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# Some Estimates of the $\varphi$-order and the $\varphi$-type of Entire and Meromorphic Functions

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**Abstract**

***In this paper, we introduce the type of entire and meromorphic function in the complex plane related to the $\varphi$-order concept given by Chyzhykov-Semochko in [8]. We establish some estimates involving those new concepts of the sum, product and the derivative of entire or meromorphic functions in the complex plane. Many previous results due to Latreuch-Belaïdi, Tu-Zeng-Xu, Chyzhykov-Semochko, will be revisited and extended.***

**Keywords:** *Entire function, Meromorphic function, $\rho_\varphi-$order, $\tau_\varphi$-type.*
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# 1 Introduction and definitions

The determination of the order and the type of entire (respectively meromorphic) functions plays a crucial role in the study of properties of solutions of linear differential equations of the form

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0, \qquad (1.1)$$

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = F(z), \qquad (1.2)$$

where $k \geq 2$, $A_0 \not\equiv 0$, $F \not\equiv 0$, the coefficients $A_j$ $(j = 0, \ldots, k-1)$ and $F$ are entire functions or meromorphic functions in the plane or in the unit disc. Through this paper, we assume familiarity of the reader with the standard notations of Nevanlinna value distribution theory of meromorphic functions (see [9, 11, 13, 18]). In addition, we mean by a meromorphic function a function which is meromorphic in the whole complex plane. To study the growth of functions, we recall the following definitions.

**Definition 1.1** *The order of a meromorphic function $f$ is defined as*

$$\rho(f) = \limsup_{r \to +\infty} \frac{\log T(r, f)}{\log r},$$

*where $T(r, f)$ is the Nevanlinna characteristic function of $f$. For $0 < \rho(f) < +\infty$, we define the type of $f$ by*

$$\tau(f) = \limsup_{r \to +\infty} \frac{T(r, f)}{r^{\rho(f)}}.$$

**Definition 1.2** *The order of an entire function $f$ is defined as*

$$\tilde{\rho}(f) = \limsup_{r \to +\infty} \frac{\log \log M(r, f)}{\log r},$$

*where $M(r, f) = \max\{|f(z)| : |z| = r\}$ is the maximum modulus of $f$. For $0 < \tilde{\rho}(f) < +\infty$, we define the type of $f$ by*

$$\tilde{\tau}(f) = \limsup_{r \to +\infty} \frac{\log M(r, f)}{r^{\tilde{\rho}(f)}}.$$

**Remark 1.3** *Note that if $f$ is an entire function, then $\tilde{\rho}(f) = \rho(f)$, but $\tilde{\tau}(f) = \tau(f)$ is not always satisfied. From Goldberg and Ostrovskii ([9]), the following estimates hold*

$$\begin{cases} \tilde{\tau}(f) \leq \frac{\pi \rho(f)}{\sin(\pi \rho(f))} \tau(f) & \text{if} \quad 0 < \rho(f) \leq \frac{1}{2}, \\ \tilde{\tau}(f) \leq \pi\, \rho(f)\, \tau(f) & \text{if} \quad \frac{1}{2} \leq \rho(f) < +\infty. \end{cases}$$

We state here two classical results investigated the order and the type of $f_1 + f_2$ and $f_1 f_2$, where $f_1$ and $f_2$ are entire (respectively meromorphic) functions.

**Theorem 1.4 ([15])** *Let $f_1$ and $f_2$ be two entire functions. Then we have*

$$\begin{aligned} \rho(f_1 + f_2) &\leq \max\{\rho(f_1), \rho(f_2)\}, \\ \rho(f_1 f_2) &\leq \max\{\rho(f_1), \rho(f_2)\} \end{aligned}$$

*and*

$$\begin{aligned} \tilde{\tau}(f_1 + f_2) &\leq \max\{\tilde{\tau}(f_1), \tilde{\tau}(f_2)\}, \\ \tilde{\tau}(f_1 f_2) &\leq \tilde{\tau}(f_1) + \tilde{\tau}(f_2). \end{aligned}$$

**Theorem 1.5 ([9])** *Let $f_1$ and $f_2$ be two meromorphic functions. If $\rho(f_1) < \rho(f_2)$, then*

$$\rho(f_1 + f_2) = \rho(f_1 f_2) = \rho(f_2).$$

In ([14]), Latreuch and Belaïdi established new estimates for the order and type of meromorphic functions and obtained the following results which improved the above two theorems.

**Theorem 1.6 ([14])** *Let $f_1$ and $f_2$ be two meromorphic functions.*

*(i) If $0 < \rho(f_1) < \rho(f_2) < +\infty$, then $\tau(f_1 + f_2) = \tau(f_1 f_2) = \tau(f_2)$.*

*(ii) If $0 < \rho(f_1) = \rho(f_2) = \rho(f_1 + f_2) = \rho(f_1 f_2) < +\infty$, then*

$$\begin{aligned} |\tau(f_1) - \tau(f_2)| \leq \quad & \tau(f_1 + f_2) & \leq \tau(f_1) + \tau(f_2), \\ |\tau(f_1) - \tau(f_2)| \leq \quad & \tau(f_1 f_2) & \leq \tau(f_1) + \tau(f_2). \end{aligned}$$

**Theorem 1.7 ([14])** *Let $f_1$ and $f_2$ be two meromorphic functions satisfying $0 < \rho(f_1) = \rho(f_2) < +\infty$ and $\tau(f_1) \neq \tau(f_2)$. Then $\rho(f_1 + f_2) = \rho(f_1 f_2) = \rho(f_1) = \rho(f_2)$.*

**Theorem 1.8 ([14])** *Let $f_1$ and $f_2$ be entire functions.*

*(i) If $0 < \rho(f_1) < \rho(f_2) < +\infty$, then $\tilde{\tau}(f_1 + f_2) = \tilde{\tau}(f_2)$ and $\tilde{\tau}(f_1 f_2) \leq \tilde{\tau}(f_2)$.*

*(ii) If $0 < \rho(f_1) = \rho(f_2) = \rho(f_1 + f_2) = \rho(f_1 f_2) < +\infty$, then*

$$\begin{aligned} \tilde{\tau}(f_1 + f_2) & \leq \max\{\tilde{\tau}(f_1), \tilde{\tau}(f_2)\}, \\ \tilde{\tau}(f_1 f_2) & \leq \tilde{\tau}(f_1) + \tilde{\tau}(f_2). \end{aligned}$$

*Furthermore, if $\tilde{\tau}(f_1) \neq \tilde{\tau}(f_2)$, then $\tilde{\tau}(f_1 + f_2) = \max\{\tilde{\tau}(f_1), \tilde{\tau}(f_2)\}$.*

**Theorem 1.9 ([14])** *Let $f_1$ and $f_2$ be entire functions. If $0 < \rho(f_1) = \rho(f_2) < +\infty$ and $\tilde{\tau}(f_1) \neq \tilde{\tau}(f_2)$, then $\rho(f_1 + f_2) = \rho(f_1) = \rho(f_2)$.*

For all $r \in \mathbb{R}$, we define $\exp_1 r = \exp r = e^r$ and $\exp_{p+1} r = \exp(\exp_p r)$, $p \in \mathbb{N} = \{1, 2, 3, \dots\}$. Inductively, for all $r$ large enough, we define $\log_1 r = \log r$ and $\log_{p+1} r = \log(\log_p r), p \in \mathbb{N}$. We also denote $\exp_0 r = r = \log_0 r$, $\exp_{-1} r = \log_1 r$ and $\log_{-1} r = \exp_1 r$.

