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Certain Applications of Differential Subordination to Analytic Functions

Sukhwinder Singh, Sushma Gupta and Sukhjit Singh

Department of Applied Sciences
Baba Banda Singh Bahadur Engineering College
Fatehgarh Sahib-140 407, Punjab, India
e-mail: ssbilling@gmail.com
Department of Mathematics
Sant Longowal Institute of Engineering & Technology
Deemed University, Longowal-148 106, Punjab, India
e-mail: sushmagupta1@yahoo.com
e-mail: sukhjit_d@yahoo.com

Abstract

In the present paper, we study two subclasses of analytic functions recently introduced by Owa et al. [3] and extend their results. Mathematica 5.2 is used to plot the extended regions. We also obtain some results in regard of problems left open by them.

Keywords: *Analytic Function, Close-to-convex function, Differential subordination, Univalent function.*

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1 Introduction

Let $\mathcal{A}$ be the class of functions f, analytic in the open unit disk $E = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$.

Let f be analytic in E, g analytic and univalent in E and $f(0) = g(0)$. Then, by the symbol $f(z) \prec g(z)$ (f subordinate to g) in E, we shall mean $f(E) \subset g(E)$.

Let $\psi : C \times C \to C$ be an analytic function, p be an analytic function in E, with $(p(z), zp'(z)) \in C \times C$ for all $z \in E$ and h be univalent in E, then the function p is said to satisfy first order differential subordination if

$$\psi(p(z), zp'(z)) \prec h(z), \ \psi(p(0), 0) = h(0). \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1), is said to be the best dominant of (1). The best dominant is unique up to a rotation of E.

A function $f \in \mathcal{A}$ is said to be close-to-convex if there is a real number α, $-\pi/2 < \alpha < \pi/2$, and a convex function g (not necessarily normalized) such that

$$\Re\left(e^{i\alpha}\frac{f'(z)}{g'(z)}\right) > 0, \ z \in E.$$

It is well-known that every close-to-convex function is univalent. In 1934/35, Noshiro [2] and Warchawski [4] obtained a simple but interesting criterion for univalence of analytic functions. They proved that if an analytic function f satisfies the condition $\Re\, f'(z) > 0$ for all z in E, then f is close-to-convex and hence univalent in E.

A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_\delta(\alpha)$ if it satisfies the condition

$$(f'(z))^\delta \prec \frac{\alpha(1-z)}{\alpha - z}, \ z \in E,$$

for some real numbers $\alpha > 1$ and $\delta > 0$. We notice that the functions in $\mathcal{S}_1(\alpha)$ satisfy the condition $\Re\, f'(z) > 0$ and therefore, are close-to-convex and hence univalent.

A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{T}_\delta(\alpha)$ if it satisfies the condition

$$\left(\frac{1}{f'(z)}\right)^\delta \prec \frac{\alpha(1-z)}{\alpha - z}, \ z \in E,$$

for some real numbers $\alpha > 1$ and $\delta > 0$.

The above defined classes are introduced by Owa et al. [3]. They proved the following results for these classes.

Theorem 1.1 *If $f \in \mathcal{A}$ satisfies*

$$\Re\frac{zf''(z)}{f'(z)} < \frac{\alpha - 1}{2\delta(\alpha + 1)}, \ z \in E,$$

for some real numbers $\alpha > 1$ and $\delta > 0$, then $f \in \mathcal{S}_\delta(\alpha)$.

Theorem 1.2 *If $f \in \mathcal{A}$ satisfies*

$$\Re\frac{zf''(z)}{f'(z)} > \frac{1-\alpha}{2\delta(\alpha+1)}, \ z \in E,$$

for some real numbers $\alpha > 1$ and $\delta > 0$, then $f \in \mathcal{T}_\delta(\alpha)$.

They also left one problem open with each of the above classes. The main objective of this paper is to extend the results of Owa et al. [3] and obtain some results in regard of open problems raised by them.

2 Preliminaries

We shall use the following lemma to prove our main result.

