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Wavelet-multipliers analysis in the framework of the q-Dunkl theory

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Abstract

Wavelet multipliers have a relatively recent development in pure and applied mathematics. Motivated by Wong's approach, we will study in this paper the two-wavelet multipliers associated with the q-Dunkl transform. Next, under suitable condition on the symbols and two admissible q-wavelets, the boundedness and compactness of these generalized two-wavelet multipliers are presented on the spaces $L^p_{\alpha,q}(\mathbb{R}_q)$, $1 \le p \le \infty$.

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1 Introduction

The q-theory, called also in some literature quantum calculus, began to arise. Interest in this theory is grown at an explosive rate by both physicists and mathematicians due to the large number of its application domains.

Very recently, many authors have been investigating the behavior of the q-theory to several problems already studied for the Fourier Analysis; for instance, sampling theorem [1], Paley-Wiener theorems [2, 3], wavelet transform [8, 22], uncertainty principles [9, 23], wavelet packets [10], Ramanujan master theorem [11], Sobolev spaces [19], Gabor transform [18, 20], localization operators [20, 21, 22], wave equation [25], Fock spaces [26] and so on.

One of the aims of the Fourier Analysis, is the study of the theory of wavelet multipliers. This theory has been initiated by He and Wong in [14], developed in the paper [7] by Du and Wong, and detailed in the book [28] by Wong.

As the q-harmonic analysis has known remarkable development, the natural question to ask whether there exists the equivalent of the theory of wavelet multipliers in the framework of the q-theory.

In our paper, we mainly concern the q-Dunkl transform under the setting of q-Dunkl operator. Our main aim in this paper is to expose and study the boundedness and compactness of two-wavelet multiplier associated with the q-Dunkl transform.

This paper is organized as follows.
In §2 we recall the main results about the harmonic analysis associated with the q-Dunkl operator and the Schatten-von Neumann classes. In §3, we introduce the two-wavelet multipliers in the setting of the q-Dunkl theory, we establish their Schatten Von Neumann properties and we show the trace and the trace class norm inequalities for trace class q-Dunkl two-wavelet multipliers. Finally, in §4, we prove the L^p-boundedness and the L^p-compactness of these q-Dunkl two-wavelet multipliers, under suitable conditions on the symbols and two admissible q-wavelets.

2 Preliminaries

For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper. We refer the reader to the general references [5, 13, 15, 16, 24, 25, 28], for the definitions, notations and properties of the q-shifted factorials, the q-hypergeometric functions, the Jackson's q-derivative, the Jackson's q-integrals, q-Dunkl operator, q-Dunkl kernel, q-Dunkl transform and the Schatten-von Neumann class. Throughout this paper, we assume that $q \in (0,1)$.

2.1 Basic symbols

For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1;\ (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k),\ n = 1,\ 2, ...;\ (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \tag{2.1}$$

and

$$\mathbb{R}_q = \Big\{ \pm q^n\ :\ n \in \mathbb{Z}\Big\},\ \ \widetilde{\mathbb{R}}_q = \Big\{ \pm q^n\ :\ n \in \mathbb{Z}\Big\} \cup \Big\{0\Big\}.$$

We also denote

$$[x]_q = \frac{1-q^x}{1-q},\ x \in \mathbb{C} \tag{2.2}$$

and

$$[n]_q! = \frac{(q;q)_n}{(1-q)^n},\ n \in \mathbb{N}.$$

2.2 Operators and elementary special functions

The q^2-analogue differential operator is given by (see [24, 25]),

$$\partial_q(f)(z) = \begin{cases} \dfrac{f(q^{-1}z)+f(-q^{-1}z)-f(qz)+f(-qz)-2f(-z)}{2(1-q)z}, & \text{if } z \neq 0 \\ \lim_{x\to 0} \partial_q(f)(x) \quad (\text{ in } \quad \mathbb{R}_q) & \text{if } z = 0. \end{cases} \tag{2.3}$$

Note that if f is differentiable at z, then $\lim_{q\to 1} \partial_q(f)(z) = f'(z)$.

The q-Gamma function is given by (see [15])

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad x \neq 0, -1, -2, \ldots.$$

It satisfies the following relations

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad \Gamma_q(1) = 1 \quad \text{and} \quad \lim_{q\to 1^-} \Gamma_q(x) = \Gamma(x),\ \mathcal{R}(x) > 0.$$

The q-trigonometric functions q-cosine and q-sine are defined by (see [24, 25])

$$\cos(x;q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n}}{[2n]_q!}$$

and

$$\sin(x;q^2) == \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n+1}}{[2n+1]_q!}.$$

The q-analogue exponential function is given by

$$e(z;q^2) = \cos(-iz;q^2) + i\sin(-iz;q^2). \tag{2.4}$$

These three functions are absolutely convergent for all z in the plane and when q tends to 1 they tend to the corresponding classical ones pointwise and uniformly on compacts.

Note that we have for all $x \in \mathbb{R}_q$

$$|\cos(x;q^2)| \leq \frac{1}{(q;q)_\infty}, \qquad |\sin(x;q^2)| \leq \frac{1}{(q;q)_\infty},$$

and

$$|e(-ix;q^2)| \leq \frac{2}{(q;q)_\infty}. \tag{2.5}$$

Here, for a function f defined on $\mathbb{R}_q$. The q-Jakson integrals are defined by (see [15, 16])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \quad \int_a^b f(x)d_qx = (1-q)\sum_{n=0}^{\infty} q^n(f(bq^n)-f(aq^n)),$$

$$\int_0^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n),$$

$$\int_{-\infty}^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} \{f(q^n)q^n + f(-q^n)q^n\},$$

provided the sums converge absolutely.

2.3 Sets and spaces

By the use of the q^2-analogue differential operator ∂_q, we note:
• $\mathcal{E}_q(\mathbb{R}_q)$ the space of functions f defined on $\mathbb{R}_q$, satisfying

$$\forall n \in \mathbb{N},\ a \geq 0,\ P_{n,a}(f) = \sup\Big\{|\partial_q^k f(x)|;\ 0 \leq k \leq n, \quad x \in [-a,a] \cap \mathbb{R}_q\Big\} < \infty$$

and

$$\lim_{x\to 0}(\partial_q^n f)(x) \quad (\text{ in } \quad \mathbb{R}_q) \ \text{ exists.}$$

We provide it with the topology defined by the semi norms $P_{n,a}$.
• $\mathcal{S}_q(\mathbb{R}_q)$ the space of function f defined on $\mathbb{R}_q$ satisfying

$$\forall n, m \in \mathbb{N}, \quad P_{n,m,q} = \sup_{x\in\mathbb{R}_q} |x^m \partial_q^n f(x)| < \infty$$

and

$$\lim_{x\to 0}(\partial_q^n f)(x) \ \ (\text{in } \ \mathbb{R}_q) \ \text{ exists.}$$

• $D_q(\mathbb{R}_q)$ the subspace of $\mathcal{S}_q(\mathbb{R}_q)$ constituted of functions with compact supports.
• $L^p_{\alpha,q}(\mathbb{R}_q), 1 \leq p \leq \infty$, the space of functions f on $\mathbb{R}_q$, satisfying

$$\begin{aligned} \|f\|_{L^p_{\alpha,q}(\mathbb{R}_q)} &= \left(\int_{-\infty}^{\infty} |f(x)|^p d_q x\right)^{1/p} < \infty, \quad 1 \leq p < \infty, \\ \|f\|_{L^\infty_\alpha(\mathbb{R}_q)} &= ess \sup_{x\in\mathbb{R}_q} |f(x)| < \infty, \quad p = \infty. \end{aligned}$$

In particular, $L^2_{\alpha,q}(\mathbb{R}_q)$ denotes the Hilbert space with the inner product

$$\langle f, g\rangle_{\alpha,q} = \int_{\mathbb{R}_q} f(x)\overline{g(x)}|x|^{2\alpha+1} d_q x.$$

2.4 Elements of q-Dunkl Harmonic Analysis

In this section, we collect some notations and results on q-Dunkl operator and q-Dunkl transform studied in [5].
For $\alpha \geq \frac{1}{2}$, the q-Dunkl transform is defined on $L^1_{\alpha,q}(\mathbb{R}_q)$ by:

$$\mathcal{F}_D^{\alpha,q}(f)(\lambda) = c_{\alpha,q} \int_{-\infty}^{\infty} f(x)\psi_{-\lambda}^{\alpha,q}(x)|x|^{2\alpha+1} d_q x, \quad \text{for all } \lambda \in \widetilde{\mathbb{R}_q}, \tag{2.6}$$

where $c_{\alpha,q} = \dfrac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)}$ and $\psi_\lambda^{\alpha,q}$ is the q-Dunkl kernel defined by

$$\psi_\lambda^{\alpha,q}(x) = j_\alpha(\lambda x; q^2) + \frac{i\lambda x}{[2\alpha+2]_q} j_{\alpha+1}(\lambda x; q^2), \tag{2.7}$$

with $j_\alpha(x; q^2)$ is the normalized third Jackson's q-Bessel function given by:

$$j_\alpha(x; q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma_{q^2}(\alpha+1) q^{n(n+1)}}{\Gamma_{q^2}(n+1)\Gamma_{q^2}(\alpha+n+1)} \Big(\frac{x}{1+q}\Big)^{2n}.$$

It was proved in [5] that for all $\lambda \in \mathbb{C}$, the function $x \mapsto \psi_\lambda^{\alpha,q}(x)$ is the unique solution of the q-differential-difference equation:

$$\begin{cases} \Lambda_{\alpha,q}(f) = i\lambda f \\ f(0) = 1, \end{cases} \tag{2.8}$$

where $\Lambda_{\alpha,q}$ is the q-Dunkl operator defined by

$$\Lambda_{\alpha,q}(f)(x) = \partial_q[f_e + q^{2\alpha+1} f_o](x) + [2\alpha+1]_q \frac{f(x) - f(-x)}{2x}, \tag{2.9}$$

with f_e and f_o are respectively the even and the odd parts of f.
We recall that the q-Dunkl operator $\Lambda_{\alpha,q}$ lives the spaces $D_q(\mathbb{R}_q)$ and $\mathcal{S}_q(\mathbb{R}_q)$ invariant.

Remark 2.1. *(i) It is easy to see that in the even case $\mathcal{F}_D^{\alpha,q}$ reduces to the q-Bessel transform and in the case $\alpha = \frac{1}{2}$, it reduces to the q^2-analogue Fourier transform.*

(ii) It is worthy to claim that letting $q \uparrow 1$ subject to the condition

$$\frac{\ln(1-q)}{\ln(q)} \in 2\mathbb{Z}, \tag{2.10}$$

$\mathcal{F}_D^{\alpha,q}$ tends, at least formally, the classical Dunkl transform. In the remainder of this paper, we assume that the condition (2.10) holds. (See [5]).

Some other properties of the q-Dunkl kernel and the q-Dunkl transform are given in the following results (see [5]).

