Some Geometric Properties for Certain Subclasses of \( p \)-valent Functions Involving Differ-Integral Operator

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Abstract

In the present paper, we aim at proving such results as inclusion relationships and convolution properties for the class \( \mathcal{S}_{p,\mu}^{(j)}(a, c; \alpha; \phi) \). Then we study the integral properties for the class \( \mathcal{S}_{p,\mu}^{(j)}(a, c; \tau) \). Also, we investigate majorization properties for subclass of analytic functions defined by differ-integral operator.

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1 Introduction

Let \( \mathcal{A}_p \) be the class of analytic and \( p \)-valent functions in the open unit disc \( \Delta = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \) which denote by

\[
f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p}z^{k+p} \quad (p \in \mathbb{N} = \{1, 2, \ldots\}). \tag{1.1}
\]

We note that, \( \mathcal{A}_1 = \mathcal{A} \) is the class of univalent and analytic functions in \( \Delta \).

Also, let \( \mathcal{P} \) denote the class of functions of the form:

\[
\mathcal{P}(z) = 1 + \sum_{k=1}^{\infty} \mathcal{P}_k z^k, \quad (z \in \Delta),
\]
which are analytic and convex in $\triangle$ and satisfy the following inequality:

$$\text{Re}\{P(z)\} > 0.$$ 

Let $f, g \in A_p$, where $f$ is given by (1.1) and $g$ is defined by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p},$$

then Hadamard product (or convolution) of the functions $f$ and $g$ is defined by

$$(f \ast g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g \ast f)(z).$$

**Definition 1.1** [6] For two functions $f$ and $g$, analytic in $\triangle$, we say that the function $f$ is subordinate to $g$ in $\triangle$, written $f \prec g$, if there exists a Schwarz function $\omega(z)$ which is analytic in $\triangle$, satisfying the following conditions:

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1, \quad (z \in \triangle),$$

such that

$$f(z) = g(\omega(z)), \quad (z \in \triangle).$$

In particular, if the function $g$ is univalent in $\triangle$, we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \triangle) \iff f(0) = g(0) \quad \text{and} \quad f(\triangle) \subset g(\triangle).$$

**Definition 1.2** [15] For two functions $f$ and $g$, analytic in $\triangle$, we say that the function $f$ is majorized by $g$ in $\triangle$, written $f \ll g$ ($z \in \triangle$), if there exists a function $\varphi(z)$ which is analytic in $\triangle$, such that

$$|\varphi(z)| < 1 \quad \text{and} \quad f(z) = \varphi(z) g(z); \quad (z \in \triangle), \quad (1.2)$$

Taking $\mu > 0$, $a, c \in \mathbb{C}$ such that $\text{Re}(c - a) \geq 0$, $\text{Re}(a) \geq -\mu p$ ($p \in \mathbb{N}$) and $f(z) \in A_p$ is given by (1.1), El-Ashwah and Drbuk [9, with $m = 0$] introduced the differ-integral operator $\mathfrak{D}_{a,c}^{p,\mu} : A_p \to A_p$ as follows:

- For $\text{Re}(c - a) > 0$ by

$$\mathfrak{D}_{a,c}^{p,\mu} f(z) = \frac{\Gamma(c + \mu \wedge)}{\Gamma(a + \mu p) \Gamma(c - a)} \int_0^1 t^{a-1}(1 - t)^{c-a-1} f(z t^\mu) dt; \quad (1.3)$$

- For $a = c$ by

$$\mathfrak{D}_{a,a}^{p,\mu} f(z) = f(z). \quad (1.4)$$
For $a = \gamma$, $c = \gamma + 1$ and $\mu = 1$, we obtain a familiar integral operator $\mathcal{H}_{\gamma,p}$ defined by [22] as follows

$$
\mathcal{H}_{\gamma,p}f(z) = \frac{p + \gamma}{z^\gamma} \int_0^z t^{\gamma - 1} f(t) dt \quad (\gamma > -p, \ p \in \mathbb{N})
$$

$$
= z^p + \sum_{k=1}^{\infty} \left( \frac{p + \gamma}{p + k + \gamma} \right) a_{k+p} z^{k+p}, \quad (1.5)
$$

It is readily verified from (1.5) that

$$
z \left[ \partial_{a,c}^{a,c} \mathcal{H}_{\gamma,p} f(z) \right]' = (\gamma + p) \partial_{a,c}^{a,c} f(z) - \gamma \partial_{a,c}^{a,c} \mathcal{H}_{\gamma,p} f(z). \quad (1.6)
$$

