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Properties for class of $\beta-$ uniformly univalent functions defined by Salagean type $q-$ difference operator

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Abstract

In this paper, using the Salagean $q-$difference operator, we define a class of $\beta-$uniformly functions and obtain coefficient estimates, modified Hadamard products, family of integral operators and $N_{k,q,\delta}(e,g)$ neighborhood of this generalized function class.

Keywords: *Analytic function, Salagean type $q-$difference, uniformly functions, modified Hadamard products, integral operator, $N_{k,q,\delta}(e,g)$ neighborhood.*

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1 Introduction

Let S be the class of analytic and univalent functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\} \tag{1}$$

and T be the subclass of S consisting of functions of the form

$$f(z)=z-\sum_{k=2}^{\infty}a_kz^k\ (a_k\geq 0; z\in\mathbb{U}). \tag{2}$$

Also let $S^*(\alpha)$ and $C(\alpha)$ denote the subclasses of S which are, respectively, starlike and convex functions of order $\alpha, 0\leq\alpha<1$, satisfying

$$S^*(\alpha)=\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha,\ 0\leq\alpha<1 \tag{3}$$

and

$$C(\alpha)=\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha,\ 0\leq\alpha<1. \tag{4}$$

For convenience, we write $S^*(0)=S^*$ and $C(0)=C$ (see Robertson [19] and Srivastava and Owa [28]).

From (3) and (4) we have

$$f(z)\in C(\alpha)\Longleftrightarrow zf'(z)\in S^*(\alpha).$$

Let

$T^*(\alpha)=S^*(\alpha)\cap T$ and $K(\alpha)=C(\alpha)\cap T$ (see Silverman [27]).

Goodman ([8] and [9]) defined the following subclasses of $S^*(C)$.

Definition 1. A function $f(z)$ is uniformly starlike (convex) in $\mathbb{U}$ if $f(z)$ is in $S^*(C)$ and has the property that for every circular are γ contained in $\mathbb{U}$, with center ζ also in $\mathbb{U}$, the arc $f(\gamma)$ is starlike (convex) with respect to $f(\zeta)$. The classes of uniformly starlike and convex functions are denoted by UST and UCV, respectively (for details see [8] and [9])).

$$f(z)\in UCV\Leftrightarrow\Re\left\{1+(z-\zeta)\frac{f''(z)}{f'(z)}\right\}\geq 0,\ (z,\zeta)\in\mathbb{U}\times\mathbb{U} \tag{5}$$

and

$$f(z)\in UST\Leftrightarrow\Re\left\{\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)}\right\}\geq 0,\ (z,\zeta)\in\mathbb{U}\times\mathbb{U}. \tag{6}$$

It is well known (see [15, 21]) that

$$f(z)\in UCV\Longleftrightarrow\left|\frac{zf''(z)}{f'(z)}\right|\leq\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\},\ z\in\mathbb{U}. \tag{7}$$

In [21], Ronning introduced the new class of starlike functions related to UCV by

$$f(z)\in S_p \Longleftrightarrow \left|\frac{zf^{'}(z)}{f(z)}-1\right| \leq \Re\left\{\frac{zf^{'}(z)}{f(z)}\right\},\ z\in\mathbb{U}. \tag{8}$$

Further Ronning [20], generalized the class S_p by introducing a parameter α by:

Definition 2. [20] A function $f(z)$ of the form (1) is in the class $S_p(\alpha)$ if it satisfies

$$\left|\frac{zf^{'}(z)}{f(z)}-1\right| \leq \Re\left\{\frac{zf^{'}(z)}{f(z)}-\alpha\right\}\ (-1\leq\alpha<1,\ z\in\mathbb{U}) \tag{9}$$

and $f(z)\in UCV(\alpha)$ if and only if $zf^{'}(z)\in S_p(\alpha)$.