Proposition 2.1. *i) For all $\lambda, x \in \mathbb{R}$, $a \in \mathbb{C}$, we have*

$$\psi_\lambda^{\alpha,q}(x) = \psi_x^{\alpha,q}(\lambda),\ \psi_{a\lambda}^{\alpha,q}(x) = \psi_\lambda^{\alpha,q}(ax),\ \overline{\psi_\lambda^{\alpha,q}(x)} = \psi_{-\lambda}^{\alpha,q}(x).$$

ii) If $\alpha = -\frac{1}{2}$, then $\psi_\lambda^{\alpha,q}(x) = e(i\lambda x; q^2)$.

iii) For $\alpha > -\frac{1}{2}$, the q-Dunkl kernel $\psi_\lambda^{\alpha,q}$ has the following q-integral representation of Mehler type

$$\forall\, x \in \mathbb{R}_q, \quad \psi_\lambda^{\alpha,q}(x) = \frac{(1+q)\Gamma_{q^2}(\alpha+1)}{2\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\alpha+\frac{1}{2})} \int_{-1}^{1} \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\alpha+1};q^2)_\infty}(1+t)e(i\lambda xt;q^2)d_qt. \tag{2.11}$$

iv) For all $\lambda \in \mathbb{R}_q$, $\psi_\lambda^{\alpha,q}$ is bounded on $\mathbb{R}_q$ and we have

$$\forall x \in \mathbb{R}_q, \quad |\psi_\lambda^{\alpha,q}(x)| \leq \frac{4}{(q;q)_\infty}. \tag{2.12}$$

v) For all $\lambda \in \mathbb{R}_q$, the function $\psi_\lambda^{\alpha,q}$ belongs to $\mathcal{S}(\mathbb{R}_q)$.

vi) The function $\psi_\lambda^{\alpha,q}$ verifies the following orthogonality relation: For all $x,\ y \in \mathbb{R}_q$

$$\int_{-\infty}^{\infty} \psi_\lambda^{\alpha,q}(x)\overline{\psi_\lambda^{\alpha,q}(y)}|\lambda|^{2\alpha+1}d_q\lambda = \frac{4(1+q)^{2\alpha}\Gamma_{q^2}^2(\alpha+1)}{(1-q)|xy|^{\alpha+1}}\delta_{x,y}. \tag{2.13}$$

vii) If $f \in L^1_{\alpha,q}(\mathbb{R}_q)$ then $\mathcal{F}_D^{\alpha,q}(f) \in L^\infty_q(\mathbb{R}_q)$ and

$$\|\mathcal{F}_D^{\alpha,q}(f)\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \leq \frac{4c_{\alpha,q}}{(q;q)_\infty}\|f\|_{L^1_{\alpha,q}(\mathbb{R}_q)}. \tag{2.14}$$

Moreover

$$\lim_{|\lambda|\to\infty} \mathcal{F}_D^{\alpha,q}(f)(\lambda) = 0,\ \lambda \in \mathbb{R}_q; \quad \lim_{|\lambda|\to 0} \mathcal{F}_D^{\alpha,q}(f)(\lambda) = \mathcal{F}_D^{\alpha,q}(f)(0),\ \lambda \in \widetilde{\mathbb{R}_q}. \tag{2.15}$$

viii) For $f \in L^1_{\alpha,q}(\mathbb{R}_q)$, *we have*

$$\mathcal{F}_D^{\alpha,q}(\Lambda_{\alpha,q} f)(\lambda) = i\lambda \mathcal{F}_D^{\alpha,q}(f)(\lambda). \tag{2.16}$$

ix) For $f,\ g \in L^1_{\alpha,q}(\mathbb{R}_q)$, *we have*

$$\int_{-\infty}^{\infty} \mathcal{F}_D^{\alpha,q}(f)(\lambda) g(\lambda) |\lambda|^{2\alpha+1} d_q\lambda = \int_{-\infty}^{\infty} f(x) \mathcal{F}_D^{\alpha,q}(g)(x) |x|^{2\alpha+1} d_q x. \tag{2.17}$$

Theorem 2.1. *For all* $f \in L^1_{\alpha,q}(\mathbb{R}_q)$, *we have*

$$\forall x \in \mathbb{R}_q,\ f(x) \quad = \quad c_{\alpha,q} \int_{-\infty}^{\infty} \mathcal{F}_D^{\alpha,q}(f)(\lambda) \psi_\lambda^{\alpha,q}(x) |\lambda|^{2\alpha+1} d_q\lambda = \overline{\mathcal{F}_D^{\alpha,q}(\overline{\mathcal{F}_D^{\alpha,q}(f)})}(x). \tag{2.18}$$

Theorem 2.2. *i) Plancherel's formula*
For $\alpha \geq -\frac{1}{2}$, *the q-Dunkl transform* $\mathcal{F}_D^{\alpha,q}$ *is an isomorphism from* $\mathcal{S}_q(\mathbb{R}_q)$ *onto itself. Moreover, for all* $f \in \mathcal{S}_q(\mathbb{R}_q)$, *we have*

$$\|\mathcal{F}_D^{\alpha,q}(f)\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = \|f\|_{L^2_{\alpha,q}(\mathbb{R}_q)}. \tag{2.19}$$

ii) Plancherel's theorem
The q-Dunkl transform can be uniquely extended to an isometric isomorphism on $L^2_{\alpha,q}(\mathbb{R}_q)$. *Its inverse transform* $(\mathcal{F}_D^{\alpha,q})^{-1}$ *is given by :*

$$(\mathcal{F}_D^{\alpha,q})^{-1}(f)(x) = c_{\alpha,q} \int_{-\infty}^{\infty} f(\lambda) \psi_\lambda^{\alpha,q}(x) |\lambda|^{2\alpha+1} d_q\lambda = \mathcal{F}_D^{\alpha,q}(f)(-x). \tag{2.20}$$

Proposition 2.2. *Parseval's formula for* $\mathcal{F}_D^{\alpha,q}$.
For all f *in* $L^2_{\alpha,q}(\mathbb{R}_q)$, *we have*

$$\int_{-\infty}^{\infty} f(\lambda) \overline{g(\lambda)} |\lambda|^{2\alpha+1} d_q\lambda = \int_{-\infty}^{\infty} \mathcal{F}_D^{\alpha,q}(f)(x) \overline{\mathcal{F}_D^{\alpha,q}(g)(x)} |x|^{2\alpha+1} d_q x. \tag{2.21}$$

By using the Riesz-Thorin theorem and relations (2.14), (2.19), we obtain the following Young's inequality:

Proposition 2.3. *Let* f *be in* $L^p_{\alpha,q}(\mathbb{R}_q)$ *and* p *belongs in* $[1, 2]$. *We have*

$$\|\mathcal{F}_D^{\alpha,q}(f)\|_{L^{p'}_{\alpha,q}(\mathbb{R}_q)} \leq \left(\frac{4c_{\alpha,q}}{(q;q)_\infty}\right)^{\frac{2-p}{p}} \|f\|_{L^p_{\alpha,q}(\mathbb{R}_q)}. \tag{2.22}$$

2.5 Schatten-von Neumann classes

Notation. We denote by

- $l^p(\mathbb{N})$ the set of all infinite sequences of real (or complex) numbers $x := (x_j)_{j\in\mathbb{N}}$, such that

$$\begin{aligned} ||x||_p &:= \Big(\sum_{j=1}^{\infty}|x_j|^p\Big)^{\frac{1}{p}} < \infty, \quad \text{if} \quad 1 \leq p < \infty, \\ ||x||_\infty &:= \sup_{j\in\mathbb{N}}|x_j| < \infty. \end{aligned}$$

  For $p = 2$, we provide this space $l^2(\mathbb{N})$ with the scalar product

$$\langle x, y\rangle_2 := \sum_{j=1}^{\infty} x_j\overline{y_j}.$$

- $B(L^p_{\alpha,q}(\mathbb{R}_q))$, $1 \leq p \leq \infty$, the space of bounded operators from $L^p_{\alpha,q}(\mathbb{R}_q)$ into itself.
- $CO(L^p_{\alpha,q}(\mathbb{R}_q))$, $1 \leq p \leq \infty$, the set of all compact operators from $L^p_{\alpha,q}(\mathbb{R}_q)$ into itself.

Definition 2.1. *(i) The singular values $(s_n(A))_{n\in\mathbb{N}}$ of a compact operator A in $B(L^2_{\alpha,q}(\mathbb{R}_q))$ are the eigenvalues of the positive self-adjoint operator $|A| = \sqrt{A^*A}$.*

(ii) For $1 \leq p < \infty$, the Schatten class S_p is the space of all compact operators whose singular values lie in $l^p(\mathbb{N})$. The space S_p is equipped with the norm

$$||A||_{S_p} := \Big(\sum_{n=1}^{\infty}(s_n(A))^p\Big)^{\frac{1}{p}}. \tag{2.23}$$

Remark 2.2. *We note that S_2 is the space of Hilbert-Schmidt operators, whereas S_1 is the space of trace class operators.*

Definition 2.2. *The trace of an operator A in S_1 is defined by*

$$tr(A) = \sum_{n=1}^{\infty}\langle Av_n, v_n\rangle_{\alpha,q} \tag{2.24}$$

where $(v_n)_n$ is any orthonormal basis of $L^2_{\alpha,q}(\mathbb{R}_q)$.

Remark 2.3. *If A is positive, then*

$$tr(A) = ||A||_{S_1}. \tag{2.25}$$

Moreover, a compact operator A on the Hilbert space $L^2_{\alpha,q}(\mathbb{R}_q)$ is Hilbert-Schmidt, if the positive operator A^*A is in the space of trace class S_1. Then

$$||A||^2_{HS} := ||A||^2_{S_2} = ||A^*A||_{S_1} = tr(A^*A) = \sum_{n=1}^{\infty}||Av_n||^2_{L^2_{\alpha,q}(\mathbb{R}_q)} \tag{2.26}$$

for any orthonormal basis $(v_n)_n$ of $L^2_{\alpha,q}(\mathbb{R}_q)$.

Definition 2.3. *We define $S_\infty := B(L^2_{\alpha,q}(\mathbb{R}_q))$, equipped with the norm,*

$$||A||_{S_\infty} := \sup_{v\in L^2_{\alpha,q}(\mathbb{R}_q):||v||_{L^2_{\alpha,q}(\mathbb{R}_q)}=1}||Av||_{L^2_{\alpha,q}(\mathbb{R}_q)}. \tag{2.27}$$

3 q-Dunkl two-wavelet multipliers

3.1 Introduction

Let $\sigma \in L^\infty_{\alpha,q}(\mathbb{R}_q)$, we define the linear operator $M_\sigma : L^2_{\alpha,q}(\mathbb{R}_q) \to L^2_{\alpha,q}(\mathbb{R}_q)$ by

$$M_\sigma(f) = (\mathcal{F}_D^{\alpha,q})^{-1}(\sigma \mathcal{F}_D^{\alpha,q}(f)). \tag{3.1}$$

This operator is called the q-Dunkl multiplier. Moreover, from Plancherel's formula (2.19), it is clair that M_σ is bounded with

$$||M_\sigma||_{S_\infty} \leq ||\sigma||_{L^\infty_{\alpha,q}(\mathbb{R}_q)}.$$

Definition 3.1. *Let u, v, σ be measurable functions on $\mathbb{R}_q$, we define the q-Dunkl two-wavelet multiplier operator noted by $\mathcal{P}_{u,v}(\sigma)$, on $L^p_{\alpha,q}(\mathbb{R}_q)$, $1 \leq p \leq \infty$, by*

$$\mathcal{P}_{u,v}(\sigma)(f)(y) = c_{\alpha,q} \int_{-\infty}^{\infty} \sigma(\xi)\mathcal{F}_D^{\alpha,q}(uf)(\xi)\psi_\xi^{\alpha,q}(y)\overline{v(y)}|\xi|^{2\alpha+1}d_q\xi, \ y \in \mathbb{R}_q. \tag{3.2}$$

In accordance with the different choices of the symbols σ and the different continuities required, we need to impose different conditions on u and v. And then we obtain an operator on $L^p_{\alpha,q}(\mathbb{R}_q)$.

It is often more convenient to interpret the definition of $\mathcal{P}_{u,v}(\sigma)$ in a weak sense, that is, for f in $L^p_{\alpha,q}(\mathbb{R}_q)$, $p \in [1, \infty]$, and g in $L^{p'}_{\alpha,q}(\mathbb{R}_q)$,

$$\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{\alpha,q} = \int_{-\infty}^{\infty} \sigma(\xi)\mathcal{F}_D^{\alpha,q}(uf)(\xi)\overline{\mathcal{F}_D^{\alpha,q}(vg)(\xi)}|\xi|^{2\alpha+1}d_q\xi. \tag{3.3}$$

In this section, we will derive a host of sufficient conditions for the boundedness and Schatten class of q-Dunkl two-wavelet multiplier operator in terms of properties of the symbol and the windows u and v. Our main results for the boundedness and compactness, of $\mathcal{P}_{u,v}(\sigma)$ on $L^2_{\alpha,q}(\mathbb{R}_q)$ are summarized in the following table.

Table 1: Boundedness and compactness of $\mathcal{P}_{u,v}(\sigma)$ on $L^2_{\alpha,q}(\mathbb{R}_q)$

Symbol	Windows		Operator
σ	u	v	$\mathcal{P}_{u,v}(\sigma)$
$L^p_{\alpha,q}(\mathbb{R}_q)$, $p \in [1,\infty]$	$L^2_{\alpha,q}(\mathbb{R}_q) \cap L^\infty_{\alpha,q}(\mathbb{R}_q)$	$L^2_{\alpha,q}(\mathbb{R}_q) \cap L^\infty_{\alpha,q}(\mathbb{R}_q)$	S_∞
$L^p_{\alpha,q}(\mathbb{R}_q)$, $p \in [1,\infty]$	$L^2_{\alpha,q}(\mathbb{R}_q) \cap L^\infty_{\alpha,q}(\mathbb{R}_q)$	$L^2_{\alpha,q}(\mathbb{R}_q) \cap L^\infty_{\alpha,q}(\mathbb{R}_q)$	S_p
$L^p_{\alpha,q}(\mathbb{R}_q)$, $p \in [1,\infty)$	$L^2_{\alpha,q}(\mathbb{R}_q) \cap L^\infty_{\alpha,q}(\mathbb{R}_q)$	$L^2_{\alpha,q}(\mathbb{R}_q) \cap L^\infty_{\alpha,q}(\mathbb{R}_q)$	$CO(L^2_{\alpha,q}(\mathbb{R}_q))$

The first line will be proved in Sec. 3.2. The general condition for membership in the Schatten class S_p will be proved in Sec. 3.3, the compactness result also is proved in Sec. 3.3.