Lemma 2.1 *([1], p.132, Theorem 3.4 h) Let q be univalent in E and let θ and ϕ be analytic in a domain D containing $q(E)$, with $\phi(w) \neq 0$, when $w \in q(E)$.*
Set $Q(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that either
(i) h is convex, or
(ii) Q is starlike.
In addition, assume that
(iii) $\Re\,\dfrac{zh'(z)}{Q(z)} > 0, \ z \in E$.
If p is analytic in E, with $p(0) = q(0), p(E) \subset D$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p \prec q$ and q is the best dominant.

3 Main Results

The following result is essentially due to Miller and Mocanu [1,p.76]. However, we present an alternative proof of the same by using Lemma 2.1.

Theorem 3.1 *Let q $(q(z) \neq 0)$ be a univalent function in E such that $\dfrac{zq'(z)}{q(z)}$ is starlike in E. If an analytic function p $(p(z) \neq 0)$ satisfies the differential subordination*

$$\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)}, \ z \in E, \tag{2}$$

then $p(z) \prec q(z)$ and q is the best dominant.

Proof. Let us define the functions θ and ϕ as follows:

$$\theta(w) = 0,$$

and

$$\phi(w) = \frac{1}{w}.$$

Obviously, the functions θ and ϕ are analytic in domain $D = C \setminus \{0\}$ and $\phi(w) \neq 0$ in D.
Now, define the functions Q and h as follows :

$$Q(z) = zq'(z)\phi(q(z)) = \frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \frac{zq'(z)}{q(z)}.$$

Given that, Q is starlike in E and we also have $\Re\frac{z\ h'(z)}{Q(z)} > 0,\ z \in E.$
Thus conditions (ii) and (iii) of Lemma 2.1, are satisfied.
In view of (2), we have

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)].$$

Therefore, the proof, now, follows from Lemma 2.1.

On writing $p(z) = (f'(z))^\delta,\ f \in \mathcal{A}$ in Theorem 3.1, we obtain the following result.

Theorem 3.2 *Suppose $\delta > 0$ is a real number. Let q $(q(z) \neq 0)$ be a univalent function in E such that $\frac{zq'(z)}{q(z)}$ is starlike in E. If $f \in \mathcal{A}$ $(f'(z) \neq 0)$ satisfies the differential subordination*

$$\frac{zf''(z)}{f'(z)} \prec \frac{1}{\delta}\frac{zq'(z)}{q(z)},\ z \in E,$$

then $(f'(z))^\delta \prec q(z)$ and q is the best dominant.

By setting $p(z) = \left(\frac{1}{f'(z)}\right)^\delta,\ f \in \mathcal{A}$ in Theorem 3.1, we have the following result.

Theorem 3.3 *Let $\delta > 0$ be a real number. Let q $(q(z) \neq 0)$ be a univalent function in E such that $\frac{zq'(z)}{q(z)}$ is starlike in E. If $f \in \mathcal{A}$ $(f'(z) \neq 0)$ satisfies the differential subordination*

$$\frac{zf''(z)}{f'(z)} \prec -\frac{1}{\delta}\frac{zq'(z)}{q(z)}, \ z \in E,$$

then $\left(\frac{1}{f'(z)}\right)^{\delta} \prec q(z)$ and q is the best dominant.

4 Applications to Analytic Functions

If we select the dominant $q(z) = \frac{\alpha(1-z)}{\alpha - z}$, a little calculation yields that $\frac{zq'(z)}{q(z)}$ is starlike in E for real $\alpha > 1$. From Theorem 3.2, we have the following result.

Theorem 4.1 *If $f \in \mathcal{A}$ $(f'(z) \neq 0)$ satisfies the differential subordination*

$$\frac{zf''(z)}{f'(z)} \prec \frac{(1-\alpha)z}{\delta(\alpha - z)(1-z)} = F_1(z), \ z \in E,$$

then $f \in \mathcal{S}_{\delta}(\alpha)$, where $\alpha > 1$ and $\delta > 0$ are some real numbers.

