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Starlikeness And Convexity of Analytic Functions Using Multiplier Transformation

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Abstract

In the present paper, we find certain results on multiplier transformation. As particular cases to our main result, we derive certain results for starlike and convex functions.

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MSC 2010 Classifications: Primary 30C45; Secondary 30C80.

1 Introduction

Let $\mathcal{H}$ denote the class of functions f, analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$. Let $\mathcal{A}_p(m)$ be the subclass of $\mathcal{H}$, consist functions of the form

$$f(z) = z^p + \sum_{k=p+m}^{\infty} a_k z^k, \text{ for } p,\ m \in \mathbb{N} = \{1,\ 2,\ 3,\ \ldots\},$$

which are analytic and $p-$valent in the open unit disk $\mathbb{E}$. A function $f \in \mathcal{A}_p(m)$ is said to be $p-$valently starlike if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{E}.$$

The class of such functions is denoted by $\mathcal{S}_p^*(m)$. A function $f \in \mathcal{A}_p(m)$ is said to be $p-$valently convex in $\mathbb{E}$, if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{E}.$$

We denote by $\mathcal{K}_p(m)$, the class of all functions $f \in \mathcal{A}_p(m)$ that are $p-$valently convex in $\mathbb{E}$. Note that $\mathcal{S}_1^*(1) = \mathcal{S}^*$ and $\mathcal{K}_1(1) = \mathcal{K}$ are the usual classes of univalent starlike functions and univalent convex functions. Also note that $\mathcal{A}_1(1) = \mathcal{A}$. For $f \in \mathcal{A}_p(1)(= \mathcal{A}_p)$, define the multiplier transformation $I_p(n,\ \lambda)$, as:

$$I_p(n,\ \lambda)f(z) = z^p + \sum_{k=p+1}^{\infty}\left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k,\ (\lambda \geq 0,\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

The special case $I_1(n,\ 0)$ of the above defined operator is the well-known Sălăgean [6] derivative operator D^n, defined for $f \in \mathcal{A}$ as given below:

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

Let f and g be two analytic functions in $\mathbb{E}$. Then we say f is subordinate to g in $\mathbb{E}$, denoted by $f \prec g$ if there exist a Schwarz function w analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{E}$ such that $f(z) = g(w(z)),\ z \in \mathbb{E}$. In case the function g is univalent, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

The differential operator $(1-\lambda)\left(\frac{f(z)}{z^p}\right)^\alpha + \lambda\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^\alpha$ was studied by Liu [3] to make certain estimates on $\left(\frac{f(z)}{z^p}\right)^\alpha$, where $\alpha > 0,\ \lambda \geq 0$ are some real numbers and $f \in \mathcal{A}_p$. In 2014, Billing [1] studied a differential inequality for $f \in \mathcal{A}_p$ involving multiplier transformation as:

$$\left|\left(\frac{I_p(n,\ \lambda)f(z)}{z^p}\right)^\beta\left[1-\alpha+\alpha\frac{I_p(n+1,\ \lambda)f(z)}{I_p(n,\ \lambda)f(z)}\right]-1\right| < M(\alpha,\ \beta,\ \lambda,\ \delta,\ p),$$

where $\alpha,\ \beta$ and δ are real numbers such that $\alpha > \dfrac{2}{1-\delta},\ 0 \leq \delta < 1,\ \beta > 0$ and $M,\ 0 < M = M(\alpha,\ \beta,\ \lambda,\ \delta,\ p) = \dfrac{[\alpha+\beta(p+\lambda)][\alpha(1-\delta)-2]}{\alpha[1+\beta(1-\delta)(p+\lambda)]},\ z \in \mathbb{E}.$

He obtained certain new criteria for starlikeness and convexity of normalized analytic functions.