Proposition 3.1. *Let $p \in [1, \infty)$. The adjoint of linear operator*

$$\mathcal{P}_{u,v}(\sigma) : L^p_{\alpha,q}(\mathbb{R}_q) \to L^p_{\alpha,q}(\mathbb{R}_q)$$

is $\mathcal{P}_{v,u}(\overline{\sigma}) : L^{p'}_{\alpha,q}(\mathbb{R}_q) \to L^{p'}_{\alpha,q}(\mathbb{R}_q)$.

Proof. For all f in $L^p_{\alpha,q}(\mathbb{R}_q)$ and g in $L^{p'}_{\alpha,q}(\mathbb{R}_q)$ it follows immediately from (3.3)

$$\begin{aligned}\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{\alpha,q} &= \overline{\int_{-\infty}^{\infty} \sigma(\xi)\mathcal{F}_D^{\alpha,q}(uf)(\xi)\overline{\mathcal{F}_D^{\alpha,q}(vg)(\xi)}|\xi|^{2\alpha+1}d_q\xi} \\ &= \overline{\int_{-\infty}^{\infty} \overline{\sigma(\xi)\mathcal{F}_D^{\alpha,q}(uf)(\xi)}\mathcal{F}_D^{\alpha,q}(vg)(\xi)|\xi|^{2\alpha+1}d_q\xi} \\ &= \overline{\langle \mathcal{P}_{v,u}(\overline{\sigma})(g), f\rangle_{\alpha,q}} = \langle f, \mathcal{P}_{v,u}(\overline{\sigma})(g)\rangle_{\alpha,q}.\end{aligned}$$

Thus we get

$$\mathcal{P}^*_{u,v}(\sigma) = \mathcal{P}_{v,u}(\overline{\sigma}). \tag{3.4}$$

□

Proposition 3.2. *Let $\sigma \in L^1_{\alpha,q}(\mathbb{R}_q) \cup L^\infty_{\alpha,q}(\mathbb{R}_q)$ and let $u, v \in L^2_{\alpha,q}(\mathbb{R}_q) \cap L^\infty_{\alpha,q}(\mathbb{R}_q)$. Then*

$$\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{\alpha,q} = \langle \bar{v}M_\sigma(uf), g\rangle_{\alpha,q}. \tag{3.5}$$

Proof. For all f, g in $L^2_{\alpha,q}(\mathbb{R}_q)$ it follows immediately from (3.3), (3.1) and Parseval's formula (2.21)

$$\begin{aligned}\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{\alpha,q} &= \int_{-\infty}^{\infty} \sigma(\xi)\mathcal{F}_D^{\alpha,q}(uf)(\xi)\overline{\mathcal{F}_D^{\alpha,q}(vg)(\xi)}|\xi|^{2\alpha+1}d_q\xi \\ &= \int_{-\infty}^{\infty} \mathcal{F}_D^{\alpha,q}(M_\sigma(uf))(\xi)\overline{\mathcal{F}_D^{\alpha,q}(vg)(\xi)}|\xi|^{2\alpha+1}d_q\xi \\ &= \int_{-\infty}^{\infty} M_\sigma(uf)(x)\overline{(vg)(x)}|x|^{2\alpha+1}d_qx = \langle \bar{v}M_\sigma(uf), g\rangle_{\alpha,q}.\end{aligned}$$

Thus the proof is complete. □

3.2 Boundedness for $\mathcal{P}_{u,v}(\sigma)$ on S_∞

The main result of this subsection is to prove that the linear operators

$$\mathcal{P}_{u,v}(\sigma) : L^2_{\alpha,q}(\mathbb{R}_q) \to L^2_{\alpha,q}(\mathbb{R}_q)$$

are bounded for all symbol $\sigma \in L^p_{\alpha,q}(\mathbb{R}_q)$, $1 \le p \le \infty$. We first consider this problem for σ in $L^1_{\alpha,q}(\mathbb{R}_q)$ and next in $L^\infty_q(\mathbb{R}_q)$ and then we conclude by using interpolation theory.

In this subsection, u and v will be any functions in $L^2_{\alpha,q}(\mathbb{R}_q) \cap L^\infty_{\alpha,q}(\mathbb{R}_q)$ such that

$$\|u\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = \|v\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = 1.$$

Proposition 3.3. *Let σ be in $L^1_{\alpha,q}(\mathbb{R}_q)$, then the q-Dunkl two-wavelet multiplier $\mathcal{P}_{u,v}(\sigma)$ is in S_∞ and we have*

$$||\mathcal{P}_{u,v}(\sigma)||_{S_\infty} \leqslant \frac{16c^2_{\alpha,q}}{(q;q)^2_\infty}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}. \tag{3.6}$$

Proof. For every functions f and g in $L^2_{\alpha,q}(\mathbb{R}_q)$, from (3.3) we have,

$$\begin{aligned}|\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{\alpha,q}| &\leqslant \int_{-\infty}^{\infty} |\sigma(\xi)|\,|\mathcal{F}_D^{\alpha,q}(uf)(\xi)\overline{\mathcal{F}_D^{\alpha,q}(vg)(\xi)}|d\nu_{\alpha,q}(\xi) \\ &\leqslant \|\mathcal{F}_D^{\alpha,q}(uf)\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|\mathcal{F}_D^{\alpha,q}(vg)\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}.\end{aligned}$$

Using relation (2.14) and the Cauchy-Schwarz inequality, we get

$$\|\mathcal{F}_D^{\alpha,q}(uf)\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \leq \frac{4c_{\alpha,q}}{(q;q)_\infty}\|u\|_{L^2_{\alpha,q}(\mathbb{R}_q)}\|f\|_{L^2_{\alpha,q}(\mathbb{R}_q)}, \quad \|\mathcal{F}_D^{\alpha,q}(vg)\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \leq \frac{4c_{\alpha,q}}{(q;q)_\infty}\|v\|_{L^2_{\alpha,q}(\mathbb{R}_q)}\|g\|_{L^2_{\alpha,q}(\mathbb{R}_q)}.$$

Hence we deduce that

$$|\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{\alpha,q}| \leqslant \frac{16c^2_{\alpha,q}}{(q;q)^2_\infty}\|f\|_{L^2_{\alpha,q}(\mathbb{R}_q)}\|g\|_{L^2_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}.$$

Thus,

$$||\mathcal{P}_{u,v}(\sigma)||_{S_\infty} \leqslant \frac{16c^2_{\alpha,q}}{(q;q)^2_\infty}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}.$$

□

Proposition 3.4. *Let σ be in $L^\infty_{\alpha,q}(\mathbb{R}_q)$, then the q-Dunkl two-wavelet multiplier operator $\mathcal{P}_{u,v}(\sigma)$ is in S_∞ and we have*

$$||\mathcal{P}_{u,v}(\sigma)||_{S_\infty} \leqslant \|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}.$$

Proof. For all functions f and g in $L^2_{\alpha,q}(\mathbb{R}_q)$, we have from Cauchy-Schwarz's inequality

$$\begin{aligned}|\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{\alpha,q}| &\leqslant \int_{-\infty}^{\infty} |\sigma(\xi)||\mathcal{F}_D^{\alpha,q}(uf)(\xi)||\overline{\mathcal{F}_D^{\alpha,q}(vg)(\xi)}||\xi|^{2\alpha+1}d_q\xi \\ &\leqslant \|\sigma\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|\mathcal{F}_D^{\alpha,q}(uf)\|_{L^2_{\alpha,q}(\mathbb{R}_q)}\|\mathcal{F}_D^{\alpha,q}(vg)\|_{L^2_{\alpha,q}(\mathbb{R}_q)}.\end{aligned}$$

Using Plancherel's formula (2.19) we get

$$|\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{\alpha,q}| \leqslant \|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|f\|_{L^2_{\alpha,q}(\mathbb{R}_q)}\|g\|_{L^2_{\alpha,q}(\mathbb{R}_q)}.$$

Thus,

$$||\mathcal{P}_{u,v}(\sigma)||_{S_\infty} \leqslant \|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}.$$

□

We can now associate a q-Dunkl two-wavelet multiplier

$$\mathcal{P}_{u,v}(\sigma) : L^2_{\alpha,q}(\mathbb{R}_q) \to L^2_{\alpha,q}(\mathbb{R}_q)$$

to every symbol σ in $L^p_{\alpha,q}(\mathbb{R}_q)$, $1 \leq p \leq \infty$ and prove that $\mathcal{P}_{u,v}(\sigma)$ is in S_∞. The precise result is the following theorem.

Theorem 3.1. *Let σ be in $L^p_{\alpha,q}(\mathbb{R}_q)$, $1 \leq p \leq \infty$. Then there exists a unique bounded linear operator*

$$\mathcal{P}_{u,v}(\sigma) : L^2_{\alpha,q}(\mathbb{R}_q) \to L^2_{\alpha,q}(\mathbb{R}_q),$$

such that

$$||\mathcal{P}_{u,v}(\sigma)||_{S_\infty} \leqslant \Big(\frac{16c^2_{\alpha,q}}{(q;q)^2_\infty}\Big)^{\frac{1}{p}}\Big(\|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\Big)^{\frac{p-1}{p}}\|\sigma\|_{L^p_{\alpha,q}(\mathbb{R}_q)}.$$

Proof. Let f be in $L^2_{\alpha,q}(\mathbb{R}_q)$. We consider the following operator

$$\mathcal{T} : L^1_{\alpha,q}(\mathbb{R}_q)\bigcap L^\infty_{\alpha,q}(\mathbb{R}_q) \rightarrow L^2_{\alpha,q}(\mathbb{R}_q),$$

given by

$$\mathcal{T}(\sigma) := \mathcal{P}_{u,v}(\sigma)(f).$$

Then by Proposition 3.3 and Proposition 3.4

$$||\mathcal{T}(\sigma)||_{L^2_{\alpha,q}(\mathbb{R}_q)} \leqslant \frac{16c^2_{\alpha,q}}{(q;q)^2_\infty}||f||_{L^2_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)} \tag{3.7}$$

and

$$||\mathcal{T}(\sigma)||_{L^2_{\alpha,q}(\mathbb{R}_q)} \leq \|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}||f||_{L^2_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}. \tag{3.8}$$

Therefore, by (3.7), (3.8) and the Riesz-Thorin interpolation theorem (see [[27], Theorem 2] and [[28], Theorem 2.11]), $\mathcal{T}$ may be uniquely extended to a linear operator on $L^p_{\alpha,q}(\mathbb{R}_q)$, $1 \leq p \leq \infty$ and we have

$$||\mathcal{P}_{u,v}(\sigma)(f)||_{L^2_{\alpha,q}(\mathbb{R}_q)} = ||\mathcal{T}(\sigma)||_{L^2_{\alpha,q}(\mathbb{R}_q)} \leq \Big(\frac{16c^2_{\alpha,q}}{(q;q)^2_\infty}\Big)^{\frac{1}{p}}\Big(\|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\Big)^{\frac{p-1}{p}}||f||_{L^2_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^p_{\alpha,q}(\mathbb{R}_q)}. \tag{3.9}$$

Since (3.9) is true for arbitrary functions f in $L^2_{\alpha,q}(\mathbb{R}_q)$, then we obtain the desired result. □

3.3 Traces of q-Dunkl two-wavelet multipliers

The main result of this subsection is to prove that, the q-Dunkl two-wavelet multiplier

$$\mathcal{P}_{u,v}(\sigma) : L^2_{\alpha,q}(\mathbb{R}_q) \rightarrow L^2_{\alpha,q}(\mathbb{R}_q)$$

is in the Schatten class S_p.

