31}$$

Thus taking $l = 1, m = 0, \alpha_1 = 1, p = j = 1$ in Theorem 3.6, we get the following interesting result contained in the corollary

Corollary 3.8. *Let $f \in \mathcal{A}$, $0 < \mu < 1$, and $0 \neq \gamma, \beta \in \mathcal{C}$. Further let $q(z)$ be*

univalent in Δ *and satisfies (3.17).*
If

$$\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left\{\mu\gamma\left[1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right]+\beta\right\}\prec\gamma zq'(z)+\beta q(z),\qquad(3.32)$$

then

$$\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\prec q(z),\qquad(3.33)$$

and $q(z)$ *is the best dominant.*

Further taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ and $\gamma=1$ in Corollary 3.8, we get

Corollary 3.9. *Let* $f\in\mathcal{A}$, $0<\mu<1$, *and* $Re(\beta)>0$. *Suppose that*

$$Re\left(\frac{1-Bz}{1+Bz}\right)>max\left\{0,-Re(\beta)\right\}.\qquad(3.34)$$

If

$$\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left\{\mu\left[1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right]+\beta\right\}\prec\frac{(A-B)z}{(1+Bz)^2}+\beta\frac{1+Az}{1+Bz},\qquad(3.35)$$

then

$$\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\prec\frac{1+Az}{1+Bz},\qquad(3.36)$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Again by setting $\beta\in R$ s.t. $\beta\ge 0$, $A=\frac{\delta}{1+\beta}$ where $0<\delta\le 1+\beta$ and $B=0$ in Corollary 3.9, we arrive at the following result.

Corollary 3.10. *Let* $f\in\mathcal{A}$ *and* $0<\mu<1$. $q(z)=1+\frac{\delta}{1+\beta}z$ *is convex univalent in* Δ. *Then*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left\{\mu\left[1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right]+\beta\right\}\prec\beta+\delta z,\qquad(3.37)$$

implies

$$\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\prec 1+\frac{\delta}{1+\beta}z,\qquad(3.38)$$

and $1+\frac{\delta}{1+\beta}z$ *is the best dominant.*

If we set $\beta=1,\gamma\in R$ s.t. $\gamma>0$ and

$$q(z)=\int_0^1\frac{1+\eta zt^{\gamma}}{1-\eta zt^{\gamma}}dt\qquad(0<\eta\le 1)\qquad(3.39)$$

in Corollary 3.8, we obtain a result by Singh [14,Theorem 3, p. 573].

4 Superordination for analytic functions

Theorem 4.1. *Let $q(z)$ be convex univalent in Δ and let $\alpha_i, \beta_k \in \mathcal{C}\backslash Z_0^-; 1 \leq i \leq l,\ 1 \leq k \leq m, 0 < \mu < 1, \lambda \in \mathcal{C}$ with $Re(\lambda) > 0$. Suppose for $1 \leq j \leq p$,*

$$\left[\frac{p!}{(p-j+1)!}\ \frac{z^{p-j+1}}{[(H_p^{(l,m)}[\alpha_1]f)(z)]^{(j-1)}}\right]^{\mu} \in \mathcal{H}[q(0),1] \cap Q, \tag{4.1}$$

and $\Phi_1(z)$ given by (3.2) is univalent in Δ. If $f \in \mathcal{A}(p)$ and

$$q(z)\ + \frac{\lambda}{\mu(p-j+1)} zq'(z) \prec \Phi_1(z), \tag{4.2}$$

then

$$q(z)\ \prec\ \left[\frac{p!}{(p-j+1)!}\ \frac{z^{p-j+1}}{[(H_p^{(l,m)}[\alpha_1]f)(z)]^{(j-1)}}\right]^{\mu}, \tag{4.3}$$

and $q(z)$ is the best subordinant.
Proof.. Define the function r by

$$r(z) := \left[\frac{p!}{(p-j+1)!}\ \frac{z^{p-j+1}}{[(H_p^{(l,m)}[\alpha_1]f)(z)]^{(j-1)}}\right]^{\mu}. \tag{4.4}$$