By $\beta-UCV$ ($0\leq\beta<\infty$), we denote the class of all $\beta-$uniformly convex functions introduced by Kanas and Wisniowska [13]. Recall that a function $f(z)\in S$ is said to be $\beta-$uniformly convex in $\mathbb{U}$ if the image of every circular *arc* contained in $\mathbb{U}$ with center at ζ, where $|\zeta|\leq\beta$, is convex. Note that the class $1-UCV$ coincides with the class UCV.

It is known that $f(z)\in\beta-UCV$ if and only if it satisfies the following condition

$$\Re\left\{1+\frac{zf^{''}(z)}{f^{'}(z)}\right\} > \beta\left|\frac{zf^{''}(z)}{f^{'}(z)}\right| \quad (z\in\mathbb{U},\ 0\leq\beta<\infty). \tag{10}$$

The class $\beta-UST$ ($0\leq\beta<\infty$), of $\beta-$uniformly starlike functions (see [14]) is associated with $\beta-UCV$ by the relation

$$f(z)\in\beta-UCV \Leftrightarrow\ zf^{'}(z)\in\beta-UST. \tag{11}$$

Thus, the class $\beta-UST$, with $0\leq\beta<\infty$, is ths subclass of S satisfies the following condition:

$$\Re\left\{\frac{zf^{'}(z)}{f(z)}\right\} > \beta\left|\frac{zf^{'}(z)}{f(z)}-1\right| \quad (z\in\mathbb{U},\ 0\leq\beta<\infty). \tag{12}$$

For $f(z)\in S$, Salagean [23] (see also [3]) defined the operator:

$$\begin{aligned} D^1f(z) &= Df(z)=zf^{'}(z), \\ D^nf(z) &= D(D^{n-1}f(z)) \\ &= z+\sum_{k=2}^{\infty}k^n a_k z^k \quad (n\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\ \mathbb{N}=\{1,2,...\}). \end{aligned} \tag{13}$$

For $0<q<1$, the Jackson's $q-$derivative of a function $f(z)\in S$ is given by (see [1, 5, 6, 7, 12, 25, 26])

$$D_q f(z)=\begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & for\quad z\neq 0, \\ f^{'}(0) & for\quad z=0, \end{cases} \tag{14}$$

and $D_q^2 f(z) = D_q(D_q f(z))$. From (14), we have

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q \, a_k z^{k-1}, \tag{15}$$

where

$$[k]_q = \frac{1-q^k}{1-q} \quad (0 < q < 1). \tag{16}$$

If $q \to 1^-$, $[k]_q \to k$. For a function $h(z) = z^k$, we obtain $D_q h(z) = D_q z^k = \frac{1-q^k}{1-q} z^{k-1} = [k]_q \, z^{k-1}$ and $\lim\limits_{q \to 1^-} D_q h(z) = k z^{k-1} = h'(z)$, where h' is the ordinary derivative of h.

Recently for $f \in S$, Govindaraj and Sivasubramanian [11] (see also [18]) defined the Salagean $q-$difference operator by:

$$\begin{aligned} D_q^0 f(z) &= f(z), \\ D_q^1 f(z) &= z D_q f(z), \\ &\vdots \\ D_q^n f(z) &= z D_q(D_q^{n-1} f(z)) \quad (n \in \mathbb{N}) \\ &= z + \sum_{k=2}^{\infty} [k]_q^n \, a_k z^k \quad (n \in \mathbb{N}_0, \ 0 < q < 1, \ z \in \mathbb{U}). \end{aligned} \tag{17}$$

We note that $\lim\limits_{q \to 1^-} D_q^n f(z) = D^n f(z)$, where $D^n f(z)$ is defined by (13).