Using (1.3), the operator $\partial_{a,c}^{a,c} f(z)$ can be expressed as follows:

$$
\partial_{a,c}^{a,c} f(z) = z^p + \frac{\Gamma(c + \mu p)}{\Gamma(a + \mu p)} \sum_{k=1}^{\infty} \frac{\Gamma(a + \mu (k + p))}{\Gamma(c + \mu (k + p))} a_{k+p} z^{k+p}, \quad (1.7)
$$

where $\mu > 0$, $a, c \in \mathbb{C}$, $\text{Re}(c - a) \geq 0, \text{Re}(a) \geq -\mu p (p \in \mathbb{N})$. It is readily verified from (1.7) that

$$
z (\partial_{a,c}^{a,c} f(z))' = \left( \frac{a + \mu p}{\mu} \right) \left( \partial_{a,c}^{a+1,c} f(z) \right) - \left( \frac{a}{\mu} \right) \left( \partial_{a,c}^{a,c} f(z) \right). \quad (1.8)
$$

We also note that the operator $\partial_{a,c}^{a,c} f(z)$ generalizes several previously studied familiar operators, and we will mention some of the interesting particular cases as follows:

(i) For $a = \beta$, $c = \alpha + \beta - \gamma + 1$ and $\mu = 1$, we obtain the operator $\mathcal{R}_{\beta,p}^{\alpha,\gamma} f(z)$ ($\gamma > 0; \alpha \geq \gamma - 1; \beta > -p$) which studied by Aouf et al. [1];

(ii) For $a = \beta$, $c = \alpha + \beta$ and $\mu = 1$, we obtain the operator $Q_{\gamma,\beta,p}^{\alpha} f(z)$ ($\alpha \geq 0; \beta > -p$) which studied by Liu and Owa [13];

(iii) For $p = 1$, we obtain the operator $I_{\mu,c}^{\alpha} f(z)$ which studied by Raina and Sharma [20];

(iv) For $p = 1$, $a = \beta$, $c = \alpha + \beta$ and $\mu = 1$, we obtain the operator $Q_{\beta,p}^{\alpha} f(z)$ ($\alpha \geq 0, \beta > -1$) which studied by Jung et al. [11];

(v) For $p = 1$, $a = \alpha - 1$, $c = \beta - 1$ and $\mu = 1$, we obtain the operator $L(\alpha, \beta) f(z)$ ($\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0$, $\mathbb{Z}_0 = \{0, -1, -2, ...\}$) which studied by Carlson and Shaffer [3];

(vi) For $p = 1$, $a = \nu - 1$, $c = \nu$ and $\mu = 1$, we obtain the operator $I_{\nu,c}^{\alpha} f(z)$ ($\nu > 0; \nu > -1$) which studied by Choi et al. [5];
(vii) For \( p = 1, a = \alpha, c = 0 \) and \( \mu = 1 \), we obtain the operator \( D^{\alpha}f(z) \) \((\alpha > -1)\) which studied by Ruscheweyh [21];

(viii) For \( p = 1, a = 1, c = n \) and \( \mu = 1 \), we obtain the operator \( I_{n}f(z) \) \((n \in \mathbb{N})\) which studied by Noor [17];

(ix) For \( p = 1, a = \beta, c = \beta + 1 \), and \( \mu = 1 \), we obtain the integral operator \( J_{\beta} \) which studied by Bernardi [2];

(x) For \( p = 1, a = 1, c = 2 \), and \( \mu = 1 \), we obtain the integral operator \( J \) which studied by Libera [12] and Livingston [14].

Note that

\[ f^{(j)}(z) = \delta(p, j)z^{(p-j)} + \sum_{k=1}^{\infty} \delta(k + p, j)a_{k+p}z^{k+p-j}, \]

where

\[ \delta(p, j) = p(p-1)(p-2)...(p-j+1). \]

By making use of the operator \( d_{a,c}^{\alpha}p,\mu \) and the above mentioned principle of subordination between analytic functions, we introduce and investigate the following subclass of the class \( A_{p} \) as follows:

**Definition 1.3** A function \( f \in A_{p} \) is said to be in the class \( S_{p,\mu}^{(j)}(a,c;\alpha;\phi) \) if it satisfies the following subordination condition:

\[ \frac{z[(1-\alpha)(d_{p,\mu}^{a,c}f)^{(j+1)}(z) + \alpha(d_{p,\mu}^{a+1,c}f)^{(j+1)}]}{(1-\alpha)(d_{p,\mu}^{a,c}f)^{(j)}(z) + \alpha(d_{p,\mu}^{a+1,c}f)^{(j)}} < (p-j)\phi(z) \quad (z \in \Delta), \quad (1.9) \]

for some \( \alpha (\alpha \geq 0) \) and \( j \ (j \in \{0,1,...,p-1\}) \) where \( \phi \in \mathcal{P} \).

