Int. J. Open Problems Complex Analysis, Vol. 2, No. 1, March 2010
ISSN 2074-2827;
www.i-csrs.org

Some Inclusion Properties for Certain Subclasses Defined by Generalised Derivative Operator and Subordination

Ma'moun Harayzeh Al-Abbadi and *Maslina Darus

School of Mathematical Sciences, Faculty of Science and Technology
Universiti Kebangsaan Malaysia, Bangi 43600 Selangor D. Ehsan, Malaysia
mamoun_nn@yahoo.com, *maslina@ukm.my(corresponding author)

Abstract

The authors in [1] have recently introduced a new generalised derivative operator $\mu_{\lambda_1,\lambda_2}^{n,m}$, which generalised many well-known operators studied earlier by many different authors. By using this operator and differential subordination, we introduce certain new subclasses of analytic function defined in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, which are defined by means of the Hadamard product (or convolution). The purpose of the present paper is to investigate various inclusion properties of these subclasses. In addition, some integral preserving properties are also considered.

Keywords: *Analytic function; Hadamard product (or convolution); Univalent function; Convex function; Starlike function; Subordination; Derivative operator.*

AMS Mathematics Subject Classification (2000): 30C45.

1 Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad a_k \quad \text{is complex number} \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let S, $S^*(\alpha)$, $K(\alpha)$, $C(\alpha)\,(0 \le \alpha < 1)$ denote the subclasses of $\mathcal{A}$

consisting of functions that are univalent, starlike of order α, convex of order α, and close-to-convex of order α in U, respectively. In particular, the classes $S^*(0) = S^*$, $K(0) = K$ and $C(0) = C$ are the familiar classes of starlike, convex and close-to-convex functions in U, respectively.

Let be given two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Then the Hadamard product (or convolution) $f * g$ of two functions f, g is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \ .$$

Next, we state basic ideas on subordination. If f and g are analytic in U, then the function f is said to be subordinate to g, and can be written as

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in U),$$

if and only if there exists the Schwarz function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in U)$.

Furthermore, if g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.([13],P.36).

Now, $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_k = \begin{cases} 1 & \text{for } k = 0,\ x \in \mathbb{C}\backslash\{0\}, \\ x(x+1)(x+2)...(x+k-1) & \text{for } k \in \mathbb{N} = \{1, 2, 3, ...\} \text{and } x \in \mathbb{C}. \end{cases}$$

Let

$$k_a(z) = \frac{z}{(1-z)^a}$$

where a is any real number. It is easy to verify that $k_a(z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(1)_{k-1}} z^k$. Thus $k_a * f$, denotes the Hadamard product of k_a with f that is

$$(k_a * f)(z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(1)_{k-1}} a_k z^k.$$

Let N denotes the class of all functions ϕ which are analytic, convex and univalent in U, with normalisation $\phi(0) = 1$ and $\text{Re}(\phi(z)) > 0$ $(z \in U)$. Making use of the principle of subordination between analytic functions, many authors investigated the subclasses $S^*(\phi)$, $K(\phi)$, and $C(\phi,\psi)$ of the class $\mathcal{A}$ for $\phi,\psi \in N$ (cf. [7, 10]), which are defined by

$$\begin{aligned} S^*(\phi) &:= \left\{ f \in \mathcal{A} : \ \frac{zf'(z)}{f(z)} \prec \phi(z) \text{ in } U \right\}, \\ K(\phi) &:= \left\{ f \in \mathcal{A} : \ 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \text{ in } U \right\}, \\ C(\phi,\psi) &:= \left\{ f \in \mathcal{A} : \ \exists g \ \in \ S^*(\phi)\, s.t. \ \frac{zf'(z)}{g(z)} \prec \psi(z) \text{ in } U \right\}. \end{aligned}$$

For $\phi(z) = \psi(z) = (1+z)/(1-z)$ in the definitions defined above, we have the well-known classes S^*, K, and C, respectively. Furthermore, for the function classes $S^*[A,B]$ and $K[A,B]$ investigated by Janowski [9] (also see [8]), it is easily seen that

$$\begin{aligned} S^*\left(\frac{1+Az}{1+Bz}\right) &= S^*[A,B] \qquad (-1 \le B < A \le 1), \\ K\left(\frac{1+Az}{1+Bz}\right) &= K[A,B] \qquad (-1 \le B < A \le 1). \end{aligned}$$

The authors in [1] have recently introduced a new generalised derivative operator $\mu^{n,m}_{\lambda_1,\lambda_2}$, as the following:

Definition 1.1. For $f \in \mathcal{A}$ the generalised derivative operator $\mu^{n,m}_{\lambda_1,\lambda_2}$ is defined by $\mu^{n,m}_{\lambda_1,\lambda_2} : \mathcal{A} \to \mathcal{A}$

$$\mu^{n,m}_{\lambda_1,\lambda_2} f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^{m}}\ c(n,k) a_k z^k, \quad (z \in U),$$

where $n,m \in \mathbb{N}_0 = \{0,1,2\dots\}$, $\lambda_2 \ge \lambda_1 \ge 0$ and $c(n,k) = \binom{n+k-1}{n} = \frac{(n+1)_{k-1}}{(1)_{k-1}}$.

Special cases of this operator includes the Ruscheweyh derivative operator in the cases $\mu^{n,1}_{\lambda_1,0} \equiv \mu^{n,m}_{0,0} \equiv \mu^{n,0}_{0,\lambda_2} \equiv R^n$ [17], the Salagean derivative operator $\mu^{0,m+1}_{1,0} \equiv S^n$ [18], the generalised Ruscheweyh derivative operator $\mu^{n,2}_{\lambda_1,0} \equiv R^n_\lambda$ [5], the generalised Salagean derivative operator introduced by Al-Oboudi $\mu^{0,m+1}_{\lambda_1,0} \equiv S^n_\beta$ [3], and the generalised Darus and Al-Shaqsi derivative operator $\mu^{n,m+1}_{\lambda_1,0} \equiv D^n_{\lambda,\beta}$ [4]. It is easily seen that $\mu^{0,1}_{\lambda_1,0} f(z) = \mu^{0,m}_{0,0} f(z) = \mu^{0,0}_{0,\lambda_2} f(z) = f(z)$ and $\mu^{1,1}_{\lambda_1,0} f(z) = \mu^{1,m}_{0,0} f(z) = \mu^{1,0}_{0,\lambda_2} f(z) = zf'(z)$ and also $\mu^{a-1,0}_{\lambda_1,0} f(z) = \mu^{a-1,m}_{0,0} f(z)$ where $a = 1,2,3,\dots$.