For $\beta \geq 0$, $-1 \leq \alpha < 1$, $0 < q < 1$ and $n \in \mathbb{N}_0$, denote by $S_q^n(\alpha, \beta)$ the subclass of S satisfying

$$\Re\left\{ \frac{z D_q(D_q^n f(z))}{D_q^n f(z)} - \alpha \right\} > \beta \left| \frac{z D_q(D_q^n f(z))}{D_q^n f(z)} - 1 \right|, \ z \in \mathbb{U}. \tag{18}$$

Let $T_q(n, \alpha, \beta) = S_q^n(\alpha, \beta) \cap T$. We note that

(i) $\lim\limits_{q \to 1^-} T_q(n, \alpha, \beta) = T(n, \alpha, \beta)$ (see Aouf [2]),

(ii) $T_q(0, \alpha, \beta) = T_q(\alpha, \beta) = \left\{ f \in T : \Re\left\{ \frac{z D_q f(z)}{f(z)} - \alpha \right\} > \beta \left| \frac{z D_q f(z)}{f(z)} \right| \right\}$;

(iii) $T_q(1, \alpha, \beta) = C_q(\alpha, \beta) = \left\{ f \in T : \Re\left\{ \frac{z D_q(D_q^1 f(z))}{D_q^1 f(z)} - \alpha \right\} > \beta \left| \frac{z D_q(D_q^1 f(z))}{D_q^1 f(z)} \right| \right\}$;

(iv) $\lim\limits_{q \to 1^-} T_q(\alpha, \beta) = T(\alpha, \beta) = \left\{ f \in T : \Re\left\{ \frac{z f'(z)}{f(z)} - \alpha \right\} > \beta \left| \frac{z f'(z)}{f(z)} - 1 \right| \right\}$;

(v) $\lim\limits_{q \to 1^-} C_q(\alpha, \beta) = C(\alpha, \beta) = \left\{ f \in T : \Re\left\{ 1 + \frac{z f''(z)}{f'(z)} - \alpha \right\} > \beta \left| \frac{z f''(z)}{f'(z)} \right| \right\}$;

(vi) $\lim\limits_{q \to 1^-} T_q(n, \alpha, \beta) = C(n, \alpha, \beta) = \left\{ f \in T : \Re\left\{ 1 + \frac{z (D^n f(z))''}{(D^n f(z))'} - \alpha \right\} > \beta \left| \frac{z (D^n f(z))''}{(D^n f(z))'} \right| \right\}$;

(vii) $T_q(0,\alpha,0) = T_q^*(\alpha) = \Re\left\{\frac{zD_qf(z)}{f(z)}\right\} > \alpha;$
$(viii)$ $T_q(1,\alpha,0) = K_q(\alpha) = \Re\left\{\frac{zD_q(D_qf(z))}{D_qf(z)}\right\} > \alpha;$
(ix) $\lim\limits_{q\to 1^-} T_q^*(\alpha) = T^*(\alpha);$
(x) $\lim\limits_{q\to 1^-} K_q(\alpha) = K(\alpha).$

2 Coefficient estimates

Unless indicated, we assume that $-1 \le \alpha < 1$, $\beta \ge 0$, $0 < q < 1$, $n \in \mathbb{N}_0$, $f(z) \in T$ and $z \in \mathbb{U}$.

Theorem 1. A function $f(z) \in T_q(n,\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty} [k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta)\right] a_k \le 1-\alpha. \tag{19}$$

proof Assume that the inequality (19) holds. Then it is suffices to show that

$$\beta\left|\frac{zD_q(D_q^n f(z))}{D_q^n f(z)} - 1\right| - \Re\left\{\frac{zD_q(D_q^n f(z))}{D_q^n f(z)} - 1\right\} \le 1-\alpha.$$

We have

$$\begin{aligned} & \beta\left|\frac{zD_q(D_q^n f(z))}{D_q^n f(z)} - 1\right| - \Re\left\{\frac{D_q(D_q^n f(z))}{D_q^n f(z)} - 1\right\} \\ \le\ & (1+\beta)\left|\frac{zD_q(zD_q^n f(z))}{D_q^n f(z)} - 1\right| \\ \le\ & \frac{(1+\beta)\sum_{k=2}^{\infty}[k]_q^n\left[[k]_q - 1\right]a_k}{1-\sum_{k=2}^{\infty}[k]_q^n a_k}. \end{aligned}$$

This last expression is bounded above by $(1-\alpha)$ since (19) holds.