For simplicity, we write

\[ S_{p,\mu}^{(j)}(a,c;0;\phi) = S_{p,\mu}^{(j)}(a,c;\phi), \]

\[ S_{p,\mu}^{(j)}\left(a,c;0;\frac{1+Az}{1+Bz}\right) = S_{p,\mu}^{(j)}(a,c;A,B) \quad (-1 \leq B < A \leq 1), \]

and

\[ S_{p,\mu}^{(j)}\left(a,c;0;\frac{1+(1-2\tau)z}{1-z}\right) = S_{p,\mu}^{(j)}(a,c;\tau) \quad (0 \leq \tau < 1). \]

**Remark 1.4** (i) Putting \( a = c = 0 \), \( \mu = 1 \) and \( \phi = \frac{1+(1-2\tau)z}{1-z} \), \((0 \leq \tau < p-j)\), the class \( S_{p,\mu}^{(j)}(a,c;\alpha;\phi) \) reduces to the class \( S(p,j,\tau) \) which studied by Chen et al. [4];
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(ii) Putting $a = c$, $\mu = 1$, $\alpha = j = 0$ and $\phi = \frac{1 + (1 - 2\tau)z}{1 - z}$, $(0 \leq \tau < p)$, the class $S_p^{(j)}(a, c; \alpha; \phi)$ reduces to the class $S_p(\tau)$ which studied by Patel and Thakare [19].

(iii) Putting $a = c = j = 0$, $\mu = \alpha = 1$ and $\phi = \frac{1 + (1 - 2\tau)z}{1 - z}$, $(0 \leq \tau < p)$, the class $S_p^{(j)}(a, c; \alpha; \phi)$ reduces to the class $K_p(\tau)$ which studied by Owa [18].

In order to establish our main results, we shall also make use of the following lemmas:

Lemma 1.5 [7] Let $\beta, \delta \in \mathbb{C}$. Suppose that $\phi(z)$ is convex and univalent in $\Delta$ with $\phi(0) = 1$ and $\text{Re}(\beta \phi(z) + \delta) > 0$ $(z \in \Delta)$. If $P(z)$ is analytic in $\Delta$ with $P(0) = 1$, then the following subordination:

$$P(z) + \frac{zP'(z)}{\beta P(z) + \delta} < \phi(z) \quad (z \in \Delta)$$

implies that

$$P(z) < \phi(z) \quad (z \in \Delta).$$

Lemma 1.6 [10] Let $w(z)$ is analytic function in $U$, with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then $z_0w'(z_0) = \zeta w(z_0)$, where $\zeta$ is a real number and $\zeta \geq 1$.

In the present paper, we aim at proving such results as inclusion relationships and convolution properties for the class $S_p^{(j)}(a, c; \alpha; \phi)$. Then we study the integral properties for the class $S_p^{(j)}(a, c; \tau)$. Also, we investigate majorization properties for subclass of analytic functions defined by differ-integral operator.

Unless otherwise mentioned, we shall assume throughout the paper that $\mu > 0$, $a$, $c \in \mathbb{R}$ such that $(c - a) \geq 0$, $a \geq -\mu p$ ($p \in \mathbb{N}$), $-1 \leq B < A < 1$ and $\alpha \geq 0$.

2 A set of inclusion relationships

We prove some inclusion relationships for the class $S_p^{(j)}(a, c; \alpha; \phi)$, which was given in the previous section.

Theorem 2.1 Let $\phi \in P$ with

$$\text{Re} \left( (p - j)\phi(z) + \frac{a + \mu p}{\alpha \mu} - p + j \right) > 0 \quad (\alpha > 0; j \in \{0, 1, ..., p-1\}; z \in \Delta),$$

then

$$S_p^{(j)}(a, c; \alpha; \phi) \subset S_p^{(j)}(a, c; \phi).$$
Proof. Let \( f \in \mathcal{G}_{p,\mu}^{(j)}(a, c; \alpha; \phi) \) and suppose that
\[
\eta(z) = \frac{z(\mathfrak{D}_{p,\mu}^{a,c}f)^{(j+1)}(z)}{(p-j)(\mathfrak{D}_{p,\mu}^{a,c}f)^{(j)}(z)} \quad (z \in \Delta).
\] (2.1)

The function \( \eta \) is analytic in \( \Delta \) and \( \eta(0) = 1 \). By using (1.8), we obtain
\[
z(\mathfrak{D}_{p,\mu}^{a,c}f)^{(j+1)}(z) = \left( \frac{a + \mu p}{\mu} \right) (\mathfrak{D}_{p,\mu}^{a+1,c}f)^{(j)}(z) - \left( \frac{a}{\mu} + j \right) (\mathfrak{D}_{p,\mu}^{a,c}f)^{(j)}(z) \quad (j \in \{0, 1, \ldots, p-1\}).
\] (2.2)