In this subsection, u and v will be any functions in $L^2_{\alpha,q}(\mathbb{R}_q) \cap L^\infty_{\alpha,q}(\mathbb{R}_q)$ such that

$$\|u\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = \|v\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = 1.$$

Proposition 3.5. *Let σ be in $L^1_{\alpha,q}(\mathbb{R}_q)$, then the q-Dunkl two-wavelet multiplier $\mathcal{P}_{u,v}(\sigma)$ is in S_2 and we have*

$$\|\mathcal{P}_{u,v}(\sigma)\|_{S_2} \leqslant \Big(\frac{4c_{\alpha,q}}{(q;q)_\infty}\Big)^2\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}.$$

Proof. Let $\{\phi_j, j = 1, 2...\}$ be an orthonormal basis for $L^2_{\alpha,q}(\mathbb{R}_q)$. Then by (3.3), Fubini's theorem, Parseval's identity and (3.4), we have

$$\begin{aligned}
\sum_{j=1}^\infty ||\mathcal{P}_{u,v}(\sigma)(\phi_j)||^2_{L^2_{\alpha,q}(\mathbb{R}_q)} &= \sum_{j=1}^\infty\langle \mathcal{P}_{u,v}(\sigma)(\phi_j), \mathcal{P}_{u,v}(\sigma)(\phi_j)\rangle_{\alpha,q} \\
&= c^2_{\alpha,q}\sum_{j=1}^\infty\int_{-\infty}^\infty \sigma(\xi)\langle\phi_j, \bar{u}\psi^{\alpha,q}_\xi\rangle_{\alpha,q}\overline{\langle \mathcal{P}_{u,v}(\sigma)(\phi_j), \bar{v}\psi^{\alpha,q}_\xi\rangle}_q|\xi|^{2\alpha+1}d_q\xi \\
&= c^2_{\alpha,q}\int_{-\infty}^\infty \sigma(\xi)\sum_{j=1}^\infty\langle \mathcal{P}^*_{u,v}(\sigma)(\bar{v}\psi^{\alpha,q}_\xi), \phi_j\rangle_{\alpha,q}\langle\phi_j, \bar{u}\psi^{\alpha,q}_\xi\rangle_{\alpha,q}|\xi|^{2\alpha+1}d_q\xi \\
&= c^2_{\alpha,q}\int_{-\infty}^\infty \sigma(\xi)\langle \mathcal{P}^*_{u,v}(\sigma)(\bar{v}\psi^{\alpha,q}_\xi), \bar{u}\psi^{\alpha,q}_\xi\rangle_{\alpha,q}|\xi|^{2\alpha+1}d_q\xi.
\end{aligned}$$

Thus from (3.6) and (2.12), we get

$$\sum_{j=1}^{\infty}||\mathcal{P}_{u,v}(\sigma)(\phi_j)||^2_{L^2_{\alpha,q}(\mathbb{R}_q)} \leq \Big(\frac{4c_{\alpha,q}}{(q;q)_\infty}\Big)^2\int_{-\infty}^{\infty}|\sigma(\xi)|\,||\mathcal{P}^*_{u,v}(\sigma)||_{S_\infty}d\nu_{\alpha,q}(\xi) \leq \Big(\frac{4c_{\alpha,q}}{(q;q)_\infty}\Big)^4\|\sigma\|^2_{L^1_{\alpha,q}(\mathbb{R}_q)} < \infty. \tag{3.10}$$

So, by (3.10) and the Proposition 2.8 in the book [28], by Wong,

$$\mathcal{P}_{u,v}(\sigma) : L^2_{\alpha,q}(\mathbb{R}_q) \to L^2_{\alpha,q}(\mathbb{R}_q)$$

is in the Hilbert-Schmidt class S_2 and hence compact. □

Proposition 3.6. *Let σ be a symbol in $L^p_{\alpha,q}(\mathbb{R}_q)$, $1 \leqslant p < \infty$. Then the q-Dunkl two-wavelet multiplier $\mathcal{P}_{u,v}(\sigma)$ is compact.*

Proof. Let σ be in $L^p_{\alpha,q}(\mathbb{R}_q)$ and let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^1_{\alpha,q}(\mathbb{R}_q)\bigcap L^\infty_{\alpha,q}(\mathbb{R}_q)$ such that $\sigma_n \to \sigma$ in $L^p_{\alpha,q}(\mathbb{R}_q)$ as $n\to\infty$. Then by Theorem 3.1

$$||\mathcal{P}_{u,v}(\sigma_n) - \mathcal{P}_{u,v}(\sigma)||_{S_\infty} \leq \Big(\frac{16c^2_{\alpha,q}}{(q;q)^2_\infty}\Big)^{\frac{1}{p}}(\|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)})^{\frac{p-1}{p}}||\sigma_n-\sigma||_{L^p_{\alpha,q}(\mathbb{R}_q)}.$$

Hence $\mathcal{P}_{u,v}(\sigma_n) \to \mathcal{P}_{u,v}(\sigma)$ in S_∞ as $n\to\infty$. On the other hand as by Proposition 3.5 $\mathcal{P}_{u,v}(\sigma_n)$ is in S_2 hence compact, it follows that $\mathcal{P}_{u,v}(\sigma)$ is compact. □

Theorem 3.2. *Let σ be in $L^1_{\alpha,q}(\mathbb{R}_q)$. Then $\mathcal{P}_{u,v}(\sigma) : L^2_{\alpha,q}(\mathbb{R}_q) \to L^2_{\alpha,q}(\mathbb{R}_q)$ is in S_1 and we have*

$$\frac{2c^2_{\alpha,q}}{\|u\|^2_{L^\infty_{\alpha,q}(\mathbb{R}_q)}+\|v\|^2_{L^\infty_{\alpha,q}(\mathbb{R}_q)}}\|\widetilde{\sigma}\|_{L^1_{\alpha,q}(\mathbb{R}_q)} \leqslant \|\mathcal{P}_{u,v}(\sigma)\|_{S_1} \leqslant \Big(\frac{4c_{\alpha,q}}{(q,q)_\infty}\Big)^2\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}, \tag{3.11}$$

where $\widetilde{\sigma}$ is given by

$$\widetilde{\sigma}(\xi) = \langle\mathcal{P}_{u,v}(\sigma)\,\psi^{\alpha,q}_\xi u,\,\psi^{\alpha,q}_\xi v\rangle_{\alpha,q},\quad \xi\in\mathbb{R}_q.$$

Proof. Since σ is in $L^1_{\alpha,q}(\mathbb{R}_q)$, by Proposition 3.5, $\mathcal{P}_{u,v}(\sigma)$ is in S_2. Using [28, Theorem 2.2], there exists an orthonormal basis $\{\phi_j,\ j=1,2...\}$ for the orthogonal complement of the kernel of the operator $\mathcal{P}_{u,v}(\sigma)$, consisting of eigenvectors of $|\mathcal{P}_{u,v}(\sigma)|$ and $\{u_j,\ j=1,2...\}$ an orthonormal set in $L^2_{\alpha,q}(\mathbb{R}_q)$, such that

$$\mathcal{P}_{u,v}(\sigma)(f) = \sum_{j=1}^{\infty}s_j\langle f,\phi_j\rangle_{\alpha,q}u_j, \tag{3.12}$$

where $s_j,\ j=1,2...$ are the positive singular values of $\mathcal{P}_{u,v}(\sigma)$ corresponding to ϕ_j. Then, we get

$$\|\mathcal{P}_{u,v}(\sigma)\|_{S_1} = \sum_{j=1}^{\infty}s_j = \sum_{j=1}^{\infty}\langle\mathcal{P}_{u,v}(\sigma)(\phi_j),u_j\rangle_{\alpha,q}.$$

Thus, by Fubini's theorem, Parseval's identity, Bessel's inequality, Cauchy-Schwarz's inequality, (2.12), and $||u||_{L^2_{\alpha,q}(\mathbb{R}_q)} = ||v||_{L^2_{\alpha,q}(\mathbb{R}_q)} = 1$, we get

$$\begin{aligned}
\|\mathcal{P}_{u,v}(\sigma)\|_{S_1} &= \sum_{j=1}^{\infty}\langle \mathcal{P}_{u,v}(\sigma)(\phi_j), u_j\rangle_{\alpha,q} \\
&= \sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\sigma(\xi)\mathcal{F}_D^{\alpha,q}(u\phi_j)(\xi)\overline{\mathcal{F}_D^{\alpha,q}(vu_j)(\xi)}|\xi|^{2\alpha+1}d_q\xi \\
&= c_{\alpha,q}^2\int_{-\infty}^{\infty}\sigma(\xi)\sum_{j=1}^{\infty}\langle \phi_j, \bar{u}\psi_\xi^{\alpha,q}\rangle_{\alpha,q}\langle \bar{v}\psi_\xi^{\alpha,q}, u_j\rangle_{\alpha,q}|\xi|^{2\alpha+1}d_q\xi \\
&\leq c_{\alpha,q}^2\int_{-\infty}^{\infty}|\sigma(\xi)|\Big(\sum_{j=1}^{\infty}|\langle \phi_j, \bar{u}\psi_\xi^{\alpha,q}\rangle_{\alpha,q}|\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^{\infty}|\langle \bar{v}\psi_\xi^{\alpha,q}, u_j\rangle_{\alpha,q}|\Big)^{\frac{1}{2}}|\xi|^{2\alpha+1}d_q\xi \\
&\leq c_{\alpha,q}^2\int_{-\infty}^{\infty}|\sigma(\xi)|\,||\bar{u}\psi_\xi^{\alpha,q}||_{L^2_{\alpha,q}(\mathbb{R}_q)}||\bar{v}\psi_\xi^{\alpha,q}||_{L^2_{\alpha,q}(\mathbb{R}_q)}|\xi|^{2\alpha+1}d_q\xi \\
&\leqslant \Big(\tfrac{4c_{\alpha,q}}{(q,q)_\infty}\Big)^2\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}.
\end{aligned}$$

Thus

$$\|\mathcal{P}_{u,v}(\sigma)\|_{S_1} \leqslant \Big(\frac{4c_{\alpha,q}}{(q,q)_\infty}\Big)^2\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}.$$

Now we prove that $\mathcal{P}_{u,v}(\sigma)$ satisfies the first member of (3.11). It is easy to see that $\widetilde{\sigma}$ belongs to $L^1_{\alpha,q}(\mathbb{R}_q)$, and using formula (3.12), we get

$$\begin{aligned}
|\widetilde{\sigma}(\xi)| &= \Big|\langle \mathcal{P}_{u,v}(\sigma)(\,\psi_\xi^{\alpha,q}u),\ \psi_\xi^{\alpha,q}v\rangle_{\alpha,q}\Big| \\
&= \Big|\sum_{j=1}^{\infty}s_j\langle\,\psi_\xi^{\alpha,q}u, \phi_j\rangle_{\alpha,q}\langle u_j,\ \psi_\xi^{\alpha,q}v\rangle_{\alpha,q}\Big| \\
&\leqslant \frac{1}{2}\sum_{j=1}^{\infty}s_j\Big(\Big|\langle\,\psi_\xi^{\alpha,q}u, \phi_j\rangle_{\alpha,q}\Big|^2 + \Big|\langle\,\psi_\xi^{\alpha,q}v, u_j\rangle_{\alpha,q}\Big|^2\Big).
\end{aligned}$$

Then using Plancherel's formula given by relation (2.19) and Fubini's theorem, we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty}|\widetilde{\sigma}(\xi)||\xi|^{2\alpha+1}d_q\xi \leq \frac{1}{2}\sum_{j=1}^{\infty}s_j\Big(&\int_{-\infty}^{\infty}|\langle\,\psi_\xi^{\alpha,q}u, \phi_j\rangle_{\alpha,q}|^2|\xi|^{2\alpha+1}d_q\xi \\
&+\int_{-\infty}^{\infty}|\langle\,\psi_\xi^{\alpha,q}v, u_j\rangle_{\alpha,q}|^2|\xi|^{2\alpha+1}d_q\xi\Big).
\end{aligned}$$

Thus

$$\int_{-\infty}^{\infty}|\widetilde{\sigma}(\xi)||\xi|^{2\alpha+1}d_q\xi \leq \frac{\|u\|^2_{L^\infty_{\alpha,q}(\mathbb{R}_q)} + \|v\|^2_{L^\infty_{\alpha,q}(\mathbb{R}_q)}}{2c_{\alpha,q}^2}\sum_{j=1}^{\infty}s_j = \frac{\|u\|^2_{L^\infty_{\alpha,q}(\mathbb{R}_q)} + \|v\|^2_{L^\infty_{\alpha,q}(\mathbb{R}_q)}}{2c_{\alpha,q}^2}\|\mathcal{P}_{u,v}(\sigma)\|_{S_1}.$$

The proof is complete. □

Corollary 3.1. *For σ in $L^1_{\alpha,q}(\mathbb{R}_q)$, we have the following trace formula*

$$tr(\mathcal{P}_{u,v}(\sigma)) = c^2_{\alpha,q}\int_{-\infty}^{\infty}\sigma(\xi)\langle\bar{v}\psi^{\alpha,q}_{\xi},\bar{u}\psi^{\alpha,q}_{\xi}\rangle_{\alpha,q}|\xi|^{2\alpha+1}d_q\xi. \tag{3.13}$$