Remark 4.1 *For $\alpha = 2, \ \delta = 1$, the constant on right hand side of Theorem 1.1 reduces to $\frac{1}{6}$. In Figure 4.1, we plot the dotted line $\Re(z) = \frac{1}{6}$ and the curve $F_1(z)$, $z \in E$. According to result of Owa et al. [3], the result in Theorem 1.1 holds only if the operator $\frac{zf''(z)}{f'(z)}$ lies in the portion of the plane left to the dotted line $\Re(z) = \frac{1}{6}$. Theorem 4.1 shows that the result holds even when the operator $\frac{zf''(z)}{f'(z)}$ lies in the portion of the plane left to the plotted curve $F_1(z)$, thus extending the region of variability of the operator $\frac{zf''(z)}{f'(z)}$ for the required implication. The extended portion lies between the dotted line $\Re(z) = \frac{1}{6}$ and the curve $F_1(z), z \in E$.*

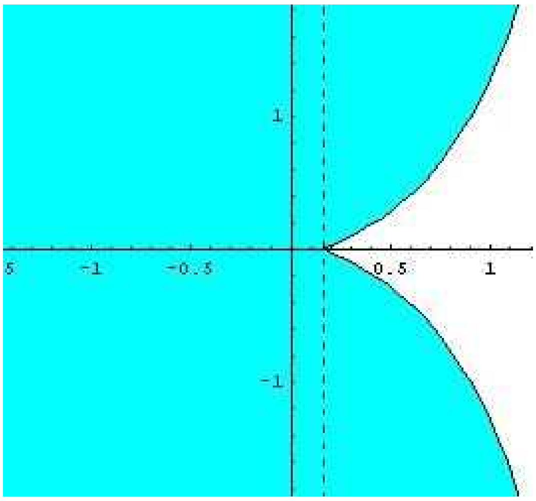

Figure 4.1 (when $\alpha = 2,\ \delta = 1$)

By setting $q(z) = \dfrac{\alpha(1-z)}{\alpha - z}$ in Theorem 3.3, we obtain the following result.

Theorem 4.2 *Suppose $f \in \mathcal{A}$ $(f'(z) \neq 0)$ satisfies the differential subordination*

$$\frac{zf''(z)}{f'(z)} \prec \frac{(\alpha-1)z}{\delta(\alpha-z)(1-z)} = F_2(z),\ z \in E,$$

then $f \in \mathcal{T}_\delta(\alpha)$, where $\alpha > 1$ and $\delta > 0$ are some real numbers.

Remark 4.2 *For $\alpha = 2,\ \delta = \frac{1}{2}$, the constant on right hand side of Theorem 1.2 reduces to $-\frac{1}{3}$. In Figure 4.2, we plot the dotted line $\Re(z) = -\frac{1}{3}$ and the curve $F_2(z)$. According to result of Owa et al. [3], the result in Theorem 1.2 holds only if the operator $\frac{zf''(z)}{f'(z)}$ takes values in the portion of the plane right to the dotted line $\Re(z) = -\frac{1}{3}$. Theorem 4.2 shows that the result holds when the operator $\frac{zf''(z)}{f'(z)}$ lies in the portion of the plane right to the plotted curve $F_2(z)$, thus extending the region of variability of the operator $\frac{zf''(z)}{f'(z)}$ for the required implication. In Figure 4.2, extended region lies between the dotted line and the curve.*

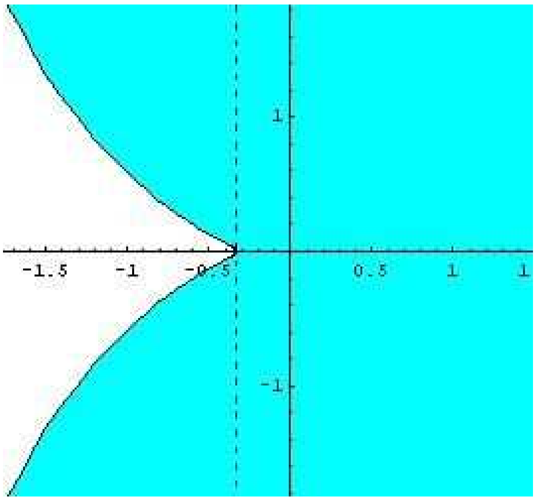

Figure 4.2 (when $\alpha = 2,\ \delta = 1/2$)

Now we obtain the results related to the problems left open by Owa et al. [3]. These are given below in Theorem 4.3 and Theorem 4.4. For this if we take $q(z) = \dfrac{1 + (1 - 2\beta)z}{1 - z}$ in Theorem 3.2, then a little calculation yields that $\dfrac{zq'(z)}{q(z)}$ is starlike in E for $0 \le \beta < 1$ and we arrive at the following result.