Let us remind the well known Carlson-Shaffer operator $L(a,c)$ [6] associated with the incomplete beta function $h(a,c;z)$, defined by

$$\begin{aligned} L(a,c) &: \mathcal{A} \to \mathcal{A}, \\ L(a,c)f(z) &:= h(a,c;z) * f(z), \ \ (z \in U), \end{aligned}$$

where

$$h(a,c;z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k,$$

a is any real number and $c \notin z_0^-$; $z_0^- = \{0,-1,-2,...\}$.

It is easily seen that

$$\begin{aligned} \mu_{0,\lambda_2}^{0,0} f(z) &= \mu_{0,0}^{0,m} f(z) = \mu_{\lambda_1,0}^{0,1} f(z) = L(0,0)f(z) = f(z), \\ \mu_{0,0}^{1,m} f(z) &= \mu_{0,\lambda_2}^{1,0} f(z) = \mu_{\lambda_1,0}^{1,1} f(z) = L(2,1)f(z) = zf'(z). \end{aligned}$$

Furthermore, we note that

$$\mu_{0,0}^{n,m} f(z) = \mu_{0,\lambda_2}^{n,0} f(z) = \mu_{\lambda_1,0}^{n,1} f(z) = L(n+1,1)f(z) = D^n f(z) \quad (n \in \mathbb{N}_0),$$

where the symbol D^n denotes the familiar Ruscheweyh derivative [17] (also, see [2]) for $n \in \mathbb{N}_0$. By using the new generalised derivative operator $\mu_{\lambda_1,\lambda_2}^{n,m}$, we introduce the following classes of analytic functions for $\phi, \psi \in N, \lambda_2 \geq \lambda_1 \geq 0$ and $n, m \in \mathbb{N}_0 = \{0,1,2... .\}$,

$$\begin{aligned} S_{\lambda_1,\lambda_2}^{n,m}(\phi) &:= \left\{ f \in \mathcal{A} : \ \mu_{\lambda_1,\lambda_2}^{n,m} f(z) \in S^*(\phi) \right\}, \\ K_{\lambda_1,\lambda_2}^{n,m}(\phi) &:= \left\{ f \in \mathcal{A} : \ \mu_{\lambda_1,\lambda_2}^{n,m} f(z) \in K(\phi) \right\}, \\ C_{\lambda_1,\lambda_2}^{n,m}(\phi,\psi) &:= \left\{ f \in \mathcal{A} : \ \mu_{\lambda_1,\lambda_2}^{n,m} f(z) \in C(\phi,\psi) \right\}. \end{aligned}$$

We note that the class

$$S_{0,0}^{a-1,m}(\phi) = S_{0,\lambda_2}^{a-1,0}(\phi) = S_a(\phi),$$

was studied by Padmanabhan and Parvatham in [14],

$$K_{0,0}^{a-1,m}(\phi) = K_{0,\lambda_2}^{a-1,0}(\phi) = K_a(\phi),$$

were studied by Padmanabhan and Manjini in [13]. Obviously, for the special choices function ϕ and variables n, m we have the following relation ships:

$$\begin{aligned} S_{0,\lambda_2}^{0,0}\left(\frac{1+z}{1-z}\right) &\equiv S_{0,0}^{0,m}\left(\frac{1+z}{1-z}\right) \equiv S^*\left(\frac{1+z}{1-z}\right), \\ K_{0,\lambda_2}^{0,0}\left(\frac{1+z}{1-z}\right) &\equiv K_{0,0}^{0,m}\left(\frac{1+z}{1-z}\right) \equiv K\left(\frac{1+z}{1-z}\right), \\ C_{0,\lambda_2}^{0,0}\left(\frac{1+z}{1-z}, \psi\right) &\equiv C\left(\frac{1+z}{1-z}, \psi\right). \end{aligned}$$

And

$$S^{0,0}_{0,\lambda_2}\left(\frac{1+(1-2\alpha)z}{1-z}\right) \equiv S^{0,m}_{0,0}\left(\frac{1+(1-2\alpha)z}{1-z}\right) \equiv S^*(\alpha),$$
$$K^{0,0}_{0,\lambda_2}\left(\frac{1+(1-2\alpha)z}{1-z}\right) \equiv K^{0,m}_{0,0}\left(\frac{1+(1-2\alpha)z}{1-z}\right) = K(\alpha) \quad (0 \le \alpha < 1).$$

In particular, we set

$$S^{n,m}_{\lambda_1,\lambda_2}\left(\frac{1+Az}{1+Bz}\right) = S^{n,m}_{\lambda_1,\lambda_2}[A,B] \quad (-1 \le B < A \le 1),$$
$$K^{n,m}_{\lambda_1,\lambda_2}\left(\frac{1+Az}{1+Bz}\right) = K^{n,m}_{\lambda_1,\lambda_2}[A,B] \quad (-1 \le B < A \le 1).$$

In this paper, we investigate several inclusion properties of the classes $S^{n,m}_{\lambda_1,\lambda_2}(\phi)$, $K^{n,m}_{\lambda_1,\lambda_2}(\phi)$ and $C^{n,m}_{\lambda_1,\lambda_2}(\phi,\psi)$. The integral preserving properties in connection with the operator $\mu^{n,m}_{\lambda_1,\lambda_2}$ are also considered. Furthermore, relevant connection of the results presented here with those obtained in earlier works are pointed out.