Proof. Let $\{\phi_j, j = 1, 2...\}$ be an orthonormal basis for $L^2_{\alpha,q}(\mathbb{R}_q)$. From Theorem 3.2, the q-Dunkl two-wavelet multiplier $\mathcal{P}_{u,v}(\sigma)$ belongs to S_1, then by the definition of the trace given by the relation (2.24), Fubini's theorem and Parseval's identity, we have

$$\begin{aligned}
tr(\mathcal{P}_{u,v}(\sigma)) &= \sum_{j=1}^{\infty}\langle\mathcal{P}_{u,v}(\sigma)(\phi_j),\phi_j\rangle_{\alpha,q}\\
&= c^2_{\alpha,q}\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\sigma(\xi)\langle\phi_j,\,\psi^{\alpha,q}_{\xi}\overline{u}\rangle_{\alpha,q}\overline{\langle\phi_j,\,\psi^{\alpha,q}_{\xi}\overline{v}\rangle}_q|\xi|^{2\alpha+1}d_q\xi\\
&= c^2_{\alpha,q}\int_{-\infty}^{\infty}\sigma(\xi)\sum_{j=1}^{\infty}\langle\phi_j,\,\psi^{\alpha,q}_{\xi}\overline{u}\rangle_{\alpha,q}\big\langle\,\psi^{\alpha,q}_{\xi}\overline{v},\phi_j\big\rangle_q|\xi|^{2\alpha+1}d_q\xi\\
&= c^2_{\alpha,q}\int_{-\infty}^{\infty}\sigma(\xi)\langle\psi^{\alpha,q}_{\xi}\overline{v},\psi^{\alpha,q}_{\xi}\overline{u}\rangle_{\alpha,q}|\xi|^{2\alpha+1}d_q\xi,
\end{aligned}$$

and the proof is complete. □

In the following we give the main result of this subsection.

Corollary 3.2. *Let σ be in $L^p_{\alpha,q}(\mathbb{R}_q)$, $1 \leqslant p \leqslant \infty$. Then, the q-Dunkl two-wavelet multiplier*

$$\mathcal{P}_{u,v}(\sigma) : L^2_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^2_{\alpha,q}(\mathbb{R}_q)$$

is in S_p and we have

$$\|\mathcal{P}_{u,v}(\sigma)\|_{S_p} \leqslant \Big(\frac{4c_{\alpha,q}}{(q,q)_\infty}\Big)^{\frac{2}{p}}(\|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)})^{\frac{p-1}{p}}\|\sigma\|_{L^p_{\alpha,q}(\mathbb{R}_q)}.$$

Proof. The result follows from Proposition 3.4, Theorem 3.2 and by interpolation (See [28, Theorem 2.10 and Theorem 2.11]). □

Remark 3.1. *If $u = v$ and if σ is a real valued and nonnegative function in $L^1_{\alpha,q}(\mathbb{R}_q)$ then $\mathcal{P}_{u,v}(\sigma) : L^2_{\alpha,q}(\mathbb{R}_q) \to L^2_{\alpha,q}(\mathbb{R}_q)$ is a positive operator. So, by (2.25) and Corollary 3.1*

$$||\mathcal{P}_{u,v}(\sigma)||_{S_1} = c^2_{\alpha,q}\int_{-\infty}^{\infty}\sigma(\xi)||\psi^{\alpha,q}_{\xi}u||^2_{L^2_{\alpha,q}(\mathbb{R}_q)}|\xi|^{2\alpha+1}d_q\xi. \tag{3.14}$$

Now we state a result concerning the trace of products of q-Dunkl two-wavelet multipliers.

Corollary 3.3. *Let σ_1 and σ_2 be any real-valued and non-negative functions in $L^1_{\alpha,q}(\mathbb{R}_q)$. We assume that $u = v$ and u is a function in $L^2_{\alpha,q}(\mathbb{R}_q)$ such that $||u||_{L^2_{\alpha,q}(\mathbb{R}_q)} = 1$. Then, the q-Dunkl two-wavelet multipliers $\mathcal{P}_{u,v}(\sigma_1)$, $\mathcal{P}_{u,v}(\sigma_2)$ are positive trace class operators and*

$$\begin{aligned}
\Big|\Big|\Big(\mathcal{P}_{u,v}(\sigma_1)\,\mathcal{P}_{u,v}(\sigma_2)\Big)^n\Big|\Big|_{S_1} &= tr\Big(\mathcal{P}_{u,v}(\sigma_1)\,\mathcal{P}_{u,v}(\sigma_2)\Big)^n\\
&\leq \Big(tr\big(\mathcal{P}_{u,v}(\sigma_1)\big)\Big)^n\Big(tr\big(\mathcal{P}_{u,v}(\sigma_2)\big)\Big)^n\\
&= \Big|\Big|\mathcal{P}_{u,v}(\sigma_1)\Big|\Big|^n_{S_1}\Big|\Big|\mathcal{P}_{u,v}(\sigma_2)\Big|\Big|^n_{S_1},
\end{aligned}$$

for all natural numbers n.

Proof. By Theorem 1 in the paper [17] by Liu we know that if A and B are in the trace class S_1 and are positive operators, then

$$\forall\, n \in \mathbb{N}, \quad tr(AB)^n \leq \Big(tr(A)\Big)^n \Big(tr(B)\Big)^n.$$

So, if we take $A = \mathcal{P}_{u,v}(\sigma_1)$, $B = \mathcal{P}_{u,v}(\sigma_2)$ and we invoke the previous remark, the proof is complete. □

3.4 The Generalized Landau-Pollak-Slepian Operator

Let R and R_1 and R_2 be positive numbers. We define the linear operators

$$Q_R : L^2_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^2_{\alpha,q}(\mathbb{R}_q),\ P_{R_1} : L^2_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^2_{\alpha,q}(\mathbb{R}_q),\ P_{R_2} : L^2_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^2_{\alpha,q}(\mathbb{R}_q),$$

by

$$Q_R f = \chi_{B_{\mathbb{R}_q}(0,R)} f,\ P_{R_1} f = (\mathcal{F}^{\alpha,q}_D)^{-1}(\chi_{B_{\mathbb{R}_q}(0,R_1)} \mathcal{F}^{\alpha,q}_D(f)),\ P_{R_2} f = (\mathcal{F}^{\alpha,q}_D)^{-1}(\chi_{B_{\mathbb{R}_q}(0,R_2)} \mathcal{F}^{\alpha,q}_D(f)),$$

where $\chi_{B_{\mathbb{R}_q}(0,s)}$ is the characteristic function of the set $B_{\mathbb{R}_q}(0,s) := (-s,s) \cap \mathbb{R}_q$.

We adapt the proof of Proposition 20.1 in the book [28] by Wong, we prove the following.

Proposition 3.7. *The linear operators Q_R, P_{R_i} $i = 1,2$, are self-adjoint projections.*

The bounded linear operator $P_{R_2} Q_R P_{R_1} : L^2_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^2_{\alpha,q}(\mathbb{R}_q)$, that it has appeared in the context of time and band-limited signals can be called the generalized Landau-Pollak-Slepian operator. We can show that the generalized Landau-Pollak-Slepian operator is in fact a q-Dunkl two-wavelet multiplier.

Theorem 3.3. *Let u and v be the functions on $\mathbb{R}_q$ defined by*

$$u = \frac{1}{\sqrt{\mu_{\alpha,q}(B_{\mathbb{R}_q}(0,R_1))}} \chi_{B_{\mathbb{R}_q}(0,R_1)}, \quad v = \frac{1}{\sqrt{\mu_{\alpha,q}(B_{\mathbb{R}_q}(0,R_2))}} \chi_{B_{\mathbb{R}_q}(0,R_2)},$$

where

$$\forall\, s > 0, \quad \mu_{\alpha,q}(B_{\mathbb{R}_q}(0,s)) := \int_{B_{\mathbb{R}_q}(0,s)} |x|^{2\alpha+1} d_q x.$$

Then the generalized Landau-Pollak-Slepian operator

$$P_{R_2} Q_R P_{R_1} : L^2_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^2_{\alpha,q}(\mathbb{R}_q)$$

is unitary equivalent to a scalar multiple of the q-Dunkl two-wavelet multiplier

$$v\mathcal{M}_{\chi_{B_{\mathbb{R}_q}(0,R)}} u : L^2_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^2_{\alpha,q}(\mathbb{R}_q).$$

In fact

$$P_{R_2} Q_R P_{R_1} = C_{\alpha,q}(R_1,R_2) (\mathcal{F}^{\alpha,q}_D)^{-1} (v\mathcal{M}_{\chi_{B_{\mathbb{R}_q}(0,R)}} u) \mathcal{F}^{\alpha,q}_D, \tag{3.15}$$

where

$$C_{\alpha,q}(R_1,R_2) := \sqrt{\mu_{\alpha,q}(B_{\mathbb{R}_q}(0,R_1) \mu_{\alpha,q}(B_{\mathbb{R}_q}(0,R_2))}.$$

Proof. It is easy to see that u and v belong to $L^2_{\alpha,q}(\mathbb{R}_q) \cap L^\infty_{\alpha,q}(\mathbb{R}_q)$ and

$$||u||_{L^2_{\alpha,q}(\mathbb{R}_q)} = ||v||_{L^2_{\alpha,q}(\mathbb{R}_q)} = 1.$$

On the other hand we have

$$\langle \mathcal{P}_{u,v}(\chi_{B_{\mathbb{R}_q}(0,R)})(f), g\rangle_{\alpha,q} = \langle vM_{\chi_{B_{\mathbb{R}_q}(0,R)}}(uf), g\rangle_{\alpha,q} = \int_{-\infty}^{\infty} M_{\chi_{B_{\mathbb{R}_q}(0,R)}}(uf)(\xi)\overline{(vg)(\xi)}|\xi|^{2\alpha+1}d_q\xi.$$

By simple calculations we find

$$\begin{aligned}
\langle \mathcal{P}_{u,v}(\chi_{B_{\mathbb{R}_q}(0,R)})(f), g\rangle_{\alpha,q} &= \langle vM_{\chi_{B_{\mathbb{R}_q}(0,R)}}(uf), g\rangle_{\alpha,q} \\
&= \frac{1}{C_{\alpha,q}(R_1,R_2)} \int_{-R}^{R} P_{R_1}((\mathcal{F}_D^{\alpha,q})^{-1}(f))(\xi)\overline{P_{R_2}((\mathcal{F}_D^{\alpha,q})^{-1}(g))(\xi)}|\xi|^{2\alpha+1}d_q\xi \\
&= \frac{1}{C_{\alpha,q}(R_1,R_2)} \int_{-\infty}^{\infty} Q_R P_{R_1}((\mathcal{F}_D^{\alpha,q})^{-1}(f))(\xi)\overline{P_{R_2}((\mathcal{F}_D^{\alpha,q})^{-1}(g))(\xi)}|\xi|^{2\alpha+1}d_q\xi \\
&= \frac{1}{C_{\alpha,q}(R_1,R_2)} \langle Q_R P_{R_1}((\mathcal{F}_D^{\alpha,q})^{-1}(f)), P_{R_2}((\mathcal{F}_D^{\alpha,q})^{-1}(g))\rangle_{\alpha,q} \\
&= \frac{1}{C_{\alpha,q}(R_1,R_2)} \langle P_{R_2} Q_R P_{R_1}((\mathcal{F}_D^{\alpha,q})^{-1}(f)), (\mathcal{F}_D^{\alpha,q})^{-1}(g)\rangle_{\alpha,q} \\
&= \frac{1}{C_{\alpha,q}(R_1,R_2)} \langle \mathcal{F}_D^{\alpha,q} P_{R_2} Q_R P_{R_1}((\mathcal{F}_D^{\alpha,q})^{-1}(f)), g\rangle_{\alpha,q}
\end{aligned}$$

for all f, g in $\mathcal{S}(\mathbb{R}_q)$ and hence the proof is complete. □

The next result gives a formula for the trace of the generalized Landau-Pollak-Slepian operator $P_{R_2}Q_RP_{R_1} : L^2_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^2_{\alpha,q}(\mathbb{R}_q)$.