Recently, Darwish *et al.* [2] studied the class $\mathcal{B}_p(m,\ \alpha,\ \lambda,\ \mu)$, for $f \in \mathcal{A}_p(m)$, defined as:

$$\left|(1-\lambda)\left(\frac{f(z)}{z^p}\right)^{\alpha} + \lambda\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha} - 1\right| < \mu,\ z \in \mathbb{E},$$

where $\alpha > 0,\ \lambda \geq 0,\ 0 < \mu \leq 1$. They obtained the conditions of starlikeness for the members of class $\mathcal{B}_p(m,\ \alpha,\ \lambda,\ \mu)$ and discussed starlikeness of certain integral operators.
In the present paper, we study the following differential inequality

$$\left|\left(\frac{I_p(n,\ \lambda)f(z)}{z^p}\right)^{\beta}\left[1-\alpha+\alpha\frac{I_p(n+1,\ \lambda)f(z)}{I_p(n,\ \lambda)f(z)}\right] - 1\right| < M(\alpha,\ \beta,\ \lambda,\ p),\ z \in \mathbb{E},$$

where $\alpha,\ \beta$ are real numbers such that $0 < \alpha \leq 1,\ \beta > 0$, to obtain certain results for starlike and convex functions.

2 Preliminaries

We shall use the following lemmas to prove our main result.

Lemma 2.1. *[4] Let h be a convex function in $\mathbb{E}$ (i.e. h is analytic and univalent in $\mathbb{E}$ and $h(\mathbb{E})$ is a convex domain), $h(0) = 1$ and let $g(z) = 1 + b_n z^n + b_{n+1}z^{n+1} + \dots$ be analytic in $\mathbb{E}$. If*

$$g(z) + \frac{1}{c}zg'(z) \prec h(z),$$

then

$$g(z) \prec \frac{c}{n}z^{-\frac{c}{n}}\int_0^z t^{\frac{c}{n}-1}h(t)dt,$$

where $c \neq 0$ and $\Re\{c\} \geq 0$.

Lemma 2.2. *[5] Let $0 < \mu_1 < \mu < 1$ and let g be analytic in $\mathbb{E}$, satisfying*

$$g(z) \prec 1 + \mu_1(z),\ g(0) = 1,$$

(a) If p is analytic in $\mathbb{E},\ p(0) = 1$ and satisfies

$$g(z)[\gamma + (1-\gamma)p(z)] \prec 1 + \mu z,$$

where

$$\gamma = \begin{cases} \dfrac{1-\mu}{1+\mu_1}, & 0 < \mu + \mu_1 \le 1 \\ \dfrac{1-(\mu^2+\mu_1^2)}{2(1-\mu_1^2)}, & \mu^2 + \mu_1^2 \le 1 \le \mu + \mu_1 \end{cases} \tag{1}$$

then $\Re\{p(z)\} > 0,\ z \in \mathbb{E}$.

(b) If w *is analytic in* $\mathbb{E}$, *with* $w(0) = 0$ *and*

$$g(z)[1 + w(z)] \prec 1 + \mu z,$$

then

$$|w(z)| \le \frac{\mu + \mu_1}{1 - \mu_1} = r \le 1, \quad \mu + 2\mu_1 \le 1. \tag{2}$$

The value of γ *given by* (1) *and the bounds* (2) *are best possible.*

3 Main Results

Theorem 3.1. *Let* β *be a real number such that* $\beta > 0$. *If* $f \in \mathcal{A}_p$ *satisfies the differential inequality*

$$\left| \left(\frac{I_p(n,\ \lambda)f(z)}{z^p} \right)^{\beta} \left[1 - \alpha + \alpha \frac{I_p(n+1,\ \lambda)f(z)}{I_p(n,\ \lambda)f(z)} \right] - 1 \right| < M(\alpha,\ \beta,\ \lambda,\ p),\ z \in \mathbb{E}, \tag{3}$$

(a) where

$$M = \begin{cases} \frac{\alpha[\beta(p+\lambda)+\alpha]}{\beta(p+\lambda)(2-\alpha)+\alpha}, & 0 < \alpha \le \sqrt{2\beta(p+\lambda) + \frac{(3\beta(p+\lambda)-1)^2}{4}} - \frac{3\beta(p+\lambda)-1}{2} \\ \frac{[\beta(p+\lambda)+\alpha]\sqrt{2\alpha-1}}{\sqrt{\alpha^2+2\alpha\beta(p+\lambda)(1+\beta(p+\lambda))}}, & \sqrt{2\beta(p+\lambda) + \frac{(3\beta(p+\lambda)-1)^2}{4}} - \frac{3\beta(p+\lambda)-1}{2} \le \alpha \le 1 \end{cases} \tag{4}$$

then

$$\Re \left(\frac{I_p(n+1,\ \lambda)f(z)}{I_p(n,\ \lambda)f(z)} \right) > 0,\ z \in \mathbb{E}.$$