Then a computation shows that

$$r(z)+\frac{\lambda}{\mu(p-j+1)} zr'(z) = \left[\frac{p!}{(p-j+1)!}\ \frac{z^{p-j+1}}{[(H_p^{(l,m)}[\alpha_1]f)(z)]^{(j-1)}}\right]^{\mu} \left\{\left(1+\frac{\alpha_1\lambda}{p-j+1}\right) - \frac{\alpha_1\lambda}{(p-j+1)}\ \frac{[(H_p^{(l,m)}[\alpha_1+1]f)(z)]^{(j-1)}}{[(H_p^{(l,m)}[\alpha_1]f)(z)]^{(j-1)}}\right\}. \tag{4.5}$$

Theorem 4.1 follows as an application of Lemma 2.5.

By considering $l = 2, m = 1, \alpha_2 = 1$ in Theorem 4.1, we get the following corollary

Corollary 4.2. *Let $q(z)$ be convex univalent in Δ and let $\alpha_1, \beta_1 \in \mathcal{C}\backslash Z_0^-, 0 < \mu < 1, \lambda \in \mathcal{C}$ with $Re(\lambda) > 0$. Suppose for $1 \leq j \leq p$,*

$$\left[\frac{p!}{(p-j+1)!}\ \frac{z^{p-j+1}}{[(L_p(\alpha_1,\beta_1)f)(z)]^{(j-1)}}\right]^{\mu} \in \mathcal{H}[q(0),1] \cap Q,$$

and $\Psi_1(z)$ defined by (3.8) is univalent in Δ. If $f \in \mathcal{A}(p)$, and

$$q(z) + \frac{\lambda}{\mu(p-j+1)} zq'(z) \prec \Psi_1(z), \tag{4.6}$$

then

$$q(z) \prec \left[\frac{p!}{(p-j+1)!}\ \frac{z^{p-j+1}}{[(L_p(\alpha_1,\beta_1)f)(z)]^{(j-1)}}\right]^{\mu}, \tag{4.7}$$

and $q(z)$ is the best subordinant.

Taking $q(z) = \frac{1+Az}{1+Bz}$ $\quad(-1 \le B < A \le 1)$ in Theorem 4.1, we get the following corollary

Corollary 4.3. *Let $q(z)$ be convex univalent in Δ. Suppose $0 \le \mu < 1,\ \lambda \in \mathcal{C}$ with $Re\,(\lambda) > 0$ and for $1 \le j \le p,\ f \in \mathcal{A}(p)$ satisfies (4.1). Also let $\Phi_1(z)$, given by (3.2) be univalent in Δ. If*

$$\frac{\lambda(A-B)z}{(p-j+1)\mu(1+Bz)^2} + \frac{1+Az}{1+Bz} \prec \Phi_1(z), \tag{4.8}$$

then

$$\frac{1+Az}{1+Bz} \prec \left[\frac{p!}{(p-j+1)!}\ \frac{z^{p-j+1}}{\left[(H_p^{(l,m)}[\alpha_1]f)(z)\right]^{(j-1)}}\right]^{\mu}, \tag{4.9}$$

and $\frac{1+Az}{1+Bz}$ is the subordinant.

For $l = 2, m = 1, j = 1, \alpha_2 = 1$ and $p = 1$ the above result reduces easily to the superordination result obtained recently by Shanmugum et al. [13]. Similarly we have