Corollary 3.4. *We have*

$$tr(P_{R_2}Q_RP_{R_1}) = c^2_{\alpha,q} \int_{B_{\mathbb{R}_q}(0,R)} \int_{B_{\mathbb{R}_q}(0,\min(R_1,R_2))} |\psi_y^{\alpha,q}(\xi)|^2 |y|^{2\alpha+1} d_q y |\xi|^{2\alpha+1} d_q\xi.$$

Proof. The result is an immediate consequence of Theorem 3.3 and Corollary 3.1. □

4 L^p-boundedness and L^p-compactness of $\mathcal{P}_{u,v}(\sigma)$

4.1 Boundedness for symbols in $L^p_{\alpha,q}(\mathbb{R}_q)$

For $1 \leq p \leq \infty$, let $\sigma \in L^1_{\alpha,q}(\mathbb{R}_q)$, $v \in L^p_{\alpha,q}(\mathbb{R}_q)$ and $u \in L^{p'}_{\alpha,q}(\mathbb{R}_q)$.

We are going to show that $\mathcal{P}_{u,v}(\sigma)$ is a bounded operator on $L^p_{\alpha,q}(\mathbb{R}_q)$. Let us start with the following propositions.

Proposition 4.1. *Let σ be in $L^1_{\alpha,q}(\mathbb{R}_q)$, $u \in L^\infty_{\alpha,q}(\mathbb{R}_q)$ and $v \in L^1_{\alpha,q}(\mathbb{R}_q)$, then the q-Dunkl two-wavelet multiplier*

$$\mathcal{P}_{u,v}(\sigma) : L^1_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^1_{\alpha,q}(\mathbb{R}_q)$$

is a bounded linear operator and we have

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^1_{\alpha,q}(\mathbb{R}_q))} \leqslant \frac{16c^2_{\alpha,q}}{(q,q)^2_\infty} \|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|v\|_{L^1_{\alpha,q}(\mathbb{R}_q)} \|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}.$$

Proof. For every function f in $L^1_{\alpha,q}(\mathbb{R}_q)$, we have from the relations (3.2), (2.14) and (2.12)

$$\begin{aligned}||\mathcal{P}_{u,v}(\sigma)(f)||_{L^1_{\alpha,q}(\mathbb{R}_q)} &\leqslant c_{\alpha,q}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\sigma(\xi)|\,|\mathcal{F}_D^{\alpha,q}(uf)(\xi)|\,|\psi_y^{\alpha,q}(\xi)v(y)||\xi|^{2\alpha+1}d_q\xi|y|^{2\alpha+1}d_qy\\ &\leqslant \frac{16c^2_{\alpha,q}}{(q,q)^2_\infty}\|f\|_{L^1_{\alpha,q}(\mathbb{R}_q)}||u||_{L^\infty_{\alpha,q}(\mathbb{R}_q)}||v||_{L^1_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}\end{aligned}$$

Thus,

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^1_{\alpha,q}(\mathbb{R}_q))} \leqslant \frac{16c^2_{\alpha,q}}{(q,q)^2_\infty}||u||_{L^\infty_{\alpha,q}(\mathbb{R}_q)}||v||_{L^1_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}.$$

□

Proposition 4.2. *Let σ be in $L^1_{\alpha,q}(\mathbb{R}_q)$, $u \in L^1_{\alpha,q}(\mathbb{R}_q)$ and $v \in L^\infty_{\alpha,q}(\mathbb{R}_q)$, then the q-Dunkl two-wavelet multipliers*

$$\mathcal{P}_{u,v}(\sigma) : L^\infty_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^\infty_{\alpha,q}(\mathbb{R}_q)$$

is a bounded linear operator and we have

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^\infty_{\alpha,q}(\mathbb{R}_q))} \leqslant \frac{16c^2_{\alpha,q}}{(q,q)^2_\infty}\|u\|_{L^1_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}.$$

Proof. Let f in $L^\infty_{\alpha,q}(\mathbb{R}_q)$. As above from the relations (3.2), (2.14) and (2.12)

$$\begin{aligned}\forall\, y \in \mathbb{R}_q,\quad |\mathcal{P}_{u,v}(\sigma)(f)(y)| &\leqslant c_{\alpha,q}\int_{-\infty}^{\infty}|\sigma(\xi)|\,|\mathcal{F}_D^{\alpha,q}(uf)(\xi)|\,|\psi_y^{\alpha,q}(\xi)v(y)||\xi|^{2\alpha+1}d_q\xi\\ &\leqslant \frac{16c^2_{\alpha,q}}{(q,q)^2_\infty}\|f\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}||u||_{L^1_{\alpha,q}(\mathbb{R}_q)}||v||_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}\end{aligned}$$

Thus,

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^\infty_{\alpha,q}(\mathbb{R}_q))} \leqslant \frac{16c^2_{\alpha,q}}{(q,q)^2_\infty}||u||_{L^1_{\alpha,q}(\mathbb{R}_q)}||v||_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}.$$

□

Remark 4.1. *Proposition 4.2 is also a corollary of Proposition 4.1, since the adjoint of*

$$\mathcal{P}_{v,u}(\overline{\sigma}) : L^1_{\alpha,q}(\mathbb{R}_q) \to L^1_{\alpha,q}(\mathbb{R}_q)$$

is $\mathcal{P}_{u,v}(\sigma) : L^\infty_{\alpha,q}(\mathbb{R}_q) \to L^\infty_{\alpha,q}(\mathbb{R}_q)$.

Using an interpolation of Propositions 4.1 and 4.2, we get the following result.

Theorem 4.1. *Let u and v be functions in $L^1_{\alpha,q}(\mathbb{R}_q)\cap L^\infty_{\alpha,q}(\mathbb{R}_q)$. Then for all σ in $L^1_{\alpha,q}(\mathbb{R}_q)$, there exists a unique bounded linear operator*

$$\mathcal{P}_{u,v}(\sigma) : L^p_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^p_{\alpha,q}(\mathbb{R}_q),\ 1 \leq p \leq \infty,$$

such that

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_{\alpha,q}(\mathbb{R}_q))} \leqslant \frac{16c^2_{\alpha,q}}{(q,q)^2_\infty}\|u\|^{\frac{1}{p'}}_{L^1_{\alpha,q}(\mathbb{R}_q)}\|v\|^{\frac{1}{p}}_{L^1_{\alpha,q}(\mathbb{R}_q)}\|u\|^{\frac{1}{p}}_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|^{\frac{1}{p'}}_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}.$$

We can give another version of the $L^p_{\alpha,q}$-boundedness. Firstly we generalize and we improve Proposition 4.2.

Proposition 4.3. *Let σ be in $L^1_{\alpha,q}(\mathbb{R}_q)$, $v \in L^p_{\alpha,q}(\mathbb{R}_q)$ and $u \in L^{p'}_{\alpha,q}(\mathbb{R}_q)$, for $1 < p \leq \infty$, then the q-Dunkl two-wavelet multiplier*

$$\mathcal{P}_{u,v}(\sigma) : L^p_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^p_{\alpha,q}(\mathbb{R}_q)$$

is a bounded linear operator, and we have

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_{\alpha,q}(\mathbb{R}_q))} \leqslant \frac{16c^2_{\alpha,q}}{(q,q)^2_\infty}\|u\|_{L^{p'}_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^p_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}.$$

Proof. For any $f \in L^p_{\alpha,q}(\mathbb{R}_q)$, consider the linear functional

$$\begin{array}{rcl} \mathcal{I}_f : L^{p'}_{\alpha,q}(\mathbb{R}_q) & \rightarrow & \mathbb{C} \\ g & \mapsto & \langle g, \mathcal{P}_{u,v}(\sigma)(f)\rangle_{\alpha,q}. \end{array}$$

From the relation (3.3)

$$\begin{aligned} |\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{\alpha,q}| &\leqslant \int_{-\infty}^{\infty} |\sigma(\xi)||\mathcal{F}^{\alpha,q}_D(uf)(\xi)|\,|\mathcal{F}^{\alpha,q}_D(vg)(\xi)||\xi|^{2\alpha+1}d_q\xi \\ &\leqslant \|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}\|\mathcal{F}^{\alpha,q}_D(uf)\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|\mathcal{F}^{\alpha,q}_D(vg)\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}. \end{aligned}$$

Using the relation (2.6), (2.12) and Hölder's inequality, we get

$$|\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{\alpha,q}| \leqslant \frac{16c^2_{\alpha,q}}{(q,q)^2_\infty}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}\|u\|_{L^{p'}_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^p_{\alpha,q}(\mathbb{R}_q)}\|f\|_{L^p_{\alpha,q}(\mathbb{R}_q)}\|g\|_{L^{p'}_{\alpha,q}(\mathbb{R}_q)}.$$

Thus, the operator $\mathcal{I}_f$ is a continuous linear functional on $L^{p'}_{\alpha,q}(\mathbb{R}_q)$, and the operator norm

$$||\mathcal{I}_f||_{B(L^{p'}_{\alpha,q}(\mathbb{R}_q))} \leqslant \frac{16c^2_{\alpha,q}}{(q,q)^2_\infty}\|u\|_{L^{p'}_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^p_{\alpha,q}(\mathbb{R}_q)}\|f\|_{L^p_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}.$$

As $\mathcal{I}_f(g) = \langle g, \mathcal{P}_{u,v}(\sigma)(f)\rangle_{\alpha,q}$, by the Riesz representation theorem, we have

$$||\mathcal{P}_{u,v}(\sigma)(f)||_{B(L^p_{\alpha,q}(\mathbb{R}_q))} = ||\mathcal{I}_f||_{B(L^{p'}_{\alpha,q}(\mathbb{R}_q))} \leqslant \frac{16c^2_{\alpha,q}}{(q,q)^2_\infty}\|u\|_{L^{p'}_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^p_{\alpha,q}(\mathbb{R}_q)}\|f\|_{L^p_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)},$$

which establishes the proposition. □

Combining Proposition 4.1 and Proposition 4.3, we have the following theorem.

Theorem 4.2. *Let σ be in $L^1_{\alpha,q}(\mathbb{R}_q)$, $v \in L^p_{\alpha,q}(\mathbb{R}_q)$ and $u \in L^{p'}_{\alpha,q}(\mathbb{R}_q)$, for $1 \leq p \leq \infty$, then the q-Dunkl two-wavelet multiplier*

$$\mathcal{P}_{u,v}(\sigma) : L^p_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^p_{\alpha,q}(\mathbb{R}_q)$$

is a bounded linear operator, and we have

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_{\alpha,q}(\mathbb{R}_q))} \leqslant \frac{16c^2_{\alpha,q}}{(q,q)^2_\infty}\|u\|_{L^{p'}_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^p_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}.$$

With a Schur technique, we can obtain an L^p-boundedness result as in the previous Theorem, but the estimate for the norm $||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_{\alpha,q}(\mathbb{R}_q))}$ is cruder.

Theorem 4.3. *Let σ be in $L^1_{\alpha,q}(\mathbb{R}_q)$, u and v in $L^1_{\alpha,q}(\mathbb{R}_q)\bigcap L^\infty_{\alpha,q}(\mathbb{R}_q)$. Then there exists a unique bounded linear operator*

$$\mathcal{P}_{u,v}(\sigma) : L^p_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^p_{\alpha,q}(\mathbb{R}_q),\ 1 \leq p \leq \infty$$

such that

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_{\alpha,q}(\mathbb{R}_q))} \leqslant \frac{16c^2_{\alpha,q}}{(q,q)^2_\infty} \max(\|u\|_{L^1_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}, \|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^1_{\alpha,q}(\mathbb{R}_q)})\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}.$$

Proof. Let $\mathcal{N}$ be the function defined on $\mathbb{R}_q \times \mathbb{R}_q$ by

$$\mathcal{N}(y,z) = c^2_{\alpha,q}\int_{-\infty}^{\infty} \sigma(\xi)\psi^{\alpha,q}_y(\xi)\overline{v(y)}\psi^{\alpha,q}_{-z}(\xi)u(z)|\xi|^{2\alpha+1}d_q\xi. \tag{4.1}$$

We have

$$\mathcal{P}_{u,v}(\sigma)(f)(y) = \int_{-\infty}^{\infty} \mathcal{N}(y,z)f(z)|z|^{2\alpha+1}d_qz.$$

By simple calculations, it is easy to see that

$$\int_{-\infty}^{\infty} |\mathcal{N}(y,z)||y|^{2\alpha+1}d_qy \leq \frac{16c^2_{\alpha,q}}{(q,q)^2_\infty}\|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^1_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)},\ z \in \mathbb{R}_q,$$

and

$$\int_{-\infty}^{\infty} |\mathcal{N}(y,z)||z|^{2\alpha+1}d_qz \leq \frac{16c^2_{\alpha,q}}{(q,q)^2_\infty}\|u\|_{L^1_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)},\ y \in \mathbb{R}_q.$$

Thus by Schur Lemma (cf. [12]), we can conclude that

$$\mathcal{P}_{u,v}(\sigma) : L^p_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^p_{\alpha,q}(\mathbb{R}_q)$$

is a bounded linear operator for $1 \leq p \leq \infty$, and we have

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_{\alpha,q}(\mathbb{R}_q))} \leqslant \frac{16c^2_{\alpha,q}}{(q,q)^2_\infty} \max(\|u\|_{L^1_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}, \|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{1,\alpha,q})\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)}.$$

□

Remark 4.2. *The previous Theorem tells us that the unique bounded linear operator on $L^p_{\alpha,q}(\mathbb{R}_q)$, $1 \leq p \leq \infty$, obtained by interpolation in Theorem 4.1 is in fact the integral operator on $L^p_{\alpha,q}(\mathbb{R}_q)$ with kernel $\mathcal{N}$ given by (4.1).*

We can now state and prove the main result in this subsection.