We first state some preliminary lemmas which shall be used in our investigation.

2 Preliminary Results

To establish our results, we recall the following:

Lemma 2.1 (Ruscheweyh and Sheil-Small ([16], p.54). If $f \in K$, $g \in S^*$, then for each analytic function ϕ in U,

$$\frac{(f * \phi g)(U)}{(f * g)(U)} \subset \overline{co}\phi(U),$$

where $\overline{co}\phi(U)$ denotes the closed convex hull of $\phi(U)$.

Lemma 2.2 (Ruscheweyh [15]). Let $0 < a \le c$. If $c \ge 2$ or $a + c \ge 3$, then the function

$$h(a,c;z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k \quad (z \in U),$$

belongs to the class K of convex functions.

Lemma 2.3 ([11]). Let ϕ be analytic, univalent, convex in U, with $\phi(0) = 1$ and

$$\mathrm{Re}(\beta\phi(z) + \gamma) > 0 \quad (\beta, \gamma \in \mathbb{C};\ z \in U).$$

If $p(z)$ is analytic in U, with $p(0) = \phi(0)$, then

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \phi(z) \Rightarrow p(z) \prec \phi(z).$$

3 Inclusion Properties Involving the Operator $\mu_{\lambda_1,\lambda_2}^{n,m}$

Our main results, are the following:

Theorem 3.1. $f(z) \in K_{\lambda_1,\lambda_2}^{n,m}(\phi)$ if and only if $zf'(z) \in S_{\lambda_1,\lambda_2}^{n,m}(\phi)$.

Proof. Consider

$$\mu_{\lambda_1,\lambda_2}^{n,m} f(z) = h_{\lambda_1,\lambda_2}^{n,m}(z) * f(z), \tag{3.1}$$

where

$$h_{\lambda_1,\lambda_2}^{n,m}(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} c(n,k) z^k. \tag{3.2}$$

By the definition of the class $K_{\lambda_1,\lambda_2}^{n,m}(\phi)$ and using the well-known property of convolution $z(f * g)'(z) = (f * zg')(z)$, we have

$$\begin{aligned}
f \in K_{\lambda_1,\lambda_2}^{n,m}(\phi) \quad &\Leftrightarrow \quad 1 + \frac{z(\mu_{\lambda_1,\lambda_2}^{n,m} f(z))''}{(\mu_{\lambda_1,\lambda_2}^{n,m} f(z))'} \prec \phi(z), \\
&\Leftrightarrow \quad \frac{\left[z(\mu_{\lambda_1,\lambda_2}^{n,m} f(z))'\right]'}{(\mu_{\lambda_1,\lambda_2}^{n,m} f(z))'} \prec \phi(z), \\
&\Leftrightarrow \quad \frac{z\left[z\left[h_{\lambda_1,\lambda_2}^{n,m}(z) * f(z)\right]'\right]'}{z\left[h_{\lambda_1,\lambda_2}^{n,m}(z) * f(z)\right]'} \prec \phi(z), \\
&\Leftrightarrow \quad \frac{z\left[h_{\lambda_1,\lambda_2}^{n,m}(z) * zf'(z)\right]'}{h_{\lambda_1,\lambda_2}^{n,m}(z) * zf'(z)} \prec \phi(z), \\
&\Leftrightarrow \quad \frac{z\left[\mu_{\lambda_1,\lambda_2}^{n,m} zf'(z)\right]'}{\mu_{\lambda_1,\lambda_2}^{n,m} zf'(z)} \prec \phi(z), \\
&\Leftrightarrow \quad zf'(z) \in S_{\lambda_1,\lambda_2}^{n,m}(\phi).
\end{aligned}$$

Theorem 3.2. Let $\phi \in N$, $\lambda_2 \geq \lambda_1 \geq 0$, $n_2 \geq n_1 \geq 0$ and $n_1, n_2, m \in \mathbb{N}_0$. If $n_2 \geq 1$ or $n_1 + n_2 \geq 1$, then

$$S_{\lambda_1,\lambda_2}^{n_2,m}(\phi) \subset S_{\lambda_1,\lambda_2}^{n_1,m}(\phi).$$

Proof. We suppose that $f \in S^{n_2,m}_{\lambda_1,\lambda_2}(\phi)$. Then there exists an analytic function w in U with $|w(z)| < 1$ $(z \in U)$ and $w(0) = 0$ such that

$$\frac{z(\mu^{n_2,m}_{\lambda_1,\lambda_2} f(z))'}{(\mu^{n_2,m}_{\lambda_1,\lambda_2} f(z))} = \phi(w(z)), \quad (z \in U). \tag{3.3}$$

Where ϕ is analytic and convex univalent with $\phi(0) = 1$ and $\mathrm{Re}(\phi(z)) > 0, \quad (z \in U)$. We set

$$\mu^{n_1,m}_{\lambda_1,\lambda_2} f(z) = \mu^{n_2,m}_{\lambda_1,\lambda_2} f(z) * \psi^{n_1}_{n_2}(z),$$

where

$$\psi^{n_1}_{n_2}(z) = z + \sum_{k=2}^{\infty} \frac{(n_1+1)_{k-1}}{(n_2+1)_{k-1}} z^k.$$

We get

$$\begin{aligned}
\frac{z(\mu^{n_1,m}_{\lambda_1,\lambda_2} f(z))'}{(\mu^{n_1,m}_{\lambda_1,\lambda_2} f(z))} &= \frac{z\left[\mu^{n_2,m}_{\lambda_1,\lambda_2} f(z) * \psi^{n_1}_{n_2}(z)\right]'}{\mu^{n_2,m}_{\lambda_1,\lambda_2} f(z) * \psi^{n_1}_{n_2}(z)} \\
&= \frac{\psi^{n_1}_{n_2}(z) * \left[z(\mu^{n_2,m}_{\lambda_1,\lambda_2} f(z))'\right]}{\psi^{n_1}_{n_2}(z) * \mu^{n_2,m}_{\lambda_1,\lambda_2} f(z)} \\
&= \frac{\psi^{n_1}_{n_2}(z) * \phi(w(z))p(z)}{\psi^{n_1}_{n_2}(z) * p(z)}, \qquad (3.4)
\end{aligned}$$

where $p(z) = \mu^{n_2,m}_{\lambda_1,\lambda_2} f(z)$.