It follows from (2.2) and (2.1) that
\[
\frac{a}{\mu} + j + (p-j)\eta(z) = \left( \frac{a + \mu p}{\mu} \right) (\mathfrak{D}_{p,\mu}^{a+1,c}f)^{(j)}(z) \quad (j \in \{0, 1, \ldots, p-1\}).
\] (2.3)

From (2.1) and (2.3), we can find that
\[
z(\mathfrak{D}_{p,\mu}^{a+1,c}f)^{(j+1)}(z) = \left( \frac{\mu(p-j)}{a + \mu p} \right) \left[ z\eta'(z) + \left\{ \frac{a}{\mu} + j + (p-j)\eta(z) \right\} \eta(z) \right] \left( \mathfrak{D}_{p,\mu}^{a,c}f \right)^{(j)}(z).
\] (2.4)

It now follows from (2.2), (2.1), (2.3) and (2.4) that
\[
\frac{z}{(p-j)} \left[ (1-\alpha)(\mathfrak{D}_{p,\mu}^{a,c}f)^{(j+1)}(z) + \alpha(\mathfrak{D}_{p,\mu}^{a+1,c}f)^{(j+1)}(z) \right] = (1-\alpha)\eta(z) + \alpha \frac{\mu}{a + \mu p} \left[ z\eta'(z) + \left\{ \frac{a}{\mu} + j + (p-j)\eta(z) \right\} \eta(z) \right] = (1-\alpha)\eta(z) + \alpha \frac{\mu}{a + \mu p} \left[ z\eta'(z) + \left\{ \frac{a}{\mu} + j + (p-j)\eta(z) \right\} \eta(z) \right] = \eta(z) + \frac{\mu}{a + \mu p} \frac{z\eta'(z)}{p + j + (p-j)\eta(z)} \prec \phi(z) \quad (z \in \Delta).
\] (2.5)

Moreover, since
\[
\Re \left( (p-j)\phi(z) + \frac{a + \mu p}{\alpha \mu} - p + j \right) > 0 \quad (\alpha > 0; j \in \{0, 1, \ldots, p-1\}; z \in \Delta),
\]
by Lemma 1.5 and (2.5), we have
\[
\eta(z) = \frac{z(\mathfrak{D}_{p,\mu}^{a,c}f)^{(j+1)}(z)}{(p-j)(\mathfrak{D}_{p,\mu}^{a,c}f)^{(j)}(z)} \prec \phi(z),
\]
that is, \( f \in \mathcal{S}_{p,\mu}^{(j)}(a, c; \phi) \). This implies that
\[
\mathcal{S}_{p,\mu}^{(j)}(a, c; \alpha; \phi) \subset \mathcal{S}_{p,\mu}^{(j)}(a, c; \phi).
\]
The proof of Theorem 2.1 is completed. \( \blacksquare \)

**Theorem 2.2** Let \( \phi \in \mathcal{P} \) with
\[
\text{Re} \left( (p - j)\phi(z) + \frac{a}{\mu} + j \right) > 0 \quad (j \in \{0, 1, \ldots, p - 1\}; z \in \Delta),
\]
then
\[
\mathcal{S}_{p,\mu}^{(j)}(a + 1, c; \phi) \subset \mathcal{S}_{p,\mu}^{(j)}(a, c; \phi).
\]

**Proof.** Let \( f \in \mathcal{S}_{p,\mu}^{(j)}(a + 1, c; \phi) \), then we obtain
\[
\frac{z(\mathfrak{d}_{p,\mu}^{a+1,c,f})^{(j+1)}(z)}{(p - j)(\mathfrak{d}_{p,\mu}^{a+1,c,f})^{(j)}(z)} \prec \phi(z) \quad (z \in \Delta). \tag{2.6}
\]
Differentiating both sides of (2.3) with respect to \( z \) logarithmically and using (2.1), we obtain
\[
\eta(z) + \frac{z\eta'(z)}{\frac{a}{\mu} + j + (p - j)\eta(z)} = \frac{z(\mathfrak{d}_{p,\mu}^{a+1,c,f})^{(j+1)}(z)}{(p - j)(\mathfrak{d}_{p,\mu}^{a+1,c,f})^{(j)}(z)} \quad (z \in \Delta). \tag{2.7}
\]
From (2.6) and (2.7), we have
\[
\eta(z) + \frac{z\eta'(z)}{\frac{a}{\mu} + j + (p - j)\eta(z)} \prec \phi(z) \quad (z \in \Delta). \tag{2.8}
\]
Moreover, since
\[
\text{Re} \left( (p - j)\phi(z) + \frac{a}{\mu} + j \right) > 0 \quad (z \in \Delta),
\]
by Lemma 1.5 and (2.8), we know that
\[
\eta(z) = \frac{z(\mathfrak{d}_{p,\mu}^{a,c,f})^{(j+1)}(z)}{(p - j)(\mathfrak{d}_{p,\mu}^{a,c,f})^{(j)}(z)} \prec \phi(z),
\]
that is, \( f \in \mathcal{S}_{p,\mu}^{(j)}(a, c; \phi) \). This implies that
\[
\mathcal{S}_{p,\mu}^{(j)}(a + 1, c; \phi) \subset \mathcal{S}_{p,\mu}^{(j)}(a, c; \phi).
\]
The proof of Theorem 2.2 is completed. \( \blacksquare \)
3 Convolution properties

In this section, we introduce some convolution properties for the class \( \mathcal{S}(j)_{p,\mu}(a,c;\phi) \).