Theorem 4.4. *Let $q(z)$ be convex univalent in the unit disk Δ, and let $l \le m+1,\ l,m \in N_0,\ \alpha_i,\beta_k \in \mathcal{C}\backslash Z_0^-; 1 \le i \le l,\ 1 \le k \le m, 0 < \mu < 1, 0 \ne \gamma,\beta \in \mathcal{C}$, and $f \in \mathcal{A}(p)$. Suppose that*

$$\left[\frac{\left[(H_p^{(l,m)}[\alpha_1+1]f)(z)\right]^{(j-1)}}{\left[(H_p^{(l,m)}[\alpha_1]f)(z)\right]^{(j-1)}}\right]^{\mu} \in \mathcal{H}[q(0),1] \cap Q \qquad (1 \le j \le p), \tag{4.10}$$

and $Re\,\{\beta q'(z)/\gamma\} > 0$. If $\Phi_2(z)$ given by (3.18) is univalent in Δ, then

$$\gamma z q'(z) + \beta q(z) \prec \Phi_2(z), \tag{4.11}$$

implies

$$q(z) \prec \left[\frac{\left[(H_p^{(l,m)}[\alpha_1+1]f)(z)\right]^{(j-1)}}{\left[(H_p^{(l,m)}[\alpha_1]f)(z)\right]^{(j-1)}}\right]^{\mu}, \tag{4.12}$$

and $q(z)$ is the best subordinant.

5 Sandwich results

Combining results of differential subordinations and superordinations, we arrive at the following "sandwich results".

Theorem 5.1. *Let $q_1(z)$ be convex univalent, and let $q_2(z)$ be univalent in Δ. Let $\alpha_i, \beta_k \in \mathcal{C}\backslash Z_0^-; 1 \leq i \leq l, 1 \leq k \leq m, 0 < \mu < 1, \lambda \in \mathcal{C}$ with $Re(\lambda) > 0$ and q_2 satisfies (3.1). If $f \in \mathcal{A}(p)$ satisfies (4.1) and $\Phi_1(z)$, given by (3.2) be univalent in Δ.*
If

$$q_1(z) + \frac{\lambda}{(p-j+1)\mu} z q_1'(z) \prec \Phi_1(z) \prec q_2(z) + \frac{\lambda}{(p-j+1)\mu} z q_2'(z), \tag{5.1}$$

then

$$q_1(z) \prec \left[\frac{p!}{(p-j+1)!} \; \frac{z^{p-j+1}}{\left[(H_p^{(l,m)}[\alpha_1]f)(z)\right]^{(j-1)}} \right]^{\mu} \prec q_2(z), \tag{5.2}$$

and $q_1(z)$ and $q_2(z)$, respectively, is the best subordinant and the best dominant.

Also by considering $l = 2, m = 1, \alpha_2 = 1, j = 1$ and $p = 1$ in the Theorem 5.1, we immediately get the sandwich result obtained by Shanmugum et al. [13].

Theorem 5.2. *Let $q_1(z)$ be convex univalent, and let $q_2(z)$ be univalent in Δ. Suppose q_1 satisfies $Re\{\beta q_1'(z)/\gamma\} > 0$ and q_2 satisfies (3.1). Let $l \leq m+1$; $l, m \in N_0$, $\alpha_i, \beta_k \in \mathcal{C}\backslash Z_0^-; 1 \leq i \leq l$, $1 \leq k \leq m$, $0 < \mu < 1$ and $\lambda \in \mathcal{C}$. Further suppose that $f \in \mathcal{A}(p)$ satisfies (4.10) and if $\Phi_2(z)$ is univalent in Δ, then*

$$\beta q_1(z) + \gamma z q_1'(z) \prec \Phi_2(z) \prec \beta q_2(z) + \gamma z q_2(z), \tag{5.3}$$

implies

$$q_1(z) \prec \left[\frac{\left[(H_p^{(l,m)}[\alpha_1+1]f)(z)\right]^{(j-1)}}{\left[(H_p^{(l,m)}[\alpha_1]f)(z)\right]^{(j-1)}} \right]^{\mu} \prec q_2(z), \tag{5.4}$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

6 The Open Problem.

The subordination, superordination and sandwich results can also be obtained by modifying suitably the class defined by (1.10) and Dziok-Srivastava operator for meromorphic multivalent functions defined in the punctured unit disk.

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