The linear measure of a set $E \subset (0, +\infty)$ is defined by $m(E) = \int_0^{+\infty} \chi_E(t)\, dt$ and the logarithmic measure of a set $F \subset (1, +\infty)$ is defined by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t}\, dt$, where $\chi_G$ is the characteristic function of a set $G$.

**Definition 1.10** *Let $p \geq 1$ be an integer. The iterated p-order of a meromorphic function $f$ is defined by*

$$\rho_p(f) = \limsup_{r \to +\infty} \frac{\log_p T(r,f)}{\log r}.$$

*If $f$ is an entire function, then*

$$\rho_p(f) = \limsup_{r \to +\infty} \frac{\log_{p+1} M(r,f)}{\log r}.$$

**Definition 1.11** *The iterated p-type of a meromorphic function $f$ with iterated p-order $(0 < \rho_p(f) < +\infty)$ is defined as*

$$\tau_p(f) = \limsup_{r \to +\infty} \frac{\log_{p-1} T(r,f)}{r^{\rho_p(f)}}.$$

*If $f$ is an entire, then its iterated p-type is defined by*

$$\tilde{\tau}_p(f) = \limsup_{r \to +\infty} \frac{\log_p M(r,f)}{r^{\rho_p(f)}}.$$

Note that $\rho_1(f)$ and $\tau_1(f)$ coincide with the usual order $\rho(f)$ and the usual type $\tau(f)$ respectively.
Several researchers (see [2, 3, 6, 7, 10, 12]) used the concept of the iterated $p$-order $\rho_p(f)$ instead of the usual order $\rho(f)$ to study the fast growing solutions of equations (1.1) and (1.2). Tu-Zeng-Hu ([17]) generalised Theorems 1.6–1.9 from the usual order to the iterated $p$-order. We state here their results.

**Theorem 1.12 ([17])** *Let $f_1$ and $f_2$ be two meromorphic functions satisfying $0 < \rho_p(f_1) = \rho_p(f_2) < +\infty$ and $\tau_p(f_1) < \tau_p(f_2)$. Then we have*

*(i)* $\rho_p(f_1 + f_2) = \rho_p(f_1\, f_2) = \rho_p(f_1) = \rho_p(f_2)$.

*(ii) If $p > 1$, then $\tau_p(f_1 + f_2) = \tau_p(f_1\, f_2) = \tau_p(f_2)$.*

*(iii) If $p = 1$, then $\alpha \leq \tau_p(f_1 + f_2) \leq \beta$ and $\alpha \leq \tau_p(f_1\, f_2) \leq \beta$, where $\alpha = \tau_p(f_2) - \tau_p(f_1)$ and $\beta = \tau_p(f_1) + \tau_p(f_2)$.*

**Theorem 1.13 ([17])** *Let $f_1$ and $f_2$ be entire functions satisfying $0 < \rho_p(f_1) = \rho_p(f_2) < +\infty$ and $\tilde{\tau}_p(f_1) < \tilde{\tau}_p(f_2)$. Then there hold:*

*(i) If $p \geq 1$, then $\rho_p(f_1 + f_2) = \rho_p(f_1) = \rho_p(f_2)$ and $\tilde{\tau}_p(f_1 + f_2) = \tilde{\tau}_p(f_2)$.*

*(ii) If $p > 1$, then $\rho_p(f_1\, f_2) = \rho_p(f_1) = \rho_p(f_2)$ and $\tilde{\tau}_p(f_1\, f_2) = \tilde{\tau}_p(f_2)$.*

Since $\rho_p(f') = \rho_p(f), p \geq 1$ and for a meromorphic function $f$ with finite iterated $p$-order, Tu-Zeng-Hu ([17]) obtained the following result for the iterated $p$-type.

**Theorem 1.14 ([17])** *Let $p > 1$ and $f$ be meromorphic function satisfying $0 < \rho_p(f) < +\infty$. Then $\tau_p(f') = \tau_p(f)$*

Recently, Chyzhykov and Semochko ([8]) showed that the iterated $p$-order does not cover an arbitrary growth, see Example 1.4 in ([8]). To avoid this disadvantage, they introduced the concept of the $\varphi$-order to measure the order of growth of entire solutions of equation (1.1). After that, Belaïdi ([4], [5]) improved the results in ([8]) for the lower $\varphi$-order and the lower $\varphi$-type.

**Definition 1.15 ([8])** *Let $\varphi$ be an increasing unbounded function on $[1, +\infty)$. The $\varphi$-orders of a meromorphic function $f$ are defined by*

$$\rho_\varphi^0(f) = \limsup_{r\to+\infty} \frac{\varphi(e^{T(r,f)})}{\log r}, \qquad \rho_\varphi^1(f) = \limsup_{r\to+\infty} \frac{\varphi(T(r,f))}{\log r}.$$

*If $f$ is an entire function, then the $\varphi$-orders are defined as*

$$\tilde{\rho}_\varphi^0(f) = \limsup_{r\to+\infty} \frac{\varphi(M(r,f))}{\log r}, \qquad \tilde{\rho}_\varphi^1(f) = \limsup_{r\to+\infty} \frac{\varphi(\log M(r,f))}{\log r}.$$

By $\Phi$ we define the class of positive unbounded increasing functions on $[1, +\infty)$ such that $\varphi(e^t)$ is slowly growing, i.e., $\forall c > 0 : \lim_{r\to+\infty} \frac{\varphi(e^{ct})}{\varphi(e^t)} = 1$. As examples, the function $\varphi(r) = \log_p r$ $(p \geq 2)$ belongs to the class $\Phi$ but $\varphi(r) = \log r \notin \Phi$.

By analogous manner, we introduce the definitions of the $\varphi$-types related to the $\varphi$-order.