**Theorem 3.1** Let \( f \in \mathcal{S}(j)_{p,\mu}(a,c;\phi) \). Then

\[
f^{(j)}(z) = \left( z^{p-j} \exp \left( (p-j) \int_0^z \frac{\phi(\omega(z)) - 1}{\zeta} d\zeta \right) \right) * \left( \sum_{k=0}^{\infty} \frac{\Gamma(a+\mu p) \Gamma(c+\mu(k+p))}{\Gamma(c+\mu p) \Gamma(a+\mu(k+p))} z^{k+p-j} \right)
\]

\((j \in \{0,1,\ldots,p-1\}; z \in \triangle), \tag{3.1}\)

where \( \omega \) is analytic in \( \triangle \) with \( \omega(0) = 0 \) and \(|\omega(z)| < 1\).

**Proof.** Suppose that \( f \in \mathcal{S}(j)_{p,\mu}(a,c;\phi) \) and from (1.9) with \((\alpha = 0)\) we have

\[
z (d^{a,c}_{p,\mu}f)^{(j+1)}(z) = \phi(\omega(z)) (z \in \triangle), \tag{3.2}\]

where \( \omega \) is analytic in \( \triangle \) with \( \omega(0) = 0 \) and \(|\omega(z)| < 1\). We can easily find that

\[
\frac{(d^{a,c}_{p,\mu}f)^{(j+1)}(z)}{(d^{a,c}_{p,\mu}f)^{(j)}(z)} = \frac{p-j}{z} = (p-j) \frac{\phi(\omega(z)) - 1}{z} (z \in \triangle), \tag{3.3}\]

upon integrating (3.3), we have

\[
(d^{a,c}_{p,\mu}f)^{(j)}(z) = z^{p-j} \exp \left( (p-j) \int_0^z \frac{\phi(\omega(\zeta)) - 1}{\zeta} d\zeta \right). \tag{3.4}\]

On the other hand, we know from (1.7) that

\[
(d^{a,c}_{p,\mu}f)^{(j)}(z) = \left( \sum_{k=0}^{\infty} \frac{\Gamma(c+\mu p) \Gamma(a+\mu(k+p))}{\Gamma(a+\mu p) \Gamma(c+\mu(k+p))} z^{k+p-j} \right) * f^{(j)}(z). \tag{3.5}\]

The assertion (3.1) of Theorem 3.1 can now easily be derived from (3.4) and (3.5). \( \blacksquare \)

**Theorem 3.2** The function \( f \in \mathcal{S}(j)_{p,\mu}(a,c;\phi) \) if and only if

\[
\frac{1}{z^{p-j}} \left[ f^{(j)}(z) * \left( \sum_{k=0}^{\infty} \frac{\Gamma(c+\mu p) \Gamma(a+\mu(k+p))}{\Gamma(a+\mu p) \Gamma(c+\mu(k+p))} (k+p-j-(p-j)\phi(e^{i\theta})) z^{k+p-j} \right) \right] \neq 0
\]

\((j \in \{0,1,\ldots,p-1\}; z \in \triangle; 0 \leq \theta < 2\pi). \tag{3.6}\)
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**Proof.** Suppose that \( f \in S_j^{(a, c; \phi)} \) and from (1.9) with \( (\alpha = 0) \) we have

\[
\frac{z(\mathcal{D}_{p,\mu}^{a,c}f)^{(j+1)}(z)}{(p-j)(\mathcal{D}_{p,\mu}^{a,c}f)^{(j)}(z)} < \phi(z) \quad (z \in \Delta),
\]

is equivalent to

\[
\frac{z(\mathcal{D}_{p,\mu}^{a,c}f)^{(j+1)}(z)}{(p-j)(\mathcal{D}_{p,\mu}^{a,c}f)^{(j)}(z)} \neq \phi(e^{i\theta}) \quad (z \in \Delta; 0 \leq \theta < 2\pi).
\]

The condition (3.8) can be written as follows:

\[
\frac{1}{z^{p-j}} \left[ z(\mathcal{D}_{p,\mu}^{a,c}f)^{(j+1)}(z) - (p-j)(\mathcal{D}_{p,\mu}^{a,c}f)^{(j)}(z)\phi(e^{i\theta}) \right] \neq 0 \quad (z \in \Delta; 0 \leq \theta < 2\pi).
\]