(b) If $\left| \dfrac{I_p(n+1,\ \lambda)f(z)}{I_p(n,\ \lambda)f(z)} - 1 \right| < r,\ z \in \mathbb{E},$ *where*

$$r = \frac{M[2\beta(p+\lambda)+\alpha]}{\alpha[\beta(p+\lambda)(1-M)+\alpha]},\ 0 < M \le \frac{\beta(p+\lambda)+\alpha}{3\beta(p+\lambda)+\alpha}, \tag{5}$$

then

$$\Re \left(\frac{I_p(n+1,\ \lambda)f(z)}{I_p(n,\ \lambda)f(z)} \right) > 0,\ z \in \mathbb{E}.$$

Proof. Let us define

$$g(z) = \left(\frac{I_p(n,\ \lambda)f(z)}{z^p}\right)^\beta,\ z \in \mathbb{E},$$

then $g(z) = 1 + \sum_{k=1}^{\infty} b_k z^k$ is analytic in $\mathbb{E}$.
On differentiating logarithmically, we obtain

$$\frac{zI_p'(n,\ \lambda)f(z)}{I_p(n,\ \lambda)f(z)} - p = \frac{zg'(z)}{\beta g(z)} \tag{6}$$

In view of the equality

$$zI_p'(n,\ \lambda)f(z) = (p+\lambda)I_p(n+1,\ \lambda)f(z) - \lambda I_p(n,\ \lambda)f(z),$$

(6) reduces to

$$\frac{I_p(n+1,\ \lambda)f(z)}{I_p(n,\ \lambda)f(z)} = 1 + \frac{zg'(z)}{\beta(p+\lambda)g(z)}$$

Therefore, in view of (3), we have

$$g(z) + \frac{\alpha}{\beta(p+\lambda)}zg'(z) \prec 1 + Mz,$$

with $h(z) = 1 + Mz$, from Lemma 2.1, we have

$$g(z) \prec 1 + \frac{\beta(p+\lambda)M}{\beta(p+\lambda)+\alpha}z = 1 + M_1 z,\ 0 \le M_1 = \frac{\beta(p+\lambda)M}{\beta(p+\lambda)+\alpha} < M < 1, \tag{7}$$

since (3) is equivalent to

$$\left(\frac{I_p(n,\ \lambda)f(z)}{z^p}\right)^\beta \left[(1-\alpha) + (1-(1-\alpha))\frac{I_p(n+1,\ \lambda)f(z)}{I_p(n,\ \lambda)f(z)}\right] \prec 1+Mz,\ z \in \mathbb{E}. \tag{8}$$

Substituting $g(z) = \left(\frac{I_p(n,\ \lambda)f(z)}{z^p}\right)^\beta$, $w(z) = \frac{I_p(n+1,\ \lambda)f(z)}{I_p(n,\ \lambda)f(z)}$ and $\gamma = (1-\alpha)$, where $\alpha,\ M,\ M_1$ and $g(z)$ satisfy the relation (4) and (7), thus conditions in Lemma 2.2(a) are satisfied.
Hence

$$\Re w(z) > 0,\ z \in \mathbb{E},\ i.e.\ \Re\left(\frac{I_p(n+1,\ \lambda)f(z)}{I_p(n,\ \lambda)f(z)}\right) > 0.$$

(b) Writing (3) as given below:

$$\left(\frac{I_p(n,\ \lambda)f(z)}{z^p}\right)^{\beta}\left[1+\alpha\left(\frac{I_p(n+1,\ \lambda)f(z)}{I_p(n,\ \lambda)f(z)}-1\right)\right]\prec 1+Mz. \tag{9}$$

Substituting $g(z)=\left(\frac{I_p(n,\ \lambda)f(z)}{z^p}\right)^{\beta}$, $w(z)=\alpha\left(\frac{I_p(n+1,\ \lambda)f(z)}{I_p(n,\ \lambda)f(z)}-1\right)$, M and M_1 as mentioned above. Using Lemma 2.2 (b), we have

$$\left|\frac{I_p(n+1,\ \lambda)f(z)}{I_p(n,\ \lambda)f(z)}-1\right|<r,$$

where r is given by (5). □

For $p=1$ and $\lambda=0$ in Theorem 3.1, we get the following result involving Sălăgean operator:

Theorem 3.2. *Let β be a real number such that $\beta>0$. If $f\in\mathcal{A}$ satisfies*

$$\left|\left(\frac{D^n f(z)}{z}\right)^{\beta}\left[1-\alpha+\alpha\frac{D^{n+1}f(z)}{D^n f(z)}\right]-1\right|<M,\ z\in\mathbb{E},$$

(a) where

$$M=\begin{cases}\dfrac{\alpha(\beta+\alpha)}{\beta(2-\alpha)+\alpha}, & 0<\alpha\leq\sqrt{2\beta+\dfrac{(3\beta-1)^2}{4}}-\dfrac{3\beta-1}{2}\\ \dfrac{(\beta+\alpha)\sqrt{2\alpha-1}}{\sqrt{\alpha^2+2\alpha\beta(1+\beta)}}, & \sqrt{2\beta+\dfrac{(3\beta-1)^2}{4}}-\dfrac{3\beta-1}{2}\leq\alpha\leq 1\end{cases}$$

then

$$\Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)>0,\ z\in\mathbb{E}.$$

(b) If $\left|\frac{D^{n+1}f(z)}{D^n f(z)}-1\right|<r,\ z\in\mathbb{E}$, *where*

$$r=\frac{M[2\beta+\alpha]}{\alpha[\beta(1-M)+\alpha]},\ 0<M\leq\frac{\beta+\alpha}{3\beta+\alpha},$$

then

$$\Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)>0,\ z\in\mathbb{E}.$$

Setting $n=\lambda=0$ in Theorem 3.1, we obtain the following result of Darwish *et al.* [2]:

Corollary 3.3. *Let β be a real number such that $\beta > 0$ and let $f \in \mathcal{A}_p$ satisfy the differential inequality*

$$\left|(1-\alpha)\left(\frac{f(z)}{z^p}\right)^{\beta} + \alpha\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\beta} - 1\right| < M,$$

(a) where

$$M = \begin{cases} \dfrac{\alpha(p\beta+\alpha)}{p\beta(2-\alpha)+\alpha}, & 0 < \alpha \leq \sqrt{2p\beta + \dfrac{(3p\beta-1)^2}{4}} - \dfrac{3p\beta-1}{2} \\ \dfrac{(p\beta+\alpha)\sqrt{2\alpha-1}}{\sqrt{\alpha^2+2p\alpha\beta(1+p\beta)}}, & \sqrt{2p\beta + \dfrac{(3p\beta-1)^2}{4}} - \dfrac{3p\beta-1}{2} \leq \alpha \leq 1 \end{cases}$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \ z \in \mathbb{E}.$$

Hence $f \in \mathcal{S}_p^*$

(b) If $\left|\dfrac{zf'(z)}{pf(z)} - 1\right| < r, \ z \in \mathbb{E},$ *where*

$$r = \frac{M(2\beta p+\alpha)}{\alpha[p\beta(1-M)+\alpha]}, \ 0 < M \leq \frac{p\beta+\alpha}{3p\beta+\alpha},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \ z \in \mathbb{E},$$

i.e. $f \in \mathcal{S}_p^*$.

On taking $\alpha = 1$ in above theorem, from part(a), we get the result of Yang [7] :

Corollary 3.4. *If $f \in \mathcal{A}_p$ satisfies*

(a)

$$\left|\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\beta} - 1\right| < \frac{p\beta+1}{\sqrt{1+2p\beta(p\beta+1)}}$$

then f is $p-$valently starlike.

(b) If $\left|\dfrac{zf'(z)}{f(z)} - 1\right| < \dfrac{M[2\beta p+1]}{\beta p(1-M)+1}, \ z \in \mathbb{E},$ *where* $0 < M \leq \dfrac{\beta p+1}{3\beta p+1},$

then $f \in \mathcal{S}_p^*$.