**Definition 1.16 ([5])** *Let $\varphi$ be an increasing unbounded function on $[1, +\infty)$. We define the $\varphi$-types of a meromorphic function $f$ with $\varphi$-order $\in (0, +\infty)$ by*

$$\tau_\varphi^0(f) = \limsup_{r\to+\infty} \frac{e^{\varphi(e^{T(r,f)})}}{r^{\rho_\varphi^0(f)}}, \qquad \tau_\varphi^1(f) = \limsup_{r\to+\infty} \frac{e^{\varphi(T(r,f))}}{r^{\rho_\varphi^1(f)}}.$$

*If $f$ is an entire function, then the $\varphi$-types are defined as*

$$\tilde{\tau}_\varphi^0(f) = \limsup_{r\to+\infty} \frac{e^{\varphi(M(r,f))}}{r^{\tilde{\rho}_\varphi^0(f)}}, \qquad \tilde{\tau}_\varphi^1(f) = \limsup_{r\to+\infty} \frac{e^{\varphi(\log M(r,f))}}{r^{\tilde{\rho}_\varphi^1(f)}}.$$

# 2 Main results

The aim of this paper is to give the counterparts of the above theorems for the $\varphi$-order and the $\varphi$-type concepts.

**Theorem 2.1** *Let $\varphi \in \Phi$ and $f_1, f_2$ be two meromorphic functions. If $\rho^j_\varphi(f_1) < \rho^j_\varphi(f_2)$ , $(j = 0, 1)$, then $\rho^j_\varphi(f_1 + f_2) = \rho^j_\varphi(f_1 f_2) = \rho^j_\varphi(f_2)$ for $j = 0, 1$.*

**Theorem 2.2** *Let $\varphi \in \Phi$ and $f_1, f_2$ be two meromorphic functions. Then*

*(i) If $0 < \rho^j_\varphi(f_1) < \rho^j_\varphi(f_2) < +\infty$ and $\tau^j_\varphi(f_1) < \tau^j_\varphi(f_2), (j = 0, 1)$, then*

$$\tau^j_\varphi(f_1 + f_2) = \tau^j_\varphi(f_1 f_2) = \tau^j_\varphi(f_2), j = 0, 1. \tag{2.1}$$

*(ii) If $0 < \rho^j_\varphi(f_1) = \rho^j_\varphi(f_2) = \rho^j_\varphi(f_1 + f_2) < +\infty, (j = 0, 1)$, then*

$$\tau^j_\varphi(f_1 + f_2) \le \max\{\tau^j_\varphi(f_1), \tau^j_\varphi(f_2)\}.$$

*Furthermore, if $\tau^j_\varphi(f_1) \ne \tau^j_\varphi(f_2)$ , $(j = 0, 1)$, then*

$$\tau^j_\varphi(f_1 + f_2) = \max\{\tau^j_\varphi(f_1), \tau^j_\varphi(f_2)\}. \tag{2.2}$$

*(iii) If $0 < \rho^j_\varphi(f_1) = \rho^j_\varphi(f_2) = \rho^j_\varphi(f_1 f_2) < +\infty, (j = 0, 1)$, then*

$$\tau^j_\varphi(f_1\, f_2) \le \max\{\tau^j_\varphi(f_1), \tau^j_\varphi(f_2)\}.$$

*Furthermore, if $\tau^j_\varphi(f_1) \ne \tau^j_\varphi(f_2)$ , $(j = 0, 1)$, then*

$$\tau^j_\varphi(f_1\, f_2) = \max\{\tau^j_\varphi(f_1), \tau^j_\varphi(f_2)\}. \tag{2.3}$$

**Corollary 2.3** *Let $\varphi \in \Phi$ and $f_1, f_2$ be two meromorphic functions.*

*(i) If $0 < \rho^j_\varphi(f_1) = \rho^j_\varphi(f_2) = \rho^j_\varphi(f_1 + f_2) < +\infty, (j = 0, 1)$, then*

$$\tau^j_\varphi(f_1) \le \max\{\tau^j_\varphi(f_1 + f_2), \tau^j_\varphi(f_2)\}, (j = 0, 1).$$

*(ii) If $0 < \rho^j_\varphi(f_1) = \rho^j_\varphi(f_2) = \rho^j_\varphi(f_1 f_2) < +\infty, (j = 0, 1)$, then*

$$\tau^j_\varphi(f_1) \le \max\{\tau^j_\varphi(f_1 f_2), \tau^j_\varphi(f_2)\}, (j = 0, 1).$$

**Theorem 2.4** *Let $\varphi \in \Phi$ and $f_1, f_2$ be two meromorphic functions. If $0 < \rho^j_\varphi(f_1) = \rho^j_\varphi(f_2) < +\infty$ and $\tau^j_\varphi(f_1) < \tau^j_\varphi(f_2), (j = 0, 1)$, then*

$$\rho^j_\varphi(f_1 + f_2) = \rho^j_\varphi(f_1\, f_2) = \rho^j_\varphi(f_1) = \rho^j_\varphi(f_2), (j = 0, 1), \tag{2.4}$$

$$\tau^j_\varphi(f_1 + f_2) = \tau^j_\varphi(f_1\, f_2) = \tau^j_\varphi(f_2). \tag{2.5}$$

**Theorem 2.5** *Let $\varphi \in \Phi$ and $f_1, f_2$ be two entire functions. Then*

*(i) If $0 < \rho_\varphi^j(f_1) < \rho_\varphi^j(f_2) < +\infty$ and $\tilde{\tau}_\varphi^j(f_1) < \tilde{\tau}_\varphi^j(f_2), (j = 0, 1)$, then*

$$\begin{aligned} \tilde{\tau}_\varphi^j(f_1 + f_2) &= \tilde{\tau}_\varphi^j(f_2), \\ \tilde{\tau}_\varphi^j(f_1 f_2) &\leq \tilde{\tau}_\varphi^j(f_2). \end{aligned}$$

*(ii) If $0 < \rho_\varphi^j(f_1) = \rho_\varphi^j(f_2) = \rho_\varphi^j(f_1 + f_2) < +\infty, (j = 0, 1)$, then*

$$\tilde{\tau}_\varphi^j(f_1 + f_2) \leq \max\{\tilde{\tau}_\varphi^j(f_1), \tilde{\tau}_\varphi^j(f_2)\}.$$

*Furthermore, if $\tilde{\tau}_\varphi^j(f_1) \neq \tilde{\tau}_\varphi^j(f_2)$, then $\tilde{\tau}_\varphi^j(f_1 + f_2) = \max\{\tilde{\tau}_\varphi^j(f_1), \tilde{\tau}_\varphi^j(f_2)\}$.*

*(ii) If $0 < \rho_\varphi^j(f_1) = \rho_\varphi^j(f_2) = \rho_\varphi^j(f_1\, f_2) < +\infty, (j = 0, 1)$, then*

$$\tilde{\tau}_\varphi^j(f_1 f_2) \leq \max\{\tilde{\tau}_\varphi^j(f_1), \tilde{\tau}_\varphi^j(f_2)\}.$$