On the other hand, we know that

\[
z(\mathcal{D}_{p,\mu}^{a,c}f)^{(j+1)}(z) = \left( \sum_{k=0}^{\infty} \frac{\Gamma(c + \mu p)\Gamma(a + \mu(k + p))}{\Gamma(a + \mu p)\Gamma(c + \mu(k + p))}(k + p - j)z^{k+p-j} \right) \ast f^{(j)}(z).
\]

Upon substituting (3.5) and (3.10) into (3.9), we can easily get the convolution property (3.6). The proof of Theorem 3.2 is completed.

## 4 A set of integral preserving properties

In this section, obtain integral preserving properties involving the integral operator \( \mathcal{H}_{\gamma,p} \) which given by (1.5). It is readily verified from (1.6) that

\[
z \left[ \mathcal{D}_{p,\mu}^{a,c} \mathcal{H}_{\gamma,p} f \right]^{(j+1)}(z) = (\gamma + p)(\mathcal{D}_{p,\mu}^{a,c}f)^{(j)}(z) - (\gamma + j)(\mathcal{D}_{p,\mu}^{a,c} \mathcal{H}_{\gamma,p} f)^{(j)}(z). \quad (4.1)
\]

**Theorem 4.1** If \( f \in S_j^{(a, c; \tau)} \), then \( \mathcal{H}_{\gamma,p} f(z) \in S_j^{(a, c; \tau)} \), where \( \mathcal{H}_{\gamma,p} f(z) \) is defined by (1.5).

**Proof.** Suppose that \( f \in S_j^{(a, c; \tau)} \) and set

\[
\frac{z(\mathcal{D}_{p,\mu}^{a,c} \mathcal{H}_{\gamma,p} f)^{(j+1)}(z)}{(p-j)(\mathcal{D}_{p,\mu}^{a,c} \mathcal{H}_{\gamma,p} f)^{(j)}(z)} = \frac{1 + (1 - 2\tau)w(z)}{1 - w(z)},
\]

where \( w(0) = 0 \). Then, by using (4.1) in (4.2), we obtain

\[
\frac{(\mathcal{D}_{p,\mu}^{a,c}f)^{(j)}(z)}{(p-j)(\mathcal{D}_{p,\mu}^{a,c} \mathcal{H}_{\gamma,p} f)^{(j)}(z)} = \frac{(\gamma + p) + [(p-j)(1-2\tau) - (\gamma + j)]w(z)}{(p-j)(\gamma + p)(1 - w(z))}.
\]

\[
(4.3)
\]
In this section, we investigate the majorization properties of subclass of analytic functions. Theorem 4.1 is completed.

Now assuming that \( \max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \) and applying Jack’s lemma, we obtain

\[ z_0 w'(z_0) = \zeta w(z_0) \quad (\zeta \in \mathbb{R}, \zeta \geq 1). \] (4.4)

If we set \( w(z_0) = e^{i\theta} (\theta \in \mathbb{R}) \) in (4.4) and observe that

\[ \text{Re} \left( (1 - \tau) \frac{1 + w(z_0)}{1 - w(z_0)} \right) = 0, \]

then we have

\[
\begin{align*}
\text{Re} \left( \frac{z(\Delta_{p,j}^{a,c}f)^{(j+1)}(z)}{(p-j)(\Delta_{p,j}^{a,c}f)^{(j)}(z)} \right) &= \frac{1}{p-j} \text{Re} \left( \frac{[(p-j)(1-2\tau) - (\gamma + j)]z_0 w'(z_0) + z_0 w'(z_0)}{(\gamma + p) + [(p-j)(1-2\tau) - (\gamma + j)]w(z_0) + 1 - w(z_0)} \right) \\
&= \frac{1}{p-j} \text{Re} \left( \frac{[(p-j)(1-2\tau) - (\gamma + j)]e^{i\theta} + \zeta e^{i\theta}}{(\gamma + p) + [(p-j)(1-2\tau) - (\gamma + j)]e^{i\theta} + 1 - e^{i\theta}} \right) \\
&= -\frac{\zeta}{2(p-j)} \frac{\tau(p-j) + (\gamma + j)}{(p-j)(1 - \tau)} < 0,
\end{align*}
\]

which obviously contradicts the hypothesis \( f \) belongs to \( \mathcal{E}_{p,j}^{(j)}(a, c; \tau) \). The proof of Theorem 4.1 is completed.