It follows from Lemma 2.2 that $\psi^{n_1}_{n_2}(z) \in K$ and it follows from the definition of the class $S^{n,m}_{\lambda_1,\lambda_2}(\phi)$ that $p(z) \in S^*$. Therefore applying Lemma 2.1 to (3.4) we obtain

$$\frac{\left\{\psi^{n_1}_{n_2}(z) * \phi(w(z))p\right\}(U)}{\left\{\psi^{n_1}_{n_2}(z) * p\right\}(U)} \subset \overline{co}\, \phi(w(U)) \subset \phi(U). \tag{3.5}$$

Since ϕ is analytic and convex univalent. Therefore from the definition of subordination and (3.5), we note that (3.4) is subordinate to ϕ in U and consequently $f(z) \in S^{n_1,m}_{\lambda_1,\lambda_2}(\phi)$. This completes the proof of Theorem 3.2.

Theorem 3.3. Let $\phi \in N$, $\lambda_2 \geq \lambda_1 \geq 0$, $n_2 \geq n_1 \geq 0$ and $n_1, n_2, m \in \mathbb{N}_0$. If $n_2 \geq 1$ or $n_1 + n_2 \geq 1$, then

$$K^{n_2,m}_{\lambda_1,\lambda_2}(\phi) \subset K^{n_1,m}_{\lambda_1,\lambda_2}(\phi).$$

Proof. Let $f(z) \in K^{n_2,m}_{\lambda_1,\lambda_2}(\phi)$ we want to show $f(z) \in K^{n_1,m}_{\lambda_1,\lambda_2}(\phi)$. Applying Theorem 3.1 and Theorem 3.2 we observe that

$$\begin{aligned}
f(z) \in K^{n_2,m}_{\lambda_1,\lambda_2}(\phi) &\Leftrightarrow zf'(z) \in S^{n_2,m}_{\lambda_1,\lambda_2}(\phi), \\
&\Rightarrow zf'(z) \in S^{n_1,m}_{\lambda_1,\lambda_2}(\phi), \\
&\Leftrightarrow f(z) \in K^{n_1,m}_{\lambda_1,\lambda_2}(\phi).
\end{aligned}$$

Theorem 3.4. Let $\phi \in N$, $\lambda_2 \geq \lambda_1 \geq 0$ and $n, m \in \mathbb{N}_0$ then

$$\begin{array}{rrcl} i) & S^{n,m}_{\lambda_1,\lambda_2}(\phi) & \subset & S^{n,m+1}_{\lambda_1,\lambda_2} f(z) \\ ii) & K^{n,m}_{\lambda_1,\lambda_2}(\phi) & \subset & K^{n,m+1}_{\lambda_1,\lambda_2}(\phi) \end{array}$$

Proof. i) We suppose that $f(z) \in S^{n,m}_{\lambda_1,\lambda_2}(\phi)$. Using similar arguments as in the proof of Theorem 3.2, and set

$$\mu^{n,m+1}_{\lambda_1,\lambda_2} f(z) = \mu^{n,m}_{\lambda_1,\lambda_2} f(z) * \psi^{\lambda_1}_{\lambda_2}(z),$$

where

$$\psi^{\lambda_1}_{\lambda_2}(z) = z + \sum_{k=2}^{\infty} \frac{1 + \lambda_1 (k-1)}{1 + \lambda_2 (k-1)} z^k.$$

We obtain

$$\begin{aligned} \frac{z(\mu^{n,m+1}_{\lambda_1,\lambda_2} f(z))'}{\mu^{n,m+1}_{\lambda_1,\lambda_2} f(z)} &= \frac{z\left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z) * \psi^{\lambda_1}_{\lambda_2}(z)\right]'}{\mu^{n,m}_{\lambda_1,\lambda_2} f(z) * \psi^{\lambda_1}_{\lambda_2}(z)}, \\ &= \frac{\psi^{\lambda_1}_{\lambda_2}(z) * z(\mu^{n,m}_{\lambda_1,\lambda_2} f(z))'}{\psi^{\lambda_1}_{\lambda_2}(z) * \mu^{n,m}_{\lambda_1,\lambda_2} f(z)}, \\ &= \frac{\psi^{\lambda_1}_{\lambda_2}(z) * \phi(w(z))p(z)}{\psi^{\lambda_1}_{\lambda_2}(z) * p(z)}. \end{aligned} \tag{3.6}$$

Where $p(z) = \mu^{n,m}_{\lambda_1,\lambda_2} f(z)$ and w is an analytic function in U with $|w(z)| < 1 \quad (z \in U)$ and $w(0) = 0$. It follows from the definition of the class $S^{n,m}_{\lambda_1,\lambda_2}(\phi)$ that $p(z) \in S^*$. And by classical results in the class of convex, the coefficients problem for convex: $|a_n| \leq 1$, and here $\frac{1+\lambda_1(k-1)}{1+\lambda_2(k-1)} \leq 1$ since $\lambda_2 \geq \lambda_1 \geq 0$, so we find $\psi^{\lambda_1}_{\lambda_2} \in K$. Hence it follows from applying Lemma 2.1 to (3.6) that

$$\frac{\left\{\psi^{\lambda_1}_{\lambda_2}(z) * \phi(w(z))p\right\}(U)}{\left\{\psi^{\lambda_1}_{\lambda_2}(z) * p\right\}(U)} \subset \overline{co}\, \phi(w(U)) \subset \phi(U). \tag{3.7}$$

Since ϕ is analytic and convex univalent. And from the definition of subordination and (3.7). Thus (3.6) is subordinate to ϕ in U and consequently $f(z) \in S^{n,m+1}_{\lambda_1,\lambda_2} f(z)$. The proof of Theorem 3.4 is complete.