*Furthermore, if $\tilde{\tau}_\varphi^j(f_1) \neq \tilde{\tau}_\varphi^j(f_2)$, then $\tilde{\tau}_\varphi^j(f_1\, f_2) = \max\{\tilde{\tau}_\varphi^j(f_1), \tilde{\tau}_\varphi^j(f_2)\}$.*

**Theorem 2.6** *Let $\varphi \in \Phi$ and $f_1, f_2$ be two entire functions. If $0 < \rho_\varphi^j(f_1) = \rho_\varphi^j(f_2) < +\infty$ and $\tilde{\tau}_\varphi^j(f_1) < \tilde{\tau}_\varphi^j(f_2)$, $(j = 0, 1)$, then*

$$\rho_\varphi^j(f_1 + f_2) = \rho_\varphi^j(f_1) = \rho_\varphi^j(f_2), \tag{2.6}$$

$$\tilde{\tau}_\varphi^j(f_1 + f_2) = \tilde{\tau}_\varphi^j(f_2). \tag{2.7}$$

**Theorem 2.7** *Let $f$ be a meromorphic function and $\varphi \in \Phi$. Then*

$$\rho_\varphi^j(f') = \rho_\varphi^j(f) \quad for\ j = 0, 1.$$

**Theorem 2.8** *Let $f$ be a meromorphic function and $\varphi \in \Phi$. Then*

$$\tau_\varphi^j(f') = \tau_\varphi^j(f) \quad for\ j = 0, 1.$$

# 3 Basic properties and lemmas

**Proposition 3.1 ([8])** *If $\varphi \in \Phi$, then*

$$\forall m > 0, \forall k \geq 0: \quad \frac{\varphi^{-1}(\log x^m)}{x^k} \longrightarrow +\infty, \quad x \longrightarrow +\infty, \tag{3.1}$$

$$\forall \delta > 0: \quad \frac{\log \varphi^{-1}((1+\delta)x)}{\log \varphi^{-1}(x)} \longrightarrow +\infty, \quad x \longrightarrow +\infty. \tag{3.2}$$

**Remark 3.2 ([8])** *We can see that (3.2) implies that*

$$\forall c > 0, \varphi(ct) \leq \varphi(t^c) \leq (1 + o(1))\varphi(t), \quad t \longrightarrow +\infty. \tag{3.3}$$

**Proposition 3.3 ([8])** *Let $\varphi \in \Phi$ and $f$ be an entire function. Then*

$$\rho_\varphi^j(f) = \tilde{\rho}_\varphi^j(f), \quad j = 0, 1.$$

Can we obtain a counterparts results of Remark 1.3 and Proposition 3.3 for the $\varphi$-types?

**Proposition 3.4** *Let $\varphi \in \Phi$ and $f$ be an entire function. Then*

$$\tau_\varphi^j(f) \leq \tilde{\tau}_\varphi^j(f), \quad j = 0, 1,$$

$$\tilde{\tau}_\varphi^j(f) \leq 2^{\rho_\varphi^j(f)} \tau_\varphi^j(f), \quad j = 0, 1.$$

**Proof.** We denote $\rho_1 = \rho_\varphi^1(f) = \tilde{\rho}_\varphi^1(f)$. By the known double inequality ([11], [13])

$$T(r, f) \leq \log M(r, f) \leq \frac{R + r}{R - r} T(R, f), 0 < r < R.$$

and the monotonicity of the function $\varphi$, we have $\tau_\varphi^1(f) \leq \tilde{\tau}_\varphi^1(f)$. Taking $R = 2r$, then by using (3.3), we get

$$\frac{e^{\varphi(\log M(r,f))}}{r^{\rho_1}} \leq \frac{e^{\varphi(3T(2r,f))}}{r^{\rho_1}} \leq \frac{e^{(1+o(1))\varphi(T(2r,f))}}{(2r)^{\rho_1} 2^{-\rho_1}}.$$

By passing to the limit as $r \longrightarrow +\infty$, we obtain $\tilde{\tau}_\varphi^1(f) \leq 2^{\rho_1} \tau_\varphi^1(f)$.

**Proposition 3.5 ([8])** *Let $\varphi \in \Phi$ and $f_1, f_2, f$ be three meromorphic functions. Then, we have*

$$\rho_\varphi^j(f_1 + f_2) \leq \max\{\rho_\varphi^j(f_1), \rho_\varphi^j(f_2)\}, j = 0, 1, \tag{3.4}$$

$$\rho_\varphi^j(f_1 \, f_2) \leq \max\{\rho_\varphi^j(f_1), \rho_\varphi^j(f_2)\}, j = 0, 1, \tag{3.5}$$

$$\rho_\varphi^j\left(\frac{1}{f}\right) = \rho_\varphi^j(f), (j = 0, 1), f \not\equiv 0. \tag{3.6}$$

By using a similar discussion as in the proof of Proposition 3.5 in ([8]) and by the properties

$$T(r, af_1) = T(r, f_1) + O(1), \ a \in \mathbb{C}^*,$$

$$T\left(r, \frac{1}{f_1}\right) = T(r, f_1) + O(1)$$

one can obtain the following results.

**Proposition 3.6** *Let $\varphi \in \Phi$ and $f$ be a meromorphic function. For $j = 0, 1$, we have*

$$\begin{aligned} \rho_\varphi^j(af) &= \rho_\varphi^j(f), a \in \mathbb{C}^*, \\ \tau_\varphi^j(af) &= \tau_\varphi^j(f), a \in \mathbb{C}^*, \\ \tau_\varphi^j\left(\frac{1}{f}\right) &= \tau_\varphi^j(f), f \not\equiv 0. \end{aligned}$$

**Lemma 3.7 ([1])** *Let $g : (0, +\infty) \to \mathbb{R}$ and $h : (0, +\infty) \to \mathbb{R}$ be monotone non-decreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set $E$ of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

**Lemma 3.8 ([18])** *If $f$ is a meromorphic function, then*

$$T(r, f) < O(T(2r, f') + \log r), \quad r \longrightarrow +\infty.$$

# 4 Proofs of the main results

## Proof of Theorem 2.1

Suppose that $\rho_\varphi^j(f_1) < \rho_\varphi^j(f_2)$. By (3.4), we have $\rho_\varphi^j(f_1+f_2) \leq \rho_\varphi^j(f_2)$. It follows again from (3.4) that

$$\rho_\varphi^j(f_2) = \rho_\varphi^j(f_1 + f_2 - f_1) \leq \max\{\rho_\varphi^j(f_1 + f_2), \rho_\varphi^j(f_1)\}.$$