5 Majorization properties for subclass of analytic functions

In this section, we investigate the majorization properties of subclass of analytic \( p \)-valent functions defined by differ-integral operator.
Theorem 5.1 Let $f \in \mathcal{A}_p$ and suppose that $g \in \mathcal{G}_{p,\mu}^{(j)}(a, c; A, B)$ and $|\frac{a+\mu p}{\mu}| > \|(A-B) + \left(\frac{a+\mu p}{\mu}\right)B\|$. If $(\mathcal{D}_{p,\mu}^{a,c} f)^{(j)}(z)$ is majorized by $(\mathcal{D}_{p,\mu}^{a,c} g)^{(j)}(z)$ in $\triangle$, then

$$\left|\left(\mathcal{D}_{p,\mu}^{a+1,c} f\right)^{(j)}(z)\right| \leq \left|\left(\mathcal{D}_{p,\mu}^{a+1,c} g\right)^{(j)}(z)\right| \text{ for } |z| \leq r_0,$$

where $r_0 = r_0(a, c, A, B, \mu, p)$ is the smallest positive real root of the equation

$$\left|\left(A-B\right) + \left(\frac{a+\mu p}{\mu}\right)B\right| r^3 - \left(\left|\frac{a+\mu p}{\mu}\right| + 2|B|\right) r^2 - \left(\left|A-B\right| + \left(\frac{a+\mu p}{\mu}\right) B + 2\right) r + \left|\frac{a+\mu p}{\mu}\right| = 0.$$ (5.2)

Proof. Since $g \in \mathcal{G}_{p,\mu}^{(j)}(a, c; A, B)$, we have

$$1 + \frac{z(\mathcal{D}_{p,\mu}^{a,c} g)^{(j+1)}(z)}{(\mathcal{D}_{p,\mu}^{a,c} g)^{(j)}(z)} - (p - j) = 1 + Aw(z) \frac{z}{1+Bw(z)},$$ (5.3)

where $w(z)$ is analytic in $\triangle$ with $w(0) = 0$ and $|w(z)| < |z| (z \in \triangle)$. From (5.3) and using (2.2), we get

$$\left|\left(\mathcal{D}_{p,\mu}^{a,c} g\right)^{(j)}(z)\right| \leq \frac{(1 + |B||z|) \left|\frac{a+\mu p}{\mu}\right| - \left(\frac{a+\mu p}{\mu}\right) B + (A-B) |z|}{\left|\mathcal{D}_{p,\mu}^{a+1,c} g\right|^j(z)}.$$ (5.4)

Next, since $(\mathcal{D}_{p,\mu}^{a,c} f)^{(j)}(z)$ is majorized by $(\mathcal{D}_{p,\mu}^{a,c} g)^{(j)}(z)$ in $\triangle$, we have

$$(\mathcal{D}_{p,\mu}^{a,c} f)^{(j)}(z) = \varphi(z)(\mathcal{D}_{p,\mu}^{a,c} g)^{(j)}(z).$$

Differentiating it with respect to $z$ and multiplying by $z$, we get

$$z(\mathcal{D}_{p,\mu}^{a,c} f)^{(j+1)}(z) = z\varphi'(z)(\mathcal{D}_{p,\mu}^{a,c} g)^{(j)}(z) + z\varphi(z)(\mathcal{D}_{p,\mu}^{a+1,c} g)^{(j+1)}(z).$$

Using (2.2) in the last equation, it yields

$$(\mathcal{D}_{p,\mu}^{a+1,c} f)^{(j)}(z) = \left(\frac{\mu}{a+\mu p}\right) z\varphi'(z)(\mathcal{D}_{p,\mu}^{a,c} g)^{(j)}(z) + \varphi(z)(\mathcal{D}_{p,\mu}^{a+1,c} g)^{(j)}(z).$$ (5.5)

Thus, noting that $\varphi(z) \in \mathcal{P}$ satisfies the inequality (see [16])

$$|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \quad (z \in \triangle),$$ (5.6)
and making use of (5.4) and (5.6) in (5.5), we get

\[
|\left( d_{a+1,c} f \right)^{(j)}(z)| \leq \left| \varphi(z) \right| + \left( \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \right) \left( \frac{1 + |B||z||z|}{|a+\mu p/\mu| - \left| \left( a+\mu p/\mu \right) B + (A - B) \right|} \right)
\]

which upon putting \(|z| = r\) and \(|\varphi(z)| = \varrho\) (0 \(\leq\) \(\varrho\) \(\leq\) 1) leads to the inequality

\[
|\left( d_{a+1,c} f \right)^{(j)}(z)| \leq \Upsilon(r, \varrho) \left| \left( d_{a+1,c} g \right)^{(j)}(z) \right|,
\]

where

\[
\Upsilon(r, \varrho) = \frac{-r(1 + |B|r)\varrho^2 + (1 - r^2) \left( \frac{a+\mu p}{\mu} \right) - \left( \frac{a+\mu p}{\mu} \right) B + (A - B) \right| r \varrho + r(1 + |B|r)}{(1 - r^2) \left( \frac{a+\mu p}{\mu} \right) - \left( \frac{a+\mu p}{\mu} \right) B + (A - B) \right|}.
\]