Conversely we show that if $f(z) \in T_q(n,\alpha,\beta)$ and z is real, then

$$\frac{1-\sum\limits_{k=2}^{\infty}[k]_q^n([k]_q)a_k z^{k-1}}{1-\sum\limits_{k=2}^{\infty}[k]_q^n a_k z^{k-1}} - \alpha \ge \beta\left|\frac{\sum\limits_{k=2}^{\infty}[k]_q^n([k]_q-1)a_k z^{k-1}}{1-\sum\limits_{k=2}^{\infty}[k]_q^n a_k z^{k-1}}\right|.$$

Letting $z \to 1^-$ along the real axis, we obtain the desired inequality (19).

Hence the proof of Theorem 1 is completed.

Corollary 1. Let the function $f(z) \in T_q(n,\alpha,\beta)$. Then

$$a_k \le \frac{1-\alpha}{[k]_q^n\left[[k]_q(1+\beta) - (\alpha+\beta)\right]} \quad (k \ge 2). \tag{20}$$

The result is sharp for

$$f(z) = z - \frac{1-\alpha}{[k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta)\right]} z^k \quad (k \geq 2). \tag{21}$$

3 Modified Hadamard products

Let the functions $f_j(z)$ be defined, for $j = 1, 2, ..., m$, by

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geq 0, \ z \in \mathbb{U}). \tag{22}$$

For $f_j(z)$ $(j = 1, 2)$ defined by (22), the modified Hadamard product is defined by

$$(f_1 * f_2)(z) = z - \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z). \tag{23}$$

Theorem 2. If $f_j(z)$ $(j = 1, 2)$ defined by (22) are in the class $T_q(n, \alpha, \beta)$, then

$$(f_1 * f_2)(z) \in T_q(n, \gamma(n, \alpha, \beta, q), \beta),$$

where

$$\gamma(n, \alpha, \beta, q) = 1 - \frac{\left([2]_q - 1\right)(1+\beta)(1-\alpha)^2}{[2]_q^n \left[[2]_q (1+\beta) - (\alpha+\beta)\right]^2 - (1-\alpha)^2}. \tag{24}$$

The result is sharp.

proof Employing the techniques used by Schild and Silverman [24], we need to find the largest $\gamma = \gamma(n, \alpha, \beta, q)$ such that

$$\sum_{k=2}^{\infty} \frac{[k]_q^n \left[[k]_q (1+\beta) - (\gamma+\beta)\right]}{1-\gamma} a_{k,1} a_{k,2} \leq 1. \tag{25}$$

Since

$$\sum_{k=2}^{\infty} \frac{[k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta)\right]}{1-\alpha} a_{k,j} \leq 1 \ (j = 1, 2), \tag{26}$$

then Cauchy-Schwarz inequality yields

$$\sum_{k=2}^{\infty} \frac{[k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta)\right]}{1-\alpha} \sqrt{a_{k,1} a_{k,2}} \leq 1. \tag{27}$$

Thus it is sufficient to show that

$$\frac{[k]_q^n\left[[k]_q(1+\beta)-(\gamma+\beta)\right]}{1-\gamma}a_{k,1}a_{k,2}\leq\frac{[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]}{1-\alpha}\sqrt{a_{k,1}a_{k,2}},\tag{28}$$

that is, that

$$\sqrt{a_{k,1}a_{k,2}}\leq\frac{\left[[k]_q(1+\beta)-(\alpha+\beta)\right](1-\gamma)}{\left[[k]_q(1+\beta)-(\gamma+\beta)\right](1-\alpha)}\quad(k\geq2).\tag{29}$$