If we suppose $\rho_\varphi^j(f_1) > \rho_\varphi^j(f_1 + f_2)$, then

$$\rho_\varphi^j(f_2) = \rho_\varphi^j(f_1 + f_2 - f_1) \leq \max\{\rho_\varphi^j(f_1 + f_2), \rho_\varphi^j(f_1)\} = \rho_\varphi^j(f_1)$$

which contradicts the assumption $\rho_\varphi^j(f_1) < \rho_\varphi^j(f_2)$. Hence $\rho_\varphi^j(f_2) \leq \rho_\varphi^j(f_1+f_2)$ and therefore $\rho_\varphi^j(f_1 + f_2) = \rho_\varphi^j(f_2)$. Now, we prove that $\rho_\varphi^j(f_1 f_2) = \rho_\varphi^j(f_2)$. Indeed, it follows from (3.5) that $\rho_\varphi^j(f_1 f_2) \leq \rho_\varphi^j(f_2)$ and by (3.6), we have

$$\rho_\varphi^j(f_2) = \rho_\varphi^j\left(f_1 f_2 \frac{1}{f_1}\right) \leq \max\left\{\rho_\varphi^j(f_1 f_2), \rho_\varphi^j\left(\frac{1}{f_1}\right)\right\} = \max\{\rho_\varphi^j(f_1 f_2), \rho_\varphi^j(f_1)\}.$$

If we suppose $\rho_\varphi^j(f_1) > \rho_\varphi^j(f_1 f_2)$, then

$$\rho_\varphi^j(f_2) = \rho_\varphi^j\left(f_1 f_2 \frac{1}{f_1}\right) \leq \max\{\rho_\varphi^j(f_1 f_2), \rho_\varphi^j(f_1)\} = \rho_\varphi^j(f_1)$$

which is a contradiction. Hence $\rho_\varphi^j(f_2) \leq \rho_\varphi^j(f_1 f_2)$ and therefore $\rho_\varphi^j(f_1 f_2) = \rho_\varphi^j(f_2)$.

## Proof of Theorem 2.2

We will prove the theorem for $j = 1$, the proofs for $j = 0$ are analogous.
(i) The definition of the $\tau^1_\varphi$-type implies that for any given $\varepsilon > 0$, there exists a sequence $\{r_n, n \geq 1\}$ tending to infinity such that

$$T(r_n, f_2) \geq \varphi^{-1}(\log(\tau^1_\varphi(f_2) - \varepsilon)r_n^{\rho^1_\varphi(f_i)})$$

and for sufficiently large $r$

$$T(r, f_1) \leq \varphi^{-1}(\log(\tau^1_\varphi(f_1) + \varepsilon)r^{\rho^1_\varphi(f_1)}).$$

We know that $T(r, f_1 + f_2) \geq T(r, f_2) - T(r, f_1) - \log 2$, then by using Proposition 3.1

$$\begin{aligned} T(r_n, f_1 + f_2) &\geq \varphi^{-1}(\log(\tau^1_\varphi(f_2) - \varepsilon)r_n^{\rho^1_\varphi(f_2)}) - \varphi^{-1}(\log(\tau^1_\varphi(f_1) + \varepsilon)r_n^{\rho^1_\varphi(f_1)}) \\ - \log 2 &\geq \varphi^{-1}(\log(\tau^1_\varphi(f_2) - 2\varepsilon)r_n^{\rho^1_\varphi(f_2)}) \end{aligned} \tag{4.1}$$

provided $\varepsilon$ such that $0 < 2\varepsilon < \tau^1_\varphi(f_2) - \tau^1_\varphi(f_1)$. It follows from Theorem 2.1 that $\rho^1_\varphi(f_1 + f_2) = \rho^1_\varphi(f_2)$, and by the monotonicity of $\varphi$ and (4.1), we get

$$\frac{e^{\varphi(T(r_n, f_1+f_2))}}{r_n^{\rho^1_\varphi(f_1+f_2)}} \geq \tau^1_\varphi(f_2) - 2\varepsilon$$

since $\varepsilon$ can be arbitrarily chosen such that $0 < 2\varepsilon < \tau^1_\varphi(f_2) - \tau^1_\varphi(f_1)$, thus

$$\tau^1_\varphi(f_1 + f_2) \geq \tau^1_\varphi(f_2). \tag{4.2}$$

It remains to prove the converse inequality. Indeed, by applying (4.2) and since

$$\rho^1_\varphi(f_1 + f_2) = \rho^1_\varphi(f_2) > \rho^1_\varphi(f_1) = \rho^1_\varphi(-f_1)$$

we obtain

$$\tau^1_\varphi(f_2) = \tau^1_\varphi(f_1 + f_2 - f_1) \geq \tau^1_\varphi(f_1 + f_2). \tag{4.3}$$

We deduce from (4.2) and ( 4.3) that $\tau^1_\varphi(f_1 + f_2) = \tau^1_\varphi(f_2)$.

Now, we prove that $\tau^1_\varphi(f_1\, f_2) = \tau^1_\varphi(f_2)$. By the property

$$T(r, f_1 f_2) \geq T(r, f_1) - T(r, f_2) + O(1) \tag{4.4}$$

and a similar discussion as in the above proof, one can easily show that

$$\tau^1_\varphi(f_1\, f_2) \geq \tau^1_\varphi(f_2). \tag{4.5}$$

Since $\rho^1_\varphi(f_1\, f_2) = \rho^1_\varphi(f_2) > \rho^1_\varphi(f_1) = \rho^1_\varphi\left(\frac{1}{f_1}\right)$, then by (4.5), we get

$$\tau^1_\varphi(f_2) = \tau^1_\varphi\left(f_1\, f_2\, \frac{1}{f_1}\right) \geq \tau^1_\varphi(f_1\, f_2)$$

and therefore $\tau_\varphi^1(f_1\, f_2) = \tau_\varphi^1(f_2)$.
(ii) The definition of the $\tau_\varphi^1$-type implies that for any given $\varepsilon > 0$ and for all $r$ sufficiently large, we have

$$T(r, f_i) \leq \varphi^{-1}(\log(\tau_\varphi^1(f_i) + \varepsilon) r^{\rho_\varphi^1(f_i)}), \;\; i = 1, 2. \tag{4.6}$$

By the assumption $0 < \rho_\varphi^1(f_1) = \rho_\varphi^1(f_2) = \rho_\varphi^1(f_1 + f_2) < +\infty$, we get by using Proposition 3.1

$$\begin{aligned} T(r, f_1 + f_2) &\leq T(r, f_1) + T(r, f_2) + O(1) \\ &\leq \varphi^{-1}(\log(\tau_\varphi^1(f_1) + \varepsilon) r^{\rho_\varphi^1(f_1+f_2)}) + \varphi^{-1}(\log(\tau_\varphi^1(f_2) + \varepsilon) r^{\rho_\varphi^1(f_1+f_2)}) \\ +O(1) &\leq \varphi^{-1}(\log(\max\{\tau_\varphi^1(f_1), \tau_\varphi^1(f_2)\} + 3\varepsilon) r^{\rho_\varphi^1(f_1+f_2)})). \end{aligned}$$