Taking $\beta = 1$ in Corollary 3.3, we obtain:

Corollary 3.5. *Let α be real number such that $0 < \alpha \leq 1$ and let $f \in \mathcal{A}_p$ satisfy the inequality*

$$\left|(1-\alpha)\frac{f(z)}{z^p} + \alpha\frac{f'(z)}{pz^{p-1}} - 1\right| < M,$$

(a) where

$$M = \begin{cases} \dfrac{\alpha(p+\alpha)}{2p+\alpha(1-p)}, & 0 < \alpha \leq \sqrt{2p + \dfrac{(3p-1)^2}{4}} - \dfrac{3p-1}{2} \\ \dfrac{(p+\alpha)\sqrt{2\alpha-1}}{\sqrt{\alpha^2+2p\alpha(1+p)}}, & \sqrt{2p + \dfrac{(3p-1)^2}{4}} - \dfrac{3p-1}{2} \leq \alpha \leq 1 \end{cases}$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{S}_p^*$.

(b) If $\left|\dfrac{zf'(z)}{pf(z)} - 1\right| < \dfrac{M[2p+\alpha]}{\alpha[p(1-M)+\alpha]},\ z \in \mathbb{E},$ *where* $0 < M \leq \dfrac{p+\alpha}{3p+\alpha}$,

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{S}_p^*$.

Selecting $n = 1,\ \lambda = 0$ in Theorem 3.1, we obtain the following result:

Corollary 3.6. *Let β be a real number such that $\beta > 0$. If $f \in \mathcal{A}_p$ satisfies the differential inequality*

$$\left|(1-\alpha)\left(\frac{f'(z)}{pz^{p-1}}\right)^\beta + \frac{\alpha}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right)\left(\frac{f'(z)}{pz^{p-1}}\right)^\beta - 1\right| < M,$$

(a) where

$$M = \begin{cases} \dfrac{\alpha(p\beta+\alpha)}{p\beta(2-\alpha)+\alpha}, & 0 < \alpha \leq \sqrt{2p\beta + \dfrac{(3p\beta-1)^2}{4}} - \dfrac{3p\beta-1}{2} \\ \dfrac{(p\beta+\alpha)\sqrt{2\alpha-1}}{\sqrt{\alpha^2+2p\alpha\beta(1+p\beta)}}, & \sqrt{2p\beta + \dfrac{(3p\beta-1)^2}{4}} - \dfrac{3p\beta-1}{2} \leq \alpha \leq 1 \end{cases}$$

then

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{K}_p$.

(b) If $\left|\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right| < \frac{M[2\beta p+\alpha]}{\alpha[\beta p(1-M)+\alpha]},\ z \in \mathbb{E},$

where

$$0 < M \leq \frac{\beta p+\alpha}{3\beta p+\alpha},$$

then $f \in \mathcal{K}_p$.

Taking $\beta = 1$ in the above corollary, we have:

Corollary 3.7. *If* $f \in \mathcal{A}_p$ *satisfies*

$$\left|(1-\alpha)\frac{f'(z)}{pz^{p-1}}+\frac{\alpha}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\frac{f'(z)}{pz^{p-1}}-1\right| < M,$$

(a) where

$$M = \begin{cases} \dfrac{\alpha(p+\alpha)}{2p+(1-p)\alpha}, & 0 < \alpha \leq \sqrt{2p+\dfrac{(3p-1)^2}{4}}-\dfrac{3p-1}{2} \\ \dfrac{(p+\alpha)\sqrt{2\alpha-1}}{\sqrt{\alpha^2+2p\alpha(1+p)}}, & \sqrt{2p+\dfrac{(3p-1)^2}{4}}-\dfrac{3p-1}{2} \leq \alpha \leq 1 \end{cases}$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{K}_p$.

(b) If $\left|\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right| < \frac{M[2p+\alpha]}{\alpha[p(1-M)+\alpha]},\ z \in \mathbb{E},$

where

$$0 < M \leq \frac{p+\alpha}{3p+\alpha},$$

then $f \in \mathcal{K}_p$.