Note that

$$\sqrt{a_{k,1}a_{k,2}}\leq\frac{(1-\alpha)}{[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]}\quad(k\geq2).\tag{30}$$

Consequently, we need only to prove that

$$\frac{(1-\alpha)}{[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]}\leq\frac{\left[[k]_q(1+\beta)-(\alpha+\beta)\right](1-\gamma)}{\left[[k]_q(1+\beta)-(\gamma+\beta)\right](1-\alpha)}\quad(k\geq2),\tag{31}$$

or, equivalently, that

$$\gamma=1-\frac{\left([k]_q-1\right)(1+\beta)(1-\alpha)^2}{[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]^2-(1-\alpha)^2}\quad(k\geq2).\tag{32}$$

Since

$$\Phi_q(k)=1-\frac{\left([k]_q-1\right)(1+\beta)(1-\alpha)^2}{[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]^2-(1-\alpha)^2},\tag{33}$$

is an increasing function of k $(k\geq2)$, letting $k=2$ in (33), we obtain

$$\gamma\leq\Phi_q(2)=1-\frac{\left([2]_q-1\right)(1+\beta)(1-\alpha)^2}{[2]_q^n\left[[2]_q(1+\beta)-(\alpha+\beta)\right]^2-(1-\alpha)^2},\tag{34}$$

which proves the main assertion of Theorem 2.

Finally, taking $f_j(z)$ $(j=1,2)$ of the form

$$f_j(z)=z-\frac{1-\alpha}{[2]_q^n\left[[2]_q(1+\beta)-(\alpha+\beta)\right]}z^2\quad(j=1,2),\tag{35}$$

we can see that the result is sharp.

Theorem 3. Let the functions $f_1(z)$ and $f_2(z)$ defined by (22) be in the classes $T_q(n,\alpha,\beta)$ and $T_q(n,\rho,\beta)$, respectively. Then

$$(f_1 * f_2)(z) \in T_q(n, \xi(n,\alpha,\rho,\beta,q),\beta),$$

where

$$\xi(n,\alpha,\rho,\beta,q) = 1 - \frac{\left([2]_q-1\right)(1+\beta)(1-\alpha)(1-\rho)}{[2]_q^n\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\left[[2]_q(1+\beta)-(\rho+\beta)\right]-(1-\alpha)(1-\rho)}. \quad (36)$$

The result is the best possible for the functions

$$f_1(z) = z - \frac{1-\alpha}{[2]_q^n\left[[2]_q(1+\beta)-(\alpha+\beta)\right]} z^2, \quad f_2(z) = z - \frac{1-\rho}{[2]_q^n\left[[2]_q(1+\beta)-(\rho+\beta)\right]} z^2. \quad (37)$$

proof Proceeding as in the proof of Theorem 2, we get

$$\xi \le B_q(k) = 1 - \frac{\left([k]_q-1\right)(1+\beta)(1-\alpha)(1-\rho)}{[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[[k]_q(1+\beta)-(\rho+\beta)\right]-(1-\alpha)(1-\rho)}, \ (k \ge 2). \quad (38)$$

Since the function $B_q(k)$ is an increasing function of k $(k \ge 2)$, setting $k=2$ in (38), we get

$$\xi \le B_q(2) = 1 - \frac{\left([2]_q-1\right)(1+\beta)(1-\alpha)(1-\rho)}{[2]_q^n\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\left[[2]_q(1+\beta)-(\rho+\beta)\right]-(1-\alpha)(1-\rho)}. \quad (39)$$

This completes the proof of Theorem 3.