By the monotonicity of $\varphi$, we obtain

$$\frac{e^{\varphi(T(r, f_1+f_2))}}{r^{\rho_\varphi^1(f_1+f_2)}} \leq \max\{\tau_\varphi^1(f_1), \tau_\varphi^1(f_2)\} + 3\varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrarily, then we get

$$\tau_\varphi^1(f_1 + f_2) \leq \max\{\tau_\varphi^1(f_1), \tau_\varphi^1(f_2)\}. \tag{4.7}$$

Without loss of generality, we may suppose $\tau_\varphi^1(f_1) < \tau_\varphi^1(f_2)$, then by (4.7) and since $\rho_\varphi^1(f_1 + f_2) = \rho_\varphi^1(f_1) = \rho_\varphi^1(-f_1)$, it follows

$$\tau_\varphi^1(f_2) = \tau_\varphi^1(f_1 + f_2 - f_1) \leq \max\{\tau_\varphi^1(f_1 + f_2), \tau_\varphi^1(f_1)\} = \tau_\varphi^1(f_1 + f_2). \tag{4.8}$$

We deduce from (4.7) and (4.8) that $\tau_\varphi^1(f_1 + f_2) = \max\{\tau_\varphi^1(f_1), \tau_\varphi^1(f_2)\}$.
(iii) By a similar discussion as in the above proof and the fact that $T(r, f_1\, f_2) \leq T(r, f_1) + T(r, f_2)$, one can prove that

$$\tau_\varphi^1(f_1\, f_2) \leq \max\{\tau_\varphi^1(f_1), \tau_\varphi^1(f_2)\}. \tag{4.9}$$

On the other hand, if we suppose that $\tau_\varphi^1(f_1) < \tau_\varphi^1(f_2)$, then by (4.9) and since $\rho_\varphi^1(f_1\, f_2) = \rho_\varphi^1(f_1) = \rho_\varphi^1\left(\frac{1}{f_1}\right)$, we get

$$\tau_\varphi^1(f_2) = \tau_\varphi^1\left(f_1\, f_2\, \frac{1}{f_1}\right) \leq \max\{\tau_\varphi^1(f_1\, f_2), \tau_\varphi^1(f_1)\} = \tau_\varphi^1(f_1\, f_2). \tag{4.10}$$

It follows from (4.9) and (4.10) that $\tau_\varphi^1(f_1\, f_2) = \max\{\tau_\varphi^1(f_1), \tau_\varphi^1(f_2)\}$.

## Proof of Corollary 2.3

The proofs follow immediately from Theorem 2.2. Indeed, since $\rho_\varphi^j(f_1 + f_2) = \rho_\varphi^j(f_2) = \rho_\varphi^j(-f_2)$, then

$$\tau_\varphi^j(f_1) = \tau_\varphi^j(f_1 + f_2 - f_2) \leq \max\{\tau_\varphi^j(f_1 + f_2), \tau_\varphi^j(f_2)\}.$$

Similarly, since $\rho_\varphi^j(f_1\, f_2) = \rho_\varphi^j(f_2) = \rho_\varphi^j\left(\frac{1}{f_2}\right)$, then

$$\tau_\varphi^j(f_1) = \tau_\varphi^j\left(f_1\, f_2\, \frac{1}{f_2}\right) \leq \max\{\tau_\varphi^j(f_1\, f_2), \tau_\varphi^j(f_2)\}.$$

## Proof of Theorem 2.4

By (3.4) and (3.5), we have

$$\rho^j_\varphi(f_1+f_2) \le \rho^j_\varphi(f_1) = \rho^j_\varphi(f_2) \quad \text{and} \quad \rho^j_\varphi(f_1\, f_2) \le \rho^j_\varphi(f_1) = \rho^j_\varphi(f_2).$$

By using similar reasoning as in the proofs of Theorem 2.2, it follows from (4.1) and (4.4) that

$$\rho^j_\varphi(f_1+f_2) \ge \rho^j_\varphi(f_2) = \rho^j_\varphi(f_1) \quad \text{and} \quad \rho^j_\varphi(f_1\, f_2) \ge \rho^j_\varphi(f_2) = \rho^j_\varphi(f_1).$$

Therefore, assumption (2.4) holds. On the other hand, we can see that assumption (2.5) follows immediately from (2.4) and Theorem 2.2.

## Proof of Theorem 2.5

We will prove the theorem for $j = 0$, the proofs for $j = 1$ are analogous.
(i) The definition of the $\tilde{\tau}^0_\varphi$-type implies that for any given $\varepsilon > 0$, there exists a sequence $\{r_n, n \ge 1\}$ tending to $+\infty$ such that

$$M(r_n, f_2) \ge \varphi^{-1}(\log(\tilde{\tau}^0_\varphi(f_2) - \varepsilon) r_n^{\tilde{\rho}^0_\varphi(f_2)}).$$

and for sufficiently large $r$, we have

$$M(r, f_1) \le \varphi^{-1}(\log(\tilde{\tau}^0_\varphi(f_1) + \varepsilon) r^{\tilde{\rho}^0_\varphi(f_1)})$$

In each circle $|z| = r_n$ we choose a sequence $\{z_n, n \ge 1\}$ with $|z_n| = r_n$ and satisfying $|f_2(z_n)| = M(r_n, f_2)$ such that by using Proposition 3.1, we obtain

$$\begin{aligned} M(r_n, f_1+f_2) &\ge |f_1(z_n) + f_2(z_n)| \ge |f_2(z_n)| - |f_1(z_n)| \\ \ge M(r_n, f_2) - M(r_n, f_1) &\ge \varphi^{-1}(\log(\tilde{\tau}^0_\varphi(f_2) - \varepsilon) r_n^{\tilde{\rho}^0_\varphi(f_2)}) \\ -\varphi^{-1}(\log(\tilde{\tau}^0_\varphi(f_1) + \varepsilon) r_n^{\tilde{\rho}^0_\varphi(f_1)}) &\ge \varphi^{-1}(\log(\tilde{\tau}^0_\varphi(f_2) - 2\varepsilon) r_n^{\tilde{\rho}^0_\varphi(f_2)}) \end{aligned} \tag{4.11}$$

provided $\varepsilon$ such that $0 < 2\varepsilon < \tilde{\tau}^0_\varphi(f_2) - \tilde{\tau}^0_\varphi(f_1)$ and $r_n \longrightarrow +\infty$. It follows from Proposition 3.3 and Theorem 2.1 that $\tilde{\rho}^0_\varphi(f_1+f_2) = \tilde{\rho}^0_\varphi(f_2)$. By the monotonicity of $\varphi$ and (4.11), we get

$$\frac{e^{\varphi(M(r,f_1+f_2))}}{r^{\tilde{\rho}^0_\varphi(f_1+f_2)}} \ge \tilde{\tau}^0_\varphi(f_2) - 2\varepsilon.$$