Theorem 4.4. *Let σ be in $L^r_{\alpha,q}(\mathbb{R}_q)$, $r \in [1,2]$, and $u, v \in L^1_{\alpha,q}(\mathbb{R}_q)\bigcap L^\infty_{\alpha,q}(\mathbb{R}_q)$ such that $\|u\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = \|v\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = 1$. Then there exists a unique bounded linear operator*

$$\mathcal{P}_{u,v}(\sigma) : L^p_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^p_{\alpha,q}(\mathbb{R}_q), \quad \text{for all } p \in [r, r'],$$

and we have

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_{\alpha,q}(\mathbb{R}_q))} \leqslant C_1^t C_2^{1-t} \|\sigma\|_{L^p_{\alpha,q}(\mathbb{R}_q)} \|u\|_{L^{p'}_{\alpha,q}(\mathbb{R}_q)} \|v\|_{L^p_{\alpha,q}(\mathbb{R}_q)}, \tag{4.2}$$

where

$$\begin{aligned} C_1 &= \Big(\frac{4c_{\alpha,q}}{(q,q)_\infty}\Big)^{\frac{2}{r}} \Big(\|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^1_{\alpha,q}(\mathbb{R}_q)}\Big)^{\frac{2}{r}-1} \Big(\|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\Big)^{\frac{1}{r'}}, \\ C_2 &= \Big(\frac{4c_{\alpha,q}}{(q,q)_\infty}\Big)^{\frac{2}{r}} \Big(\|u\|_{L^1_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\Big)^{\frac{2}{r}-1} \Big(\|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\Big)^{\frac{1}{r'}}, \end{aligned}$$

and

$$\frac{t}{r} + \frac{1-t}{r'} = \frac{1}{p}.$$

Proof. Consider the linear functional

$$\begin{aligned} \mathcal{I} : \Big(L^1_{\alpha,q}(\mathbb{R}_q) \cap L^2_{\alpha,q}(\mathbb{R}_q)\Big) \times \Big(L^1_{\alpha,q}(\mathbb{R}_q) \cap L^2_{\alpha,q}(\mathbb{R}_q)\Big) &\to L^1_{\alpha,q}(\mathbb{R}_q) \cap L^2_{\alpha,q}(\mathbb{R}_q) \\ (\sigma, f) &\mapsto \mathcal{P}_{u,v}(\sigma)(f). \end{aligned}$$

Then by Proposition 4.1 and Theorem 3.1

$$||\mathcal{I}(\sigma,f)||_{L^1_{\alpha,q}(\mathbb{R}_q)} \leqslant \frac{16c^2_{\alpha,q}}{(q,q)^2_\infty} \|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|v\|_{L^1_{\alpha,q}(\mathbb{R}_q)} ||f||_{L^1_{\alpha,q}(\mathbb{R}_q)} \|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)} \tag{4.3}$$

and

$$||\mathcal{I}(\sigma,f)||_{L^2_{\alpha,q}(\mathbb{R}_q)} \leq \frac{4c_{\alpha,q}}{(q,q)_\infty} \sqrt{\|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}} ||f||_{L^2_{\alpha,q}(\mathbb{R}_q)} \|\sigma\|_{L^2_{\alpha,q}(\mathbb{R}_q)}. \tag{4.4}$$

Therefore, by (4.3), (4.4) and the the multi-linear interpolation theory, see Section 10.1 in [6] for reference, we get a unique bounded linear operator

$$\mathcal{I}(\sigma,f) : L^r_{\alpha,q}(\mathbb{R}_q) \times L^r_{\alpha,q}(\mathbb{R}_q) \to L^r_{\alpha,q}(\mathbb{R}_q)$$

such that

$$||\mathcal{I}(\sigma,f)||_{L^r_{\alpha,q}(\mathbb{R}_q)} \leq C_1 ||f||_{L^r_{\alpha,q}(\mathbb{R}_q)} \|\sigma\|_{L^r_{\alpha,q}(\mathbb{R}_q)}, \tag{4.5}$$

where

$$C_1 = \Big(\frac{4c_{\alpha,q}}{(q,q)_\infty}\Big)^{\theta+1} \Big(\|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^1_{\alpha,q}(\mathbb{R}_q)}\Big)^{\theta} \Big(\|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\Big)^{\frac{1-\theta}{2}}$$

and

$$\frac{\theta}{1} + \frac{1-\theta}{2} = \frac{1}{r}.$$

By the definition of $\mathcal{I}$, we have

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^r_{\alpha,q}(\mathbb{R}_q))} \leqslant C_1 \|\sigma\|_{L^r_{\alpha,q}(\mathbb{R}_q)}.$$

As the adjoint of $\mathcal{P}_{u,v}(\sigma)$ is $\mathcal{P}_{v,u}(\overline{\sigma})$, so $\mathcal{P}_{u,v}(\sigma)$ is a bounded linear map on $L^{r'}_{\alpha,q}(\mathbb{R}_q)$ with its operator norm

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^{r'}_{\alpha,q}(\mathbb{R}_q))} = ||\mathcal{P}_{v,u}(\overline{\sigma})||_{B(L^r_{\alpha,q}(\mathbb{R}_q))} \leq C_2 \|\sigma\|_{L^r_{\alpha,q}(\mathbb{R}_q)}, \tag{4.6}$$

where

$$C_2 = \Big(\frac{4c_{\alpha,q}}{(q,q)_\infty}\Big)^{\theta+1}\Big(\|u\|_{L^1_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\Big)^{\theta}\Big(\|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\Big)^{\frac{1-\theta}{2}}.$$

Using an interpolation of (4.5) and (4.6), we have that, for any $p \in [r, r']$,

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_{\alpha,q}(\mathbb{R}_q))} \leqslant C_1^t C_2^{1-t}\|\sigma\|_{L^p_{\alpha,q}(\mathbb{R}_q)}\|u\|_{L^{p'}_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^p_{\alpha,q}(\mathbb{R}_q)},$$

with

$$\frac{t}{r} + \frac{1-t}{r'} = \frac{1}{p}.$$

□

Theorem 4.5. *Let σ be in $L^r_{\alpha,q}(\mathbb{R}_q)$, $r \in [1, 2)$, and $u, v \in L^r_{\alpha,q}(\mathbb{R}_q) \bigcap L^\infty_{\alpha,q}(\mathbb{R}_q)$. Then there exists a unique bounded linear operator*

$$\mathcal{P}_{u,v}(\sigma) : L^p_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^p_{\alpha,q}(\mathbb{R}_q),\ \text{for all } p \in [r, r'],$$

and we have

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_{\alpha,q}(\mathbb{R}_q))} \leqslant (\frac{4c_{\alpha,q}}{(q;q)_\infty})^{\frac{2}{r}}(\|u\|_{L^r_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)})^t(\|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^r_{\alpha,q}(\mathbb{R}_q)})^{1-t}\|\sigma\|_{L^r_{\alpha,q}(\mathbb{R}_q)}, \tag{4.7}$$

where

$$t = \frac{r-p}{p(r-2)}.$$

Proof. For every functions f in $L^{r'}_{\alpha,q}(\mathbb{R}_q)$ and g in $L^r_{\alpha,q}(\mathbb{R}_q)$, from (3.3) we have,

$$\begin{aligned}|\langle\mathcal{P}_{u,v}(\sigma)(f), g\rangle_{L^2_{\alpha,q}(\mathbb{R}_q)}| \quad &\leqslant \int_{-\infty}^{\infty}|\sigma(\xi)|\,|\mathcal{F}_D^{\alpha,q}(uf)(\xi)\overline{\mathcal{F}_D^{\alpha,q}(vg)(\xi)}|d_q\xi\\ &\leqslant \|\mathcal{F}_D^{\alpha,q}(uf)\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|\mathcal{F}_D^{\alpha,q}(vg)\|_{L^{r'}_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^r_{\alpha,q}(\mathbb{R}_q)}.\end{aligned}$$

Using relation (2.14), Hölder's inequality and relation (2.22), we get

$$\|\mathcal{F}_D^{\alpha,q}(uf)\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \leq \frac{4c_{\alpha,q}}{(q;q)_\infty}\|u\|_{L^r_{\alpha,q}(\mathbb{R}_q)}\|f\|_{L^{r'}_{\alpha,q}(\mathbb{R}_q)}$$

and

$$\|\mathcal{F}_D^{\alpha,q}(vg)\|_{L^{r'}_{\alpha,q}(\mathbb{R}_q)} \leq (\frac{4c_{\alpha,q}}{(q;q)_\infty})^{\frac{2-r}{r}}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|g\|_{L^r_{\alpha,q}(\mathbb{R}_q)}.$$

Hence we deduce that

$$|\langle\mathcal{P}_{u,v}(\sigma)(f), g\rangle_{L^2_{\alpha,q}(\mathbb{R}_q)}| \quad \leqslant (\frac{4c_{\alpha,q}}{(q;q)_\infty})^{\frac{2}{r}}\|u\|_{L^r_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|f\|_{L^{r'}_{\alpha,q}(\mathbb{R}_q)}\|g\|_{L^r_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^r_{\alpha,q}(\mathbb{R}_q)}.$$

Thus,

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^{r'}_{\alpha,q}(\mathbb{R}_q))} \leqslant (\frac{4c_{\alpha,q}}{(q;q)_\infty})^{\frac{2}{r}}\|u\|_{L^r_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^r_{\alpha,q}(\mathbb{R}_q)}. \tag{4.8}$$

As the adjoint of $\mathcal{P}_{u,v}(\sigma)$ is $\mathcal{P}_{v,u}(\overline{\sigma})$, so $\mathcal{P}_{u,v}(\sigma)$ is a bounded linear map on $L^r_{\alpha,q}(\mathbb{R}_q)$ with its operator norm

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^r_{\alpha,q}(\mathbb{R}_q))} \leq (\frac{4c_{\alpha,q}}{(q;q)_\infty})^{\frac{2}{r}}\|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|v\|_{L^r_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^r_{\alpha,q}(\mathbb{R}_q)}. \tag{4.9}$$

Using an interpolation of (4.8) and (4.9), we deduce the result. □

Theorem 4.6. *Let σ be in $L^r_{\alpha,q}(\mathbb{R}_q)$, $r \in [1,\infty]$, and $u, v \in L^1_{\alpha,q}(\mathbb{R}_q) \bigcap L^\infty_{\alpha,q}(\mathbb{R}_q)$ such that $\|u\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = \|v\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = 1$. Then there exists a unique bounded linear operator*

$$\mathcal{P}_{u,v}(\sigma) : L^p_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^p_{\alpha,q}(\mathbb{R}_q), \quad \text{for all } p \in [\frac{2r}{r+1}, \frac{2r}{r-1}],$$

and we have

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_{\alpha,q}(\mathbb{R}_q))} \leqslant C_3^{\frac{t}{r}} C_4^{\frac{1-t}{r}} \|u\|^{\frac{1}{r'}}_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|v\|^{\frac{1}{r'}}_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|\sigma\|_{L^r_{\alpha,q}(\mathbb{R}_q)}, \tag{4.10}$$

where

$$\begin{array}{rcl} C_3 & = & \frac{16c^2_{\alpha,q}}{(q;q)^2_\infty} \|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|u\|_{L^1_{\alpha,q}(\mathbb{R}_q)}, \\ C_4 & = & \|v\|_{L^1_{\alpha,q}(\mathbb{R}_q)} \|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}, \end{array}$$

and

$$t = \frac{r+1}{2} - \frac{r}{p}.$$

For prove this theorem we need the following lemmas.