Corollary 2. Let the functions $f_j(z)$ $(j=1,2,3)$ defined by (22) be in the class $T_q(n,\alpha,\beta)$. Then

$$(f_1 * f_2 * f_3)(z) \in T_q(n, \delta(n,\alpha,\beta,q),\beta),$$

where

$$\delta(n,\alpha,\beta,q) = 1 - \frac{\left([2]_q-1\right)^2(1+\beta)^2(1-\alpha)^3}{[2]_q^{2n}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]^3-(1-\alpha)^3}. \quad (40)$$

The result is the best possible for the functions $f_j(z)$ given by (35); $j=1,2,3$.

proof From Theorem 2, we have $(f_1 * f_2)(z) \in T_q(n,\gamma(n,\alpha,\beta,q),\beta)$, where γ is given by (24). By using Theorem 3, we get $(f_1 * f_2 * f_3)(z) \in T_q(n,\delta(n,\alpha,\beta,q),\beta)$, where

$$\begin{aligned} \delta(n,\alpha,\beta,q) &= 1 - \frac{\left([2]_q-1\right)(1+\beta)(1-\alpha)(1-\gamma)}{[2]_q^n\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\left[[2]_q(1+\beta)-(\gamma+\beta)\right]-(1-\alpha)(1-\gamma)} \\ &= 1 - \frac{\left([2]_q-1\right)^2(1+\beta)^2(1-\alpha)^3}{[2]_q^{2n}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]^3-(1-\alpha)^3}. \end{aligned}$$

This completes the proof of Corollary 2.

Theorem 4. Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (22) be in the class $T_q(n, \alpha, \beta)$. Then the function

$$h(z) = z - \sum_{k=2}^{\infty}(a_{k,1}^2 + a_{k,2}^2)z^k, \tag{41}$$

belongs to the class $T_q(n, \tau(n, \alpha, \beta, q), \beta)$, where

$$\tau(n, \alpha, \beta, q) = 1 - \frac{2\big([2]_q - 1\big)(1+\beta)(1-\alpha)^2}{[2]_q^n\big[[2]_q(1+\beta) - (\alpha+\beta)\big]^2 - 2(1-\alpha)^2}. \tag{42}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ defined by (35).

proof By virtue of Theorem 1, we obtain

$$\sum_{k=2}^{\infty}\left[\frac{[k]_q^n\Big[[k]_q(1+\beta) - (\alpha+\beta)\Big]}{1-\alpha}\right]^2 a_{k,j}^2$$
$$\leq \left[\sum_{k=2}^{\infty}\frac{[k]_q^n\Big[[k]_q(1+\beta) - (\alpha+\beta)\Big]}{1-\alpha}a_{k,j}\right]^2 \leq 1 \quad (j = 1, 2), \tag{43}$$

It follows that

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{[k]_q^n\Big[[k]_q(1+\beta) - (\alpha+\beta)\Big]}{1-\alpha}\right]^2\left(a_{k,1}^2 + a_{k,2}^2\right) \leq 1. \tag{44}$$

Therefore, we need to find the largest $\tau = \tau(n, \alpha, \beta, q)$ such that

$$\frac{[k]_q^n\big[[k]_q(1+\beta) - (\tau+\beta)\big]}{1-\tau} \leq \frac{1}{2}\left[\frac{[k]_q^n\big[[k]_q(1+\beta) - (\alpha+\beta)\big]}{1-\alpha}\right]^2 \quad (k \geq 2), \tag{45}$$

that is,

$$\tau \leq 1 - \frac{2\big([k]_q - 1\big)(1+\beta)(1-\alpha)^2}{[k]_q^n\big[[k]_q(1+\beta) - (\alpha+\beta)\big]^2 - 2(1-\alpha)^2} \quad (k \geq 2). \tag{46}$$

Since

$$Q_q(k) = 1 - \frac{2\big([k]_q - 1\big)(1+\beta)(1-\alpha)^2}{[k]_q^n\big[[k]_q(1+\beta) - (\alpha+\beta)\big]^2 - 2(1-\alpha)^2}, \tag{47}$$

is an increasing function of k $(k \geq 2)$, we readily have

$$\tau \leq Q_q(2) = 1 - \frac{2\big([2]_q - 1\big)(1+\beta)(1-\alpha)^2}{[2]_q^n\big[[2]_q(1+\beta) - (\alpha+\beta)\big]^2 - 2(1-\alpha)^2}, \tag{48}$$

and Theorem 9 follows at once.