Since $\varepsilon$ can be arbitrarily chosen such that $0 < 2\varepsilon < \tilde{\tau}^0_\varphi(f_2) - \tilde{\tau}^0_\varphi(f_1)$, then we obtain

$$\tilde{\tau}^0_\varphi(f_1+f_2) \ge \tilde{\tau}^0_\varphi(f_2). \tag{4.12}$$

Since $\rho^0_\varphi(f_1+f_2)=\rho^0_\varphi(f_2)>\rho^0_\varphi(f_1)=\rho^0_\varphi(-f_1)$, then we obtain by applying (4.12 ) that

$$\tilde{\tau}^0_\varphi(f_2)=\tilde{\tau}^0_\varphi(f_1+f_2-f_1)\le\tilde{\tau}^0_\varphi(f_1+f_2)$$

and therefore $\tilde{\tau}^0_\varphi(f_1+f_2)=\tilde{\tau}^0_\varphi(f_2)$. Now, we prove that $\tau^1_\varphi(f_1f_2)\le\tau^1_\varphi(f_2)$. We have

$$\begin{aligned}M(r,f_1f_2) &\le M(r,f_1)\,M(r,f_2)\\ &\le \varphi^{-1}(\log(\tilde{\tau}^0_\varphi(f_1)+\varepsilon)r^{\tilde{\rho}^0_\varphi(f_1)})\,\varphi^{-1}(\log(\tilde{\tau}^0_\varphi(f_2)+\varepsilon)r^{\tilde{\rho}^0_\varphi(f_2)})\\ &\le [\varphi^{-1}(\log(\tilde{\tau}^0_\varphi(f_2)+\varepsilon)r^{\tilde{\rho}^0_\varphi(f_2)})]^2.\end{aligned}$$

By the monotonicity of $\varphi$ and (3.3), we obtain

$$\varphi(M(r,f_1f_2))\le(1+o(1))\log(\tilde{\tau}^0_\varphi(f_2)+\varepsilon)r^{\tilde{\rho}^0_\varphi(f_2)}\le\log(\tilde{\tau}^0_\varphi(f_2)+2\varepsilon)r^{\tilde{\rho}^0_\varphi(f_2)}.$$

Since $\tilde{\rho}^0_\varphi(f_2)=\tilde{\rho}^0_\varphi(f_1\,f_2)$, one can deduce that $\tilde{\tau}^0_\varphi(f_1\,f_2)\le\tilde{\tau}^0_\varphi(f_2)$.
(ii) The definition of the $\tilde{\tau}^0_\varphi$-type implies that for any given $\varepsilon>0$ and for all $r$ sufficiently large, we have

$$M(r,f_i)\le\varphi^{-1}(\log(\tilde{\tau}^0_\varphi(f_i)+\varepsilon)r^{\tilde{\rho}^0_\varphi(f_i)}),\ i=1,2.$$

By the assumption $0<\rho^0_\varphi(f_1)=\rho^0_\varphi(f_2)=\rho^0_\varphi(f_1+f_2)<+\infty$, we get by using Proposition 3.1

$$\begin{aligned}M(r,f_1+f_2) &\le M(r,f_1)+M(r,f_2)\\ &\le \varphi^{-1}(\log(\tilde{\tau}^0_\varphi(f_1)+\varepsilon)r^{\tilde{\rho}^0_\varphi(f_1+f_2)})+\varphi^{-1}(\log(\tilde{\tau}^0_\varphi(f_2)+\varepsilon)r^{\tilde{\rho}^0_\varphi(f_1+f_2)})\\ &\le \varphi^{-1}(\log(\max\{\tilde{\tau}^0_\varphi(f_1),\tilde{\tau}^0_\varphi(f_2)\}+2\varepsilon)r^{\tilde{\rho}^0_\varphi(f_1+f_2)}).\end{aligned}$$

From this inequality and the monotonicity of $\varphi$, we obtain

$$\frac{e^{\varphi(M(r,f_1+f_2))}}{r^{\tilde{\rho}^0_\varphi(f_1+f_2)}}\le\max\{\tilde{\tau}^0_\varphi(f_1),\tilde{\tau}^0_\varphi(f_2)\}+2\varepsilon.$$

Since $\varepsilon>0$ can be arbitrarily chosen, then we get

$$\tilde{\tau}^0_\varphi(f_1+f_2)\le\max\{\tilde{\tau}^0_\varphi(f_1),\tilde{\tau}^0_\varphi(f_2)\}. \tag{4.13}$$

Without loss of generality, we may suppose $\tilde{\tau}^0_\varphi(f_1)<\tilde{\tau}^0_\varphi(f_2)$, then by (4.13) and since $\rho^0_\varphi(f_1+f_2)=\rho^0_\varphi(f_1)=\rho^0_\varphi(-f_1)$, it follows that

$$\tilde{\tau}^0_\varphi(f_2)=\tilde{\tau}^0_\varphi(f_1+f_2-f_1)\le\max\{\tilde{\tau}^0_\varphi(f_1+f_2),\tilde{\tau}^0_\varphi(f_1)\}=\tilde{\tau}^0_\varphi(f_1+f_2). \tag{4.14}$$

We deduce from (4.13) and (4.14) that $\tilde{\tau}^0_\varphi(f_1+f_2)=\max\{\tilde{\tau}^0_\varphi(f_1),\tilde{\tau}^0_\varphi(f_2)\}$.

(iii) Since $0<\rho_\varphi^0(f_1)=\rho_\varphi^0(f_2)=\rho_\varphi^0(f_1f_2)<+\infty$, then

$$\begin{aligned} M(r,f_1\,f_2) &\le M(r,f_1)\,M(r,f_2)\\ &\le \varphi^{-1}(\log(\tilde\tau_\varphi^0(f_1)+\varepsilon)r^{\tilde\rho_\varphi^0(f_1f_2)})\varphi^{-1}(\log(\tilde\tau_\varphi^0(f_2)+\varepsilon)r^{\tilde\rho_\varphi^0(f_1f_2)})\\ &\le [\varphi^{-1}(\log(\max\{\tilde\tau_\varphi^0(f_1),\tilde\tau_\varphi^0(f_2)\}+\varepsilon)r^{\tilde\rho_\varphi^0(f_1f_2)})]^2. \end{aligned}$$

By the monotonicity of $\varphi$ and (3.3), we obtain

$$\begin{aligned} \varphi(M(r,f_1f_2)) &\le (1+o(1))\log(\max\{\tilde\tau_\varphi^0(f_1),\tilde\tau_\varphi^0(f_2)\}+\varepsilon)r^{\tilde\rho_\varphi^0(f_1f_2)}\\ &\le \log(\max\{\tilde\tau_\varphi^0(f_1),\tilde\tau_\varphi^0(f_2)\}+2\varepsilon)r^{\tilde\rho_\varphi^0(f_1f_2)}. \end{aligned}$$