In order to determine \(r_0\), we note that

\[
r_0 = \max\{r \in [0, 1] : \Upsilon(r, \varrho) \leq 1 \ \forall \varrho \in [0, 1] \}
= \max\{r \in [0, 1] : \Psi(r, \varrho) \geq 0 \ \forall \varrho \in [0, 1] \},
\]

where

\[
\Psi(r, \varrho) = (1 - r^2) \left( \frac{a+\mu p}{\mu} \right) - \left( \frac{a+\mu p}{\mu} \right) B + (A - B) \right| r \varrho - (1 - \varrho^2)r(1 + |B|r).
\]

A simple calculation shows that the inequality \(\Psi(r, \varrho) \geq 0\) is equivalent to

\[
v(r, \varrho) = (1 - r^2) \left( \frac{a+\mu p}{\mu} \right) - \left( \frac{a+\mu p}{\mu} \right) B + (A - B) \right| r - (1 + \varrho^2)r(1 + |B|r) \geq 0.
\]

Obviously the function \(v(r, \varrho)\) takes its minimum value at \(\varrho = 1\), we conclude that (5.1) holds true for \(|z| \leq r_0 = r_0(a, c, A, B, \mu, p)\) where \(r_0(a, c, A, B, \mu, p)\) is the smallest positive real root of (5.2). The proof of Theorem 5.1 is completed.

\[\blacksquare\]

Setting \(A = 1 - 2\tau\) and \(B = -1\) in Theorem 5.1, we will get the following result:
Corollary 5.2 Let $f \in A_p$ and suppose that $g \in S_{(j) p}^{(a,c)}(1-2\tau,-1)$. If $(d_{p,\mu}^{a,c} f)^{(j)}(z)$ is majorized by $(d_{p,\mu}^{a,c} g)^{(j)}(z)$ in $\Delta$, then
\[
| (d_{p,\mu}^{a+1,c} f)^{(j)}(z) | \leq | (d_{p,\mu}^{a+1,c} g)^{(j)}(z) | \quad \text{for } |z| \leq r_1,
\]
where $r_1 = r_1(a,c,1-2\tau,-1,\mu,p)$ is the smallest positive real root of the equation
\[
- \left( \frac{a + \mu p}{\mu} \right) + 2(1-\tau) r^3 - \left( \frac{a + \mu p}{\mu} \right) + 2 \left( - \left( \frac{a + \mu p}{\mu} \right) + 2(1-\tau) \right) r^2 - \left( - \left( \frac{a + \mu p}{\mu} \right) + 2(1-\tau) \right) r + \frac{a + \mu p}{\mu} = 0.
\]

Setting $\tau = 0$ in Corollary 5.2, we will get the following result:

Corollary 5.3 Let $f \in A_p$ and suppose that $g \in S_{(j) p}^{(a,c)}(1,1)$. If $(d_{p,\mu}^{a,c} f)^{(j)}(z)$ is majorized by $(d_{p,\mu}^{a,c} g)^{(j)}(z)$ in $\Delta$, then
\[
| (d_{p,\mu}^{a+1,c} f)^{(j)}(z) | \leq | (d_{p,\mu}^{a+1,c} g)^{(j)}(z) | \quad \text{for } |z| \leq r_2,
\]
where $r_2 = r_2(a,c,\mu,p)$ is the smallest positive real root of the equation
\[
r_2(a,c,\mu,p) = \frac{\kappa - \sqrt{\kappa^2 - 4|\nu||2-\nu|}}{2|2-\nu|},
\]
where $\nu = \frac{a + \mu p}{\mu}$, $\kappa = |2 + \nu| + |2 - \nu|$, $p \in \mathbb{N}$.

Remark 5.4

• Putting $a = c = 0$, $\mu = 1$ and $j = 0$ in Corollary 5.3, we obtain the results which obtained by El-Ashwah and Aouf. [8, Corollary 2.4 with $\gamma = 1$];

• Putting $a = c = 0$, $\mu = 1$, $j = 0$ and $p = 1$ in Corollary 5.3, we obtain the results which obtained by MacGregor [15].

6 Open problem

Discussing some results as inclusion relationships and convolution properties for the class $S_{(j) p}^{(a,c)}(a,c;\alpha;A,B)$, $(-1 \leq B < A \leq 1$, $j \in \{0,1,...,p-1\}$, $\alpha \geq 0$, $\mu > 0$, $a,c \in \mathbb{R}$, $(c-a) \geq 0$, $a \geq -\mu p$, $p \in \mathbb{N}$).
References