Theorem 4.3 *If $f \in \mathcal{A}$ $(f'(z) \neq 0)$ satisfies the differential subordination*

$$\frac{zf''(z)}{f'(z)} \prec \frac{2(1-\beta)z}{\delta(1 + (1-2\beta)z)(1-z)},\ \ z \in E,$$

then $(f'(z))^{\delta} \prec \dfrac{1 + (1 - 2\beta)z}{1 - z}$, for some real numbers $0 \le \beta < 1$ and $\delta > 0$.

In view of Theorem 4.3, we have the following result.

Corollary 4.1 *Let $0 \le \beta < 1$ and $\delta > 0$ be real numbers. If $f \in \mathcal{A}$ $(f'(z) \neq 0)$ satisfies the condition*

$$\Re\, \frac{zf''(z)}{f'(z)} > \begin{cases} \frac{1}{2\delta}\left(1 - \frac{1}{1-\beta}\right), & 0 \le \beta \le 1/2, \\ \frac{1}{2\delta}\left(1 - \frac{1}{\beta}\right), & 1/2 \le \beta < 1, \end{cases}$$

then $\Re\, (f'(z))^{\delta} > \beta$.

Remark 4.3 *Setting $\beta = \delta = 1/2$ in above Corollary, we obtain the following result of Miller and Mocanu [1,p.57]:*

For $f \in \mathcal{A}$ $(f'(z) \neq 0)$,

$$\Re \left(\frac{zf''(z)}{f'(z)} + 1 \right) > 0 \Rightarrow \Re \sqrt{f'(z)} > 1/2, z \in E.$$

By setting $q(z) = \dfrac{1 + (1-2\beta)z}{1-z}$ in Theorem 3.3, we obtain the following result.

Theorem 4.4 *Suppose* $f \in \mathcal{A}$ $(f'(z) \neq 0)$ *satisfies the differential subordination*

$$\frac{zf''(z)}{f'(z)} \prec \frac{2(\beta - 1)z}{\delta(1 + (1-2\beta)z)(1-z)}, \;\; z \in E,$$

then $\left(\dfrac{1}{f'(z)}\right)^{\delta} \prec \dfrac{1 + (1-2\beta)z}{1-z}$, *where* $0 \leq \beta < 1$ *and* $\delta > 0$ *are some real numbers.*

In view of Theorem 4.4, we obtain the following result.

Corollary 4.2 *Let* $0 \leq \beta < 1$ *and* $\delta > 0$ *be real numbers. If* $f \in \mathcal{A}$ $(f'(z) \neq 0)$, *satisfies the condition*

$$\Re \frac{zf''(z)}{f'(z)} < \begin{cases} \frac{1}{2\delta}\left(\frac{1}{1-\beta} - 1\right), & 0 \leq \beta \leq 1/2, \\ \frac{1}{2\delta}\left(\frac{1}{\beta} - 1\right), & 1/2 \leq \beta < 1, \end{cases}$$

then $\Re \left(\dfrac{1}{f'(z)}\right)^{\delta} > \beta$.

5 Open Problem

It would be of interest to raise the following problems.

(i) For $0 \leq \beta < 1, \; \delta > 0$ and $f \in \mathcal{A}$, does there exist a best value of $\gamma > 0$ such that the condition

$$\Re \frac{zf''(z)}{f'(z)} < \gamma, \;\; z \in E,$$

implies that $\Re(f'(z))^{\delta} > \beta$?

(ii) For $0 \leq \beta < 1, \; \delta > 0$ and $f \in \mathcal{A}$, does there exist a best value of $\gamma < 0$ such that the condition

$$\Re \frac{zf''(z)}{f'(z)} > \gamma, \;\; z \in E,$$

implies that $\Re \left(\dfrac{1}{f'(z)}\right)^{\delta} > \beta$?

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