ii) Let $f(z) \in K^{n,m}_{\lambda_1,\lambda_2}(\phi)$. We shall show $f(z) \in K^{n,m+1}_{\lambda_1,\lambda_2}(\phi)$. By applying Theorem 3.1, and Theorem 3.4 (i), we have

$$\begin{aligned} f(z) \in K^{n,m}_{\lambda_1,\lambda_2}(\phi) &\Leftrightarrow zf'(z) \in S^{n,m}_{\lambda_1,\lambda_2}(\phi), \\ &\Rightarrow zf'(z) \in S^{n,m+1}_{\lambda_1,\lambda_2} f(z), \\ &\Leftrightarrow f(z) \in K^{n,m+1}_{\lambda_1,\lambda_2}(\phi). \end{aligned}$$

Theorem 3.5. Let $\phi \in N$, $\lambda_3 \geq \lambda_2 \geq \lambda_1 \geq 0$ and $n, m \in \mathbb{N}_0$, then

$$S^{n,m}_{\lambda_1,\lambda_2}(\phi) \quad \subset \quad S^{n,m}_{\lambda_1,\lambda_3}(\phi).$$

Proof. Let $f \in S^{n,m}_{\lambda_1,\lambda_2}(\phi)$. Applying the definition of the class $S^{n,m}_{\lambda_1,\lambda_2}(\phi)$, then the setting,

$$\mu^{n,m}_{\lambda_1,\lambda_3} f(z) = \mu^{n,m}_{\lambda_1,\lambda_2} f(z) * \psi^m_{\lambda_2,\lambda_3}(z),$$

where

$$\psi^m_{\lambda_2,\lambda_3}(z) = z + \sum_{k=2}^{\infty} \left(\frac{1 + \lambda_2 (k-1)}{1 + \lambda_3 (k-1)} \right)^m z^k, \ (z \in U).$$

And by classical results in the class of convex, the coefficients problem for convex: $|a_n| \leq 1$, and here

$$\left(\frac{1 + \lambda_2 (k-1)}{1 + \lambda_3 (k-1)} \right)^m \leq 1, \text{ since } \lambda_3 \geq \lambda_2 \geq 0,$$

so we get $\psi^m_{\lambda_2,\lambda_3}(z) \in K$. After that, using the same arguments as in the Theorem 3.4 we obtain the desired result.

Theorem 3.6. Let $\phi \in N$, $\lambda_3 \geq \lambda_2 \geq \lambda_1 \geq 0$ and $n \in \mathbb{N}_0$, then.

$$\text{If } \ m \in \mathbb{N} \ \text{ then } \ S^{n,m}_{\lambda_2,\lambda_3}(\phi) \quad \subset \quad S^{n,m}_{\lambda_1,\lambda_3}(\phi).$$

Proof. Let $f(z) \in S^{n,m}_{\lambda_2,\lambda_3}(\phi)$. Applying the definition of the class $S^{n,m}_{\lambda_2,\lambda_3}(\phi)$, then the setting.

$$\mu^{n,m}_{\lambda_1,\lambda_3} f(z) = \mu^{n,m}_{\lambda_2,\lambda_3} f(z) * \psi^m_{\lambda_1,\lambda_2}(z),$$

where

$$\psi^m_{\lambda_1,\lambda_2}(z) = z + \sum_{k=2}^{\infty} \left(\frac{1 + \lambda_1 (k-1)}{1 + \lambda_2 (k-1)} \right)^{m-1} z^k, \ (z \in U).$$

And using the same arguments as in the Theorem 3.4 we obtain the desired result.

Corollary 3.1. Let $\phi \in N$, $\lambda_3 \geq \lambda_2 \geq \lambda_1 \geq 0$ and $n, m \in \mathbb{N}_0$, then

$$\begin{array}{ll} i) & K^{n,m}_{\lambda_1,\lambda_2}(\phi) \subset K^{n,m}_{\lambda_1,\lambda_3}(\phi) \\ ii) & \text{If} \quad m \in \mathbb{N} \quad \text{then} \quad K^{n,m}_{\lambda_2,\lambda_3}(\phi) \subset K^{n,m}_{\lambda_1,\lambda_3}(\phi). \end{array}$$

Taking

$$\phi(z) = \frac{1 + Az}{1 + Bz} \ \ (-1 \leq B < A \leq 1, \ z \in U).$$

In Corollary 3.1 and Theorems 3.5 and 3.6 we have the following Corollary.

Corollary 3.2. Let $\lambda_3 \geq \lambda_2 \geq \lambda_1 \geq 0$ and $n, m \in \mathbb{N}_0$, then

$$\begin{aligned} S^{n,m}_{\lambda_1,\lambda_2}[A,B] &\subset S^{n,m}_{\lambda_1,\lambda_3}[A,B], \quad (-1 \leq B < A \leq 1), \\ K^{n,m}_{\lambda_1,\lambda_2}[A,B] &\subset K^{n,m}_{\lambda_1,\lambda_3}[A,B] \ \ (-1 \leq B < A \leq 1), \end{aligned}$$

if $m \in \mathbb{N}$ then

$$\begin{aligned} S^{n,m}_{\lambda_2,\lambda_3}[A,B] &\subset S^{n,m}_{\lambda_1,\lambda_3}[A,B] \ \ (-1 \leq B < A \leq 1), \\ K^{n,m}_{\lambda_2,\lambda_3}[A,B] &\subset K^{n,m}_{\lambda_1,\lambda_3}[A,B] \ \ (-1 \leq B < A \leq 1). \end{aligned}$$

Theorem 3.7. If $f(z) \in S^{n,m}_{\lambda_1,\lambda_2}(\phi)$ for $n, m \in \mathbb{N}_0$. Then $F_\mu(f) \in S^{n,m}_{\lambda_1,\lambda_2}(\phi)$. Where F_μ is the integral operator defined by

$$F_\mu(f) = F_\mu(f)(z) := \frac{\mu+1}{z^\mu} \int_0^z t^{\mu-1} f(t)dt \quad (\mu \geq 0). \tag{3.8}$$