Theorem 5. Let the function $f_1(z) = z - \sum_{k=2}^{\infty} a_{k,1} z^k$ be in the class $T_q(n, \alpha, \beta)$ and $f_2(z) = z - \sum_{k=2}^{\infty} |a_{k,2}| z^k$ with $|a_{k,2}| \leq 1$, $k = 2, 3, ...$. Then $(f_1 * f_2)(z) \in T_q(n, \alpha, \beta)$.

proof Since

$$\begin{aligned} & \sum_{k=2}^{\infty} [k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta) \right] |a_{k,1} a_{k,2}| \\ = & \sum_{k=2}^{\infty} [k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta) \right] |a_{k,2}| \, a_{k,1} \\ \leq & \sum_{k=2}^{\infty} [k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta) \right] a_{k,1} \\ \leq & \; 1 - \alpha, \end{aligned}$$

by Theorem 1, it follows that $(f_1 * f_2)(z) \in T_q(n, \alpha, \beta)$.

4 A family of integral operators

Theorem 6. Let the function $f(z)$ defined by (2) be in the class $T_q(n, \alpha, \beta)$ and let $c > -1$ be real number. Then the function $F(z)$ defined by

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (c > -1), \tag{49}$$

also belongs to the class $T_q(n, \alpha, \beta)$.

proof It follows from (49) that

$$F(z) = z - \sum_{k=2}^{\infty} b_k z^k, \tag{50}$$

where

$$b_k = \frac{c+1}{c+k} a_k \leq a_k. \tag{51}$$

Therefore,

$$\begin{aligned} & \sum_{k=2}^{\infty} [k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta) \right] b_k \\ = & \sum_{k=2}^{\infty} [k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta) \right] \left(\frac{c+1}{c+k} \right) a_k \\ \leq & \sum_{k=2}^{\infty} [k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta) \right] a_k \leq 1 - \alpha, \end{aligned} \tag{52}$$

since $f(z) \in T_q(n,\alpha,\beta)$. Hence, by Theorem 1, $F(z) \in T_q(n,\alpha,\beta)$.

Theorem 7. Let the function $F(z) = z - \sum_{k=2}^{\infty} a_k z^k \ \ (a_k \geq 0)$, be in the class $T_q(n,\alpha,\beta)$ and $c > -1$ be real number. Then the function $f(z)$ given by (49) is univalent in $|z| < R^*$, where

$$R^* = \inf_k \left[\frac{[k]_q^n \left[[k]_q(1+\beta) - (\alpha+\beta)\right](c+1)}{k(1-\alpha)(c+k)} \right]^{\frac{1}{k-1}} \ (k \geq 2). \tag{53}$$

The result is sharp.

proof From (49), we have

$$f(z) = \frac{z^{1-c}(z^c F(z))'}{c+1} = z - \sum_{k=2}^{\infty} \left(\frac{c+k}{c+1} \right) a_k z^k. \tag{54}$$

In order to obtain the required result, it suffices to show that $\left|f'(z) - 1\right| < 1$ whenever $|z| < R^*$, where R^* is given by (53). Now

$$\left|f'(z) - 1\right| \leq \sum_{k=2}^{\infty} k \left(\frac{c+k}{c+1} \right) a_k |z|^{k-1}.$$

Thus $\left|f'(z) - 1\right| < 1$, if

$$\sum_{k=2}^{\infty} k \left(\frac{c+k}{c+1} \right) a_k |z|^{k-1} < 1. \tag{55}$$