Taking $p = 1$ in Corollary 3.3, we get;

Corollary 3.8. *Let* β *be a real number such that* $\beta > 0$ *and let* $f \in \mathcal{A}$ *satisfies*

$$\left|(1-\alpha)\left(\frac{f(z)}{z}\right)^{\beta}+\alpha\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\beta}-1\right| < M,$$

(a) *where*

$$M = \begin{cases} \dfrac{\alpha(\beta+\alpha)}{\beta(2-\alpha)+\alpha}, & 0<\alpha\leq\sqrt{2\beta+\dfrac{(3\beta-1)^2}{4}}-\dfrac{3\beta-1}{2} \\ \dfrac{(\beta+\alpha)\sqrt{2\alpha-1}}{\sqrt{\alpha^2+2\alpha\beta(1+\beta)}}, & \sqrt{2\beta+\dfrac{(3\beta-1)^2}{4}}-\dfrac{3\beta-1}{2}\leq\alpha\leq 1 \end{cases}$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0,\ \ z\in\mathbb{E}.$$

Hence $f\in\mathcal{S}^*$

(b) *If* $\left|\dfrac{zf'(z)}{f(z)}-1\right| < r,\ \ z\in\mathbb{E}$, *where*

$$r = \frac{M(2\beta+\alpha)}{\alpha[\beta(1-M)+\alpha]},\ \ 0<M\leq\frac{\beta+\alpha}{3\beta+\alpha},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0,\ \ z\in\mathbb{E},$$

i.e. $f\in\mathcal{S}^*$.

For $p=1$ in Corollary 3.6, we have:

Corollary 3.9. *Let* β *be a real number such that* $\beta>0$. *If* $f\in\mathcal{A}$ *satisfies the inequality*

$$\left|(f'(z))^\beta\left[1+\alpha\frac{zf''(z)}{f'(z)}\right]-1\right| < M,$$

(a) *where*

$$M = \begin{cases} \dfrac{\alpha(\beta+\alpha)}{\beta(2-\alpha)+\alpha}, & 0<\alpha\leq\sqrt{2\beta+\dfrac{(3\beta-1)^2}{4}}-\dfrac{3\beta-1}{2} \\ \dfrac{(\beta+\alpha)\sqrt{2\alpha-1}}{\sqrt{\alpha^2+2\alpha\beta(1+p\beta)}}, & \sqrt{2\beta+\dfrac{(3\beta-1)^2}{4}}-\dfrac{3\beta-1}{2}\leq\alpha\leq 1 \end{cases}$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > 0,\ \ z\in\mathbb{E},$$

i.e. $f\in\mathcal{K}$.

(b) *If*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{M(2\beta+\alpha)}{\alpha[\beta(1-M)+\alpha]},\ \ \textit{where } 0<M\leq\frac{\beta+\alpha}{3\beta+\alpha},$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0,\ z\in\mathbb{E},$$

i.e. $f\in\mathcal{K}$.

Putting $\lambda=p=1$ and $n=0$ in Theorem 3.1, we get:

Corollary 3.10. *Let β be a real number such that $\beta>0$ and let $f\in\mathcal{A}$ satisfy*

$$\left|\left(1-\frac{\alpha}{2}\right)\left(\frac{f(z)}{z}\right)^{\beta}+\frac{\alpha}{2}\frac{f'(z)(f(z))^{\beta-1}}{z^{\beta-1}}-1\right|<M,$$

where

(a) where

$$M=\begin{cases}\dfrac{\alpha(2\beta+\alpha)}{2\beta(2-\alpha)+\alpha}, & 0<\alpha\leq\sqrt{4\beta+\dfrac{(6\beta-1)^2}{4}}-\dfrac{6\beta-1}{2}\\ \dfrac{(2\beta+\alpha)\sqrt{2\alpha-1}}{\sqrt{\alpha^2+4\alpha\beta(1+2\beta)}}, & \sqrt{4\beta+\dfrac{(6\beta-1)^2}{4}}-\dfrac{6\beta-1}{2}\leq\alpha\leq 1\end{cases}$$

then

$$\Re\left(1+\frac{zf'(z)}{f(z)}\right)>0,\ z\in\mathbb{E}.$$

(b) If $\left|\dfrac{zf'(z)}{f(z)}\right|<r,\ z\in\mathbb{E},$ *where*

$$r=\frac{M[4\beta+\alpha]}{\alpha[2\beta(1-M)+\alpha]},\ 0<M\leq\frac{2\beta+\alpha}{6\beta+\alpha},$$

then

$$\Re\left(1+\frac{zf'(z)}{f(z)}\right)>0,\ z\in\mathbb{E}.$$

4 Open Problem

In the present paper, the parameter α varies from zero to one. The results are still open for the rest range of real number α.

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