Proof. Let $f(z) \in S^{n,m}_{\lambda_1,\lambda_2}(\phi)$ and

$$p(z) = \frac{z\left[\mu^{n,m}_{\lambda_1,\lambda_2} F_\mu(f)\right]'}{\mu^{n,m}_{\lambda_1,\lambda_2} F_\mu(f)}.$$

From (3.8), we have

$$z(F_\mu(f))'(z) + \mu F_\mu(f)(z) = (\mu+1)f(z)$$

and so

$$(h^{n,m}_{\lambda_1,\lambda_2} * z(F_\mu(f))')(z) + \mu(h^{n,m}_{\lambda_1,\lambda_2} * F_\mu(f))(z) = (\mu+1)(h^{n,m}_{\lambda_1,\lambda_2} * f)(z).$$

Using the fact

$$z(h^{n,m}_{\lambda_1,\lambda_2} * F_\mu(f))'(z) = (h^{n,m}_{\lambda_1,\lambda_2} * zF'_\mu(f))(z),$$

and setting (3.1), we obtain

$$z(\mu^{n,m}_{\lambda_1,\lambda_2} F_\mu(f))'(z) + \mu(\mu^{n,m}_{\lambda_1,\lambda_2} F_\mu(f))(z) = (\mu+1)\mu^{n,m}_{\lambda_1,\lambda_2} f(z). \tag{3.9}$$

Differentiating (3.9), we have

$$p(z) + \mu = (\mu+1)\left[\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{\mu^{n,m}_{\lambda_1,\lambda_2} F_\mu(f)}\right]. \tag{3.10}$$

Making use of the logarithmic differentiation on both sides of (3.10) and multiplying the resulting equation by z, we have

$$p(z)+\frac{zp'(z)}{p(z)+\mu}=\frac{z(\mu^{n,m}_{\lambda_1,\lambda_2}f(z))'}{\mu^{n,m}_{\lambda_1,\lambda_2}f(z)}. \tag{3.11}$$

By applying Lemma 2.3 to (3.11), it follows that $p \prec \phi$ in U, that is $F_\mu(f) \in S^{n,m}_{\lambda_1,\lambda_2}(\phi)$.

Remark 3.1. special case of this operator $\mu^{n,m}_{\lambda_1,\lambda_2}$ include the Bernardi integral operator $F_\mu(f)$ in two cases, for $n=0, m=0$, and $\lambda_1=\frac{1}{1+\mu}$ and also for $n=0, m=1$ and $\lambda_2=\frac{1}{1+\mu}$.

Theorem 3.8. If $f \in S^{n,m}_{\lambda_1,\lambda_2}(\phi)$. Then

$$\psi=\int_0^z \frac{f(t)}{t}dt \in K^{n,m}_{\lambda_1,\lambda_2}(\phi).$$

Proof. Let $f \in S^{n,m}_{\lambda_1,\lambda_2}(\phi)$. Now $z\psi'=f$ that is $z\psi' \in S^{n,m}_{\lambda_1,\lambda_2}(\phi)$, applying Theorem 3.1, we see

$$z\psi' \in S^{n,m}_{\lambda_1,\lambda_2}(\phi) \Leftrightarrow \psi \in K^{n,m}_{\lambda_1,\lambda_2}(\phi).$$

This proves the Theorem.

To prove the Theorems below, we need the following Lemma.

Lemma 3.1. Let $\phi \in N$. If $f \in K$ and $q \in S^*(\phi)$, then $f * q \in S^*(\phi)$.

Proof. Let $q \in S^*(\phi)$. Then there exists an analytic function w in U with $|w(z)|<1,\quad (z \in U)$ and $w(0)=0$ such that

$$\frac{zq'(z)}{q(z)}=\phi(w(z)), \quad (z \in U).$$

Where ϕ is analytic and convex univalent with $\phi(0)=1$ and $\mathrm{Re}(\phi(z))>0, \quad (z \in U)$. Thus we have

$$\begin{aligned}\frac{z\,(f(z)*q(z))'}{f(z)*q(z)} &= \frac{f(z)*zq'(z)}{f(z)*q(z)}\\ &= \frac{f(z)*\phi(w(z))q(z)}{f(z)*q(z)}, \quad (z \in U).\end{aligned} \tag{3.12}$$

By using similar arguments to those used in the proof of Theorem 3.2, we conclude that (3.12) is subordinate to ϕ in U and so $f * q \in S^*(\phi)$.

Theorem 3.9. Let $\phi, \psi \in N$, $\lambda_2 \geq \lambda_1 \geq 0$ and $n_1, n_2, m \in \mathbb{N}_0$. If $n_2 \geq \min\{1, 1 - n_1\}$, then

$$C_{\lambda_1,\lambda_2}^{n_2,m}(\phi,\psi) \subset C_{\lambda_1,\lambda_2}^{n_1,m}(\phi,\psi) \subset C_{\lambda_1,\lambda_2}^{n_1,m+1}(\phi,\psi).$$

Proof. First of all, we show that

$$C_{\lambda_1,\lambda_2}^{n_2,m}(\phi,\psi) \subset C_{\lambda_1,\lambda_2}^{n_1,m}(\phi,\psi). \tag{3.13}$$

Let $f \in C_{\lambda_1,\lambda_2}^{n_2,m}(\phi,\psi)$. Then there exist a function $q \in S^*(\phi)$ such that

$$\frac{z\left(\mu_{\lambda_1,\lambda_2}^{n_2,m} f(z)\right)'}{q(z)} \prec \psi(z) \quad (z \in U). \tag{3.14}$$