It follows that

$$\frac{e^{\varphi(M(r,f_1\,f_2))}}{r^{\tilde\rho_\varphi^0(f_1\,f_2)}}\le \max\{\tilde\tau_\varphi^0(f_1),\tilde\tau_\varphi^0(f_2)\}+2\varepsilon.$$

By arbitrariness of $\varepsilon>0$, we deduce that

$$\tilde\tau_\varphi^0(f_1\,f_2)\le \max\{\tilde\tau_\varphi^0(f_1),\tilde\tau_\varphi^0(f_2)\}. \tag{4.15}$$

On the other hand, if we suppose $\tilde\tau_\varphi^0(f_1)<\tilde\tau_\varphi^0(f_2)$, then by (4.15) and since $\rho_\varphi^0(f_1\,f_2)=\rho_\varphi^0(f_1)=\rho_\varphi^0\left(\frac{1}{f_1}\right)$, we get

$$\tilde\tau_\varphi^0(f_2)=\tilde\tau_\varphi^0\left(f_1\,f_2\,\frac{1}{f_1}\right)\le \max\{\tilde\tau_\varphi^0(f_1\,f_2),\tilde\tau_\varphi^0(f_1)\}=\tilde\tau_\varphi^0(f_1\,f_2). \tag{4.16}$$

It follows from (4.15) and (4.16) that $\tilde\tau_\varphi^0(f_1\,f_2)=\max\{\tilde\tau_\varphi^0(f_1),\tilde\tau_\varphi^0(f_2)\}$.

## Proof of Theorem 2.6

By Proposition 3.3 and (3.4) we have $\rho_\varphi^j(f_1+f_2)\le\rho_\varphi^j(f_1)=\rho_\varphi^j(f_2)$. By (4.11) and Proposition 3.3, we have $\rho_\varphi^j(f_1+f_2)\ge\rho_\varphi^j(f_2)=\rho_\varphi^j(f_1)$. Therefore, assumption (2.6) holds. On the other hand, we can see that assumption (2.7) follows immediately from (2.6) and Theorem 2.5.

## Proof of Theorem 2.7

The inequality $\rho_\varphi^1(f')\le\rho_\varphi^1(f)$ was proved in [8, Proposition 3.4]. It remains to prove the converse inequality. The definition of $\rho_\varphi^1(f'):=\rho_1'$ implies that for any given $\varepsilon>0$ and for all $r$ sufficiently large, we have

$$T\,(r,f')\le\varphi^{-1}(\log r^{\rho_1'+\varepsilon}).$$

By (3.1) and Lemma 3.8, we get that

$$T(r,f)<O\{\varphi^{-1}(\log(2r)^{\rho_1'+\varepsilon})+\log r\}=O\{\varphi^{-1}(\log(2r)^{\rho_1'+2\varepsilon})\},\ r\longrightarrow+\infty.$$

In view of (3.3), we have $\varphi(T(r,f))\le(1+o\,(1))\,(\rho_1'+2\varepsilon)\log 2r\le(\rho_1'+3\varepsilon)\log 2r$. Since $\varepsilon>0$ is an arbitrary number, we obtain $\rho_\varphi^1(f)\le\rho_1':=\rho_\varphi^1(f')$ and therefore $\rho_\varphi^1(f')=\rho_\varphi^1(f)$. Similar proof for $j=0$.

## Proof of Theorem 2.8

We denote $\rho^1_\varphi(f') = \rho^1_\varphi(f) = \rho_1$. The definition of the $\tau^1_\varphi$-type implies that for any given $\varepsilon > 0$ and for all $r$ sufficiently large, we have

$$T(r,f) \leq \varphi^{-1}(\log(\tau^1_\varphi(f) + \varepsilon)r^{\rho_1}).$$

By the lemma of logarithmic derivative ([11], [13]), we have

$$T(r,f') \leq 2T(r,f) + O(\log T\,(r,f) + \log r),\ r \notin E,$$

where $E \subset [0,+\infty)$ is a set of finite linear measure. Then, in view of (3.1), we get

$$T(r,f') \leq O\left(\varphi^{-1}(\log \tau^1_\varphi(f) + 2\varepsilon)r^{\rho_1}\right),\ r \notin E.$$

It follows from Lemma 3.7, for any given $\alpha > 1$ and sufficiently large $r$

$$T(r,f') \leq O\left(\varphi^{-1}\left(\log(\tau^1_\varphi(f) + 2\varepsilon)(\alpha r)^{\rho_1}\right)\right).$$

In view of (3.3), we have for sufficiently large $r$

$$\varphi(T(r,f')) \leq (1 + o\,(1))\left(\log(\tau^1_\varphi(f) + 2\varepsilon)(\alpha r)^{\rho_1}\right) \leq \log(\tau^1_\varphi(f) + 3\varepsilon)(\alpha r)^{\rho_1},$$

so

$$e^{\varphi(T(r,f'))} \leq \left(\tau^1_\varphi(f) + 3\varepsilon\right)(\alpha r)^{\rho_1}.$$

By arbitrariness of $\varepsilon > 0$, we obtain $\tau^1_\varphi(f') \leq \alpha^{\rho_1}\tau^1_\varphi(f)$. On the other hand, by the definition of the $\tau^1_\varphi$-type implies that for any given $\varepsilon > 0$ and for all $r$ sufficiently large, we have

$$T(r,f') \leq \varphi^{-1}(\log(\tau^1_\varphi(f') + \varepsilon)r^{\rho_1}).$$

By (3.1) and Lemma 3.8, we get that

$$T(r,f) < O\{\varphi^{-1}(\log(\tau^1_\varphi(f') + \varepsilon)\,(2r)^{\rho_1}) + \log r\}$$

$$= O\{\varphi^{-1}(\log(\tau^1_\varphi(f') + 2\varepsilon)\,(2r)^{\rho_1})\},\ r \longrightarrow +\infty.$$

In view of (3.3), we have for sufficiently large $r$

$$\varphi(T(r,f)) \leq (1 + o\,(1))\left(\log(\tau^1_\varphi(f) + 2\varepsilon)(2r)^{\rho_1}\right) \leq \log(\tau^1_\varphi(f) + 3\varepsilon)(2r)^{\rho_1}.$$

From this, we can easily obtain $\tau^1_\varphi(f) \leq 2^{\rho_1}\tau^1_\varphi(f')$ and hence we deduce that

$$\frac{1}{2^{\rho^1_\varphi(f)}}\tau^1_\varphi(f) \leq \tau^1_\varphi(f') \leq \alpha^{\rho^1_\varphi(f)}\tau^1_\varphi(f) \quad (\alpha > 1)\,.$$

# 5 Open Problem

It is interesting to study the growth of entire and meromorphic functions by using the concept of $\alpha, \beta$-order introduced by Sheremeta ([16]).

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