Lemma 4.1. *Let σ be in $L^r_{\alpha,q}(\mathbb{R}_q)$, $r \in [1,\infty]$, $u \in L^2_{\alpha,q}(\mathbb{R}_q) \cap L^\infty_{\alpha,q}(\mathbb{R}_q)$, and v belongs in $L^1_{\alpha,q}(\mathbb{R}_q) \bigcap L^\infty_{\alpha,q}(\mathbb{R}_q)$ such that $\|u\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = \|v\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = 1$. Then there exists a unique bounded linear operator*

$$\mathcal{P}_{u,v}(\sigma) : L^{\frac{2r}{r+1}}_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^{\frac{2r}{r+1}}_{\alpha,q}(\mathbb{R}_q),$$

and we have

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^{\frac{2r}{r+1}}_{\alpha,q}(\mathbb{R}_q))} \leqslant (\frac{16c^2_{\alpha,q}}{(q;q)^2_\infty})^{\frac{1}{r}} \|u\|^{\frac{1}{r}}_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|v\|^{\frac{1}{r}}_{L^1_{\alpha,q}(\mathbb{R}_q)} \|u\|^{\frac{1}{r'}}_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|v\|^{\frac{1}{r'}}_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|\sigma\|_{L^r_{\alpha,q}(\mathbb{R}_q)}. \tag{4.11}$$

Proof. Consider the linear functional

$$\begin{array}{rcl} \mathcal{I} : L^1_{\alpha,q}(\mathbb{R}_q) \cap L^\infty_{\alpha,q}(\mathbb{R}_q) & \to & B(L^1_{\alpha,q}(\mathbb{R}_q) \cap B(L^2_{\alpha,q}(\mathbb{R}_q)) \\ \sigma & \mapsto & \mathcal{P}_{u,v}(\sigma). \end{array}$$

Then by Proposition 4.1 and Theorem 3.1

$$||\mathcal{I}||_{B(L^1_{\alpha,q}(\mathbb{R}_q), B(L^1_{\alpha,q}(\mathbb{R}_q)))} \leqslant \frac{16c^2_{\alpha,q}}{(q;q)^2_\infty} \|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|v\|_{L^1_{\alpha,q}(\mathbb{R}_q)} \tag{4.12}$$

and

$$||\mathcal{I}||_{B(L^\infty_{\alpha,q}(\mathbb{R}_q), B(L^2_{\alpha,q}(\mathbb{R}_q)))} \leqslant \|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}, \tag{4.13}$$

where $||\ ||_{B(L^p_{\alpha,q}(\mathbb{R}_q), B(L^q_{\alpha,q}(\mathbb{R}_q)))}$ denotes the norm in the Banach space of the bounded linear operators from $L^p_{\alpha,q}(\mathbb{R}_q)$ into $B(L^q_{\alpha,q}(\mathbb{R}_q)$, $1 \leq p, q \leq \infty$. Using an interpolation of (4.12) and (4.13) we get the result. □

Lemma 4.2. *Let σ be in $L^r_{\alpha,q}(\mathbb{R}_q)$, $r \in [1,\infty]$, $v \in L^2_{\alpha,q}(\mathbb{R}_q) \cap L^\infty_{\alpha,q}(\mathbb{R}_q)$, and u belongs in $L^1_{\alpha,q}(\mathbb{R}_q) \bigcap L^\infty_{\alpha,q}(\mathbb{R}_q)$ such that $\|u\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = \|v\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = 1$. Then there exists a unique bounded linear operator*

$$\mathcal{P}_{u,v}(\sigma) : L^{\frac{2r}{r-1}}_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^{\frac{2r}{r-1}}_{\alpha,q}(\mathbb{R}_q),$$

and we have

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^{\frac{2r}{r-1}}_{\alpha,q}(\mathbb{R}_q))} \leqslant (\frac{16c^2_{\alpha,q}}{(q;q)^2_\infty})^{\frac{1}{r}} \|v\|^{\frac{1}{r}}_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|u\|^{\frac{1}{r}}_{L^1_{\alpha,q}(\mathbb{R}_q)} \|u\|^{\frac{1}{r'}}_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|v\|^{\frac{1}{r'}}_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|\sigma\|_{L^r_{\alpha,q}(\mathbb{R}_q)}. \tag{4.14}$$

Proof. As the adjoint of $\mathcal{P}_{u,v}(\sigma)$ is $\mathcal{P}_{v,u}(\overline{\sigma})$, so the result follows from duality and previous lemma. □

Proof. of Theorem 4.6. Using an interpolation of (4.11) and (4.14), we have that, for any $p \in [\frac{2r}{r+1}, \frac{2r}{r-1}]$,

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_{\alpha,q}(\mathbb{R}_q))} \leqslant C_3^{\frac{t}{r}} C_4^{\frac{1-t}{r}} \|u\|^{\frac{1}{r'}}_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|v\|^{\frac{1}{r'}}_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|\sigma\|_{L^r_{\alpha,q}(\mathbb{R}_q)},$$

with

$$t = \frac{r+1}{2} - \frac{r}{p}.$$

□

Proposition 4.4. *Let $p, r \in [1,\infty]$ be such that $p \in [\frac{2r}{r+1}, 2]$. Let σ be in $L^r_{\alpha,q}(\mathbb{R}_q)$, u belongs to $L^2_{\alpha,q}(\mathbb{R}_q) \bigcap L^\infty_{\alpha,q}(\mathbb{R}_q)$, and $v \in L^1_{\alpha,q}(\mathbb{R}_q) \bigcap L^\infty_{\alpha,q}(\mathbb{R}_q)$. Then there exists a unique bounded linear operator*

$$\mathcal{P}_{u,v}(\sigma) : L^p_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^p_{\alpha,q}(\mathbb{R}_q),$$

and we have

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_{\alpha,q}(\mathbb{R}_q))} \leqslant C_5^t C_6^{1-t} \|\sigma\|_{L^r_{\alpha,q}(\mathbb{R}_q)}, \tag{4.15}$$

where

$$C_5 = (\|u\|_{L^2_{\alpha,q}(\mathbb{R}_q)} \|v\|_{L^2_{\alpha,q}(\mathbb{R}_q)})^{\frac{1}{s}} (\|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|v\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)})^{\frac{1}{s'}}, \; C_6 = \|u\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|v\|_{L^1_{\alpha,q}(\mathbb{R}_q)}$$

and

$$t = \frac{(r-1)s}{(s-1)r}, \quad \text{with} \quad s = \frac{(2p-2)r}{p-(2-p)r}.$$

Proof. The proof follows from Theorem 4.1 and Theorem 3.1 with $p = 1$, s instead of p, and interpolation theory. □

4.2 L^p-compactness of $\mathcal{P}_{u,v}(\sigma)$

Proposition 4.5. *Under the same hypothesis of Theorem 4.1, the q-Dunkl two-wavelet multiplier*

$$\mathcal{P}_{u,v}(\sigma) : L^1_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^1_{\alpha,q}(\mathbb{R}_q)$$

is compact.

Proof. Let $(f_n)_{n\in\mathbb{N}} \in L^1_{\alpha,q}(\mathbb{R}_q)$ such that $f_n \rightharpoonup 0$ weakly in $L^1_{\alpha,q}(\mathbb{R}_q)$ as $n \to \infty$. It is enough to prove that

$$\lim_{n\to\infty} ||\mathcal{P}_{u,v}(\sigma)(f_n)||_{L^1_{\alpha,q}(\mathbb{R}_q)} = 0.$$

We have

$$||\mathcal{P}_{u,v}(\sigma)(f_n)||_{L^1_{\alpha,q}(\mathbb{R}_q)} \leqslant c^2_{\alpha,q} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\sigma(\xi)|\,|\langle f_n, \psi^{\alpha,q}_{\xi} u\rangle_{\alpha,q}|\,|\psi^{\alpha,q}_{y}(\xi)v(y)||\xi|^{2\alpha+1}d_q\xi|y|^{2\alpha+1}d_qy. \tag{4.16}$$

Now using the fact that $f_n \rightharpoonup 0$ weakly in $L^1_{\alpha,q}(\mathbb{R}_q)$, we deduce that

$$\forall\, \xi, y \in \mathbb{R}_q, \quad \lim_{n\to\infty} |\sigma(\xi)|\,|\langle f_n, \psi^{\alpha,q}_{\xi} u\rangle_{\alpha,q}|\,|\psi^{\alpha,q}_{y}(\xi)v(y)| = 0. \tag{4.17}$$

On the other hand as $f_n \rightharpoonup 0$ weakly in $L^1_{\alpha,q}(\mathbb{R}_q)$ as $n \to \infty$, then there exists a positive constant C such that $||f_n||_{L^1_{\alpha,q}(\mathbb{R}_q)} \leq C$. Hence by simple calculations we get

$$\forall\, \xi, y \in \mathbb{R}_q, \quad |\sigma(\xi)|\,|\langle f_n, \psi^{\alpha,q}_{\xi} u\rangle_{\alpha,q}|\,|\psi^{\alpha,q}_{y}(\xi)v(y)| \leq C\Big(\frac{4}{(q,q)_\infty}\Big)^2 |\sigma(\xi)|\,||u||_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\,|v(y)|. \tag{4.18}$$

Moreover, from Fubini's theorem and relation (2.12), we have

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\sigma(\xi)|\,|\langle f_n, \psi^{\alpha,q}_{\xi} u\rangle_{\alpha,q}|\,|\psi^{\alpha,q}_{y}(\xi)v(y)||\xi|^{2\alpha+1}d_q\xi|y|^{2\alpha+1}d_qy \\ &\leq C\Big(\tfrac{4}{(q,q)_\infty}\Big)^2 ||u||_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \int_{-\infty}^{\infty}|\sigma(\xi)|\int_{-\infty}^{\infty}|v(y)||y|^{2\alpha+1}d_qy|\xi|^{2\alpha+1}d_q\xi \\ &\leq C\Big(\tfrac{4}{(q,q)_\infty}\Big)^2 ||u||_{L^\infty_{\alpha,q}(\mathbb{R}_q)}||v||_{L^1_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^1_{\alpha,q}(\mathbb{R}_q)} < \infty. \end{aligned} \tag{4.19}$$

Thus from the relations (4.16), (4.17), (4.18), (4.19) and the Lebesgue dominated convergence theorem we deduce that

$$\lim_{n\to\infty} ||\mathcal{P}_{u,v}(\sigma)(f_n)||_{L^1_{\alpha,q}(\mathbb{R}_q)} = 0$$

and the proof is complete. □

In the following we give three results for compactness of the q-Dunkl two-wavelet multiplier operators.

Theorem 4.7. *Under the hypothesis of Theorem 4.1, the bounded linear operator*

$$\mathcal{P}_{u,v}(\sigma) : L^p_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^p_{\alpha,q}(\mathbb{R}_q)$$

is compact for $1 \leq p \leq \infty$.

Proof. From the previous proposition, we only need to show that the conclusion holds for $p = \infty$. In fact, the operator

$$\mathcal{P}_{u,v}(\sigma) : L^\infty_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^\infty_{\alpha,q}(\mathbb{R}_q)$$

is the adjoint of the operator $\mathcal{P}_{v,u}(\overline{\sigma}) : L^1_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^1_{\alpha,q}(\mathbb{R}_q)$, which is compact by the previous Proposition. Thus by the duality property, $\mathcal{P}_{u,v}(\sigma) : L^\infty_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^\infty_{\alpha,q}(\mathbb{R}_q)$ is compact. Finally, by an interpolation of the compactness on $L^1_{\alpha,q}(\mathbb{R}_q)$ and on $L^\infty_{\alpha,q}(\mathbb{R}_q)$ such as the one given on pages 202 and 203 of the book [4] by Bennett and Sharpley, the proof is complete. □

The following result is an analogue of Theorem 4.4 for compact operators.

Theorem 4.8. *Under the hypotheses of Theorem 4.4, the bounded linear operator*

$$\mathcal{P}_{u,v}(\sigma) : L^p_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^p_{\alpha,q}(\mathbb{R}_q)$$

is compact for all $p \in [r, r']$.

Proof. The result is an immediate consequence of an interpolation of Corollary 3.2 and Proposition 4.5. See again pages 202 and 203 of the book [4] by Bennett and Sharpley for the interpolation used. □

Remark 4.3. *Using Remark 2.1, we obtain all results of this paper, for the two-wavelet multipliers associated with the q-Bessel transform,* q^2*-analogue Fourier transform and, at least formally, the classical Dunkl transform.*

5 Open Problem

As perspective, involving the ε-concentration of the q-Dunkl two-wavelet multipliers, we will prove an uncertainty principle of Donoho-Stark type for the q-Dunkl transform. Moreover, we will study functions whose time-frequency content are concentrated in a region with finite measure in phase space using the phase space restriction operators as a main tool. We claim to obtain approximation inequalities for such functions using a finite linear combination of eigenfunctions of these operators.

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