Then there exists analytic function w in U with $|w(z)| < 1 \quad (z \in U)$, and $w(0) = 0$ such that

$$z\left(\mu_{\lambda_1,\lambda_2}^{n_2,m} f(z)\right)' = \psi(w(z))q(z) \quad (z \in U),$$

where ψ is analytic and convex univalent with $\psi(0) = 1$ and $\mathrm{Re}(\psi(z)) > 0, \quad (z \in U)$. We setting

$$\mu_{\lambda_1,\lambda_2}^{n_1,m} f(z) = \mu_{\lambda_1,\lambda_2}^{n_2,m} f(z) * \psi_{n_2}^{n_1}(z),$$

where

$$\psi_{n_2}^{n_1}(z) = z + \sum_{k=2}^{\infty} \frac{(n_1+1)_{k-1}}{(n_2+1)_{k-1}} z^k.$$

By virtue of Lemma 2.2, $\psi_{n_2}^{n_1} \in K$ and using Lemma 3.1, we see that $\psi_{n_2}^{n_1}(z) * q(z)$ belongs to $S^*(\phi)$. Then we have

$$\begin{aligned} \frac{z\left(\mu_{\lambda_1,\lambda_2}^{n_1,m} f(z)\right)'}{g(z)} &= \frac{z\left[\mu_{\lambda_1,\lambda_2}^{n_2,m} f(z) * \psi_{n_2}^{n_1}(z)\right]'}{\psi_{n_2}^{n_1}(z) * q(z)}, \\ &= \frac{\psi_{n_2}^{n_1}(z) * z\left(\mu_{\lambda_1,\lambda_2}^{n_2,m} f(z)\right)'}{\psi_{n_2}^{n_1}(z) * q(z)}, \\ &= \frac{\psi_{n_2}^{n_1}(z) * \psi(w(z))q(z)}{\psi_{n_2}^{n_1}(z) * q(z)}. \end{aligned} \tag{3.15}$$

By using similar arguments to those used in the proof of Theorem 3.2, we conclude that (3.15) is subordinated to ψ in U. This implies that $f \in C_{\lambda_1,\lambda_2}^{n_1,m}(\phi,\psi)$.

Moreover, the proof of the second part is similar to that of the first part and so we omit the details involved.

4 Convolution Results and Inclusion Properties Involving Various Operators

The next theorem shows that the classes

$$S^{n,m}_{\lambda_1,\lambda_2}(\phi),\ K^{n,m}_{\lambda_1,\lambda_2}(\phi) \text{ and } C^{n,m}_{\lambda_1,\lambda_2}(\phi,\psi)$$

are invariant under convolution with convex functions.

Theorem 4.1. Let $\phi, \psi \in N$, $\lambda_2 \geq \lambda_1 \geq 0$, $n, m \in \mathbb{N}_0$ and let $g \in K$. Then

$$\begin{array}{ll} i) & f \in S^{n,m}_{\lambda_1,\lambda_2}(\phi) \Rightarrow g * f \in S^{n,m}_{\lambda_1,\lambda_2}(\phi), \\ ii) & f \in K^{n,m}_{\lambda_1,\lambda_2}(\phi) \Rightarrow g * f \in K^{n,m}_{\lambda_1,\lambda_2}(\phi), \\ iii) & f \in C^{n,m}_{\lambda_1,\lambda_2}(\phi,\psi) \Rightarrow g * f \in C^{n,m}_{\lambda_1,\lambda_2}(\phi,\psi). \end{array}$$

Proof. i) We begin by assuming $f \in S^{n,m}_{\lambda_1,\lambda_2}(\phi)$ and $g \in K$. In the proof we use the same techniques as in the proof of Theorem 3.2. Let

$$\frac{z(\mu^{n,m}_{\lambda_1,\lambda_2} f(z))'}{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)} = \phi(w(z)), \quad (z \in U).$$

and

$$p(z) = \mu^{n,m}_{\lambda_1,\lambda_2} f(z).$$

Using the following equality

$$z\left(h^{n,m}_{\lambda_1,\lambda_2} * f\right)'(z) = \left(h^{n,m}_{\lambda_1,\lambda_2} * zf'\right)(z),$$

from (3.1) we write

$$\begin{aligned} \frac{z\left[\mu^{n,m}_{\lambda_1,\lambda_2}(g * f)\right]'(z)}{\mu^{n,m}_{\lambda_1,\lambda_2}(g * f)(z)} &= \frac{z\left[h^{n,m}_{\lambda_1,\lambda_2}(z) * (g * f)(z)\right]'}{h^{n,m}_{\lambda_1,\lambda_2}(z) * (g * f)(z)}, \\ &= \frac{g(z) * z(\mu^{n,m}_{\lambda_1,\lambda_2} f(z))'}{g(z) * \mu^{n,m}_{\lambda_1,\lambda_2} f(z)}, \\ &= \frac{g(z) * \phi(w(z))p(z)}{g(z) * p(z)} \prec \phi(z). \end{aligned}$$

Consequently $g * f \in S^{n,m}_{\lambda_1,\lambda_2}(\phi)$.

ii) Let $f \in K^{n,m}_{\lambda_1,\lambda_2}(\phi)$. Then by Theorem 3.1 and from Theorem 4.1(i), we have

$$\begin{aligned} f \in K^{n,m}_{\lambda_1,\lambda_2}(\phi) &\Leftrightarrow zf'(z) \in S^{n,m}_{\lambda_1,\lambda_2}(\phi), \\ &\Rightarrow g * [zf'(z)] \in S^{n,m}_{\lambda_1,\lambda_2}(\phi), \\ &\Leftrightarrow z(g * f)' \in S^{n,m}_{\lambda_1,\lambda_2}(\phi), \\ &\Leftrightarrow g * f \in K^{n,m}_{\lambda_1,\lambda_2}(\phi). \end{aligned}$$

iii) Let $f \in C^{n,m}_{\lambda_1,\lambda_2}(\phi,\psi)$. Then there exists a function $q \in S^*(\phi)$ such that

$$z\left(\mu^{n,m}_{\lambda_1,\lambda_3} f(z)\right)' = \psi(w(z))q(z) \quad (z \in U),$$

where w is an analytic function in U with $|w(z)| < 1 \quad (z \in U)$ and $w(0) = 0$. From Lemma 3.1, we have that $g * q \in S^*(\phi)$. Applying the same method in the proof of Theorem 3.9 and using the fact that $z(f*g)'(z) = (f * zg')(z)$ we have

$$\begin{aligned} \frac{z\left[\mu^{n,m}_{\lambda_1,\lambda_2}(g*f)\right]'(z)}{(g*q)(z)} &= \frac{g(z) * z(\mu^{n,m}_{\lambda_1,\lambda_2} f(z))'}{g(z)*q(z)}, \\ &= \frac{g(z) * \psi(w(z))q(z)}{g(z)*q(z)} \prec \psi(z) \quad (z \in U). \end{aligned}$$

We obtain (iii).

Now we consider the following operators [17, 12] given by

$$\begin{aligned} \psi_1(z) &= \sum_{k=1}^{\infty} \frac{1+c}{k+c} z^k \ \ (\text{Re}\,\{c\} \geq 0; \ (z \in U), \\ \psi_2(z) &= \frac{1}{1-x} \log\left[\frac{1-xz}{1-z}\right] \quad (Log\, 1 = 0; \ |x| \leq 1, \ x \neq 1; \ z \in U). \end{aligned} \tag{4.1}$$

It is well known [17] that the operators ψ_1 and ψ_2 are convex univalent in U. Therefore, we have the following result, which can be obtained from Theorem 4.1 immediately.

Corollary 4.1. *Let* $\lambda_2 \geq \lambda_1 \geq 0$, $n, m \in \mathbb{N}_0$, ϕ, $\psi \in N$ and let $\psi_i\,(i = 1,2)$ be defined by (4.1). Then

$$\begin{array}{ll} i) & f \in S^{n,m}_{\lambda_1,\lambda_2}(\phi) \Rightarrow \psi_i * f \in S^{n,m}_{\lambda_1,\lambda_2}(\phi), \\ ii) & f \in K^{n,m}_{\lambda_1,\lambda_2}(\phi) \Rightarrow \psi_i * f \in K^{n,m}_{\lambda_1,\lambda_2}(\phi), \\ iii) & f \in C^{n,m}_{\lambda_1,\lambda_2}(\phi,\psi) \Rightarrow \psi_i * f \in C^{n,m}_{\lambda_1,\lambda_2}(\phi,\psi). \end{array}$$

5 Open Problem

The generalised derivative operator for the three classes $S^{n,m}_{\lambda_1,\lambda_2}(\phi)$, $K^{n,m}_{\lambda_1,\lambda_2}(\phi)$ and $C^{n,m}_{\lambda_1,\lambda_2}(\phi,\psi)$ can be applied in the subordination and superordination theorem. In fact, basic properties such as the coefficient estimates are yet to be solved for these type of classes.

Acknowledgement: This work is fully supported by UKM-GUP-TMK-07-02-107, Malaysia.

References

[1] M. H. Al-Abbadi and M. Darus, Differential subordination for new generalised derivative operator, Acta. Univ. Apul., No 20/2009, 265-280.

[2] H. S. Al-Amiri, On Ruscheweyh derivatives, Ann. Polon. Math., vol. 38, no. 1, pp, (1980), 88-94.

[3] F. M. Al-Oboudi, On univalent functions defined by a generalised Salagean Operator, Int, J. Math. Math. Sci., **27**(2004), 1429-1436.

[4] M.Darus and K.Al-Shaqsi, Differential sandwich theorem with generalised derivative operator, Int. J. Math. Sci., **2**(2), (2008), 75-78.

[5] K. Al-Shaqsi and M. Darus, On univalent functions with respect to k-summetric points defined by a generalization ruscheweyh derivative operators, Jour. Anal. Appl., **7**(1), (2009), 53-61.

[6] B. C. Carlson and D.B.Shaffer, Starlike and prestarlike hypergeometric function, SIAM J. Math. Anal., **15**, (1984), 737-745.

[7] J. H. Choi, M. Saigo, and H. M. Srivastava, Some inclusion properties of a certain family of integral operators, Jour. Math. Anal. Appl., vol. 276, no. 1, pp. (2002), 432-445.

[8] R. M. Goel and B. S. Mehrok, On the coefficients of a subclass of starlike functions, Indian J. Pure Appl. Math., vol. 12, no. 5, pp. (1981), 634-647.

[9] W. Janowski, Some extremal problems for certain families of analytic functions. I, Bulletin de I' Academie Polonaise des sciences. Serie des sciences mathematiques, Astronomiques et Physiques., vol. 21, pp. (1973), 17-25.

[10] W. Ma and D. Minda, An internal geometric characterization of strongly starlike functions, Ann. Uni. Mariae Curie-Sklod. Sectio A., vol. 45, pp. (1991), 89-97.

[11] S. S. Miller and P.T.Mocanu, Differential subordinations: Theory and applications, in: Series on monographs and textbooks in pure and applied mathematics., **225**, Marcel Dekker, New York, 2000.

[12] S. Owa and H. M. Srivastava, Some applications of the generalized Libera integral operator, Proc. Japan Acad. Series A., vol. 62, no. 4, pp. (1986), 125-128.

[13] K. S. Padmanabhan and R. Manjini, Certain applications of differential subordination, Publ. Inst. Math., (N.S.), Tome 39 (53), (1986), 107-118.

[14] K. S. Padmanabhan and R.Parvatham, Some applications of differential subordination, Bull. Austral. Math. Soc., **32**, (1985), 321-330.

[15] S. Ruscheweyh, Convolutions in geometric function theory, Les Presses de I'Univ.de Montreal., 1982.

[16] S. Ruscheweyh and T.Sheil-small, Hadamard product of schlicht functions and the Polya-Schoenberg conjecture, Comment. Math. Helv., **48** (1973), 119 - 135.

[17] S. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc., Vol. **49**, pp. (1975), 109-115.

[18] G. S. Salagean, Subclasses of univalent functions, Lecture Notes in Math. (Springer-Verlag)., **1013**,(1983), 362-372.