But Theorem 1 confirms that

$$\sum_{k=2}^{\infty} \frac{[k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta)\right] a_k}{1-\alpha} \leq 1. \tag{56}$$

Hence (55) will be satisfied if

$$k \left(\frac{c+k}{c+1} \right) |z|^{k-1} < \frac{[k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta)\right]}{1-\alpha},$$

that is, if

$$|z| < \left[\frac{[k]_q^n \left[[k]_q(1+\beta) - (\alpha+\beta)\right](c+1)}{k(1-\alpha)(c+k)} \right]^{\frac{1}{k-1}} \ (k \geq 2). \tag{57}$$

Therefore the function $f(z)$ given by (49) is univalent in $|z| < R^*$. The sharpness of the result follows if we take

$$f(z) = z - \frac{(1-\alpha)(c+k)}{[k]_q^n \left\{[k]_q(1+\beta) - (\alpha+\beta)\right\}(c+1)} z^k \ \ (k \geq 2;\ c > -1). \tag{58}$$

This completes the proof Theorem 7.

5 Inclusion relations involving $N_{k,q,\delta}(e)$

In this section following the works of Goodman [10] and Ruscheweyh [22], we define the k,δ neighborhood of function $f(z)\in T$ by (see also [3], [4], [16] and [17])

$$N_{k,\delta}(f;g)=\left\{g\in T: g(z)=z-\sum_{k=2}^{\infty}b_k z^k \quad \text{and} \quad \sum_{k=2}^{\infty}k\left|a_k-b_k\right|\le\delta\right\}. \tag{59}$$

In particular, for the identity function $e(z)=z$, we have

$$N_{k,\delta}(e;g)=\left\{g\in T: g(z)=z-\sum_{k=2}^{\infty}b_k z^k \quad \text{and} \quad \sum_{k=2}^{\infty}k\left|b_k\right|\le\delta\right\}. \tag{60}$$

Now we define the k,q,δ neighborhood of function $f(z)\in T$ by

$$N_{k,q,\delta}(f;g)=\left\{g\in T: g(z)=z-\sum_{k=2}^{\infty}b_k z^k \quad \text{and} \quad \sum_{k=2}^{\infty}[k]_q\left|a_k-b_k\right|\le\delta_q\right\}, \tag{61}$$

In particular, for the identity function $e(z)=z$, we have

$$N_{k,q,\delta}(e;g)=\left\{g\in T: g(z)=z-\sum_{k=2}^{\infty}b_k z^k \quad \text{and} \quad \sum_{k=2}^{\infty}[k]_q\left|b_k\right|\le\delta_q\right\}. \tag{62}$$

Theorem 8. Let

$$\delta_q=\frac{[2]_q(1-\alpha)}{[2]_q^n\left[[2]_q(1+\beta)-(\alpha+\beta)\right]}. \tag{63}$$

Then $T_q(n,\alpha,\beta)\subset N_{k,q,\delta}(e)$.

proof For $f\in T_q(n,\alpha,\beta)$, Theorem 1, yields

$$[2]_q^n\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\sum_{k=2}^{\infty}a_k\le 1-\alpha,$$

so that

$$\sum_{k=2}^{\infty}a_k\le\frac{1-\alpha}{[2]_q^n\left[[2]_q(1+\beta)-(\alpha+\beta)\right]}. \tag{64}$$

On the other hand, we also find from (19) and (64) that

$$\begin{aligned}
[2]_q^n (1+\beta) \sum_{k=2}^{\infty} [k]_q a_k &\leq 1-\alpha+[2]_q^n (\alpha+\beta) \sum_{k=2}^{\infty} a_k \\
&\leq 1-\alpha+\frac{[2]_q^n (\alpha+\beta)(1-\alpha)}{[2]_q^n \left[[2]_q (1+\beta)-(\alpha+\beta)\right]} \\
&\leq \frac{[2]_q (1+\beta)(1-\alpha)}{[2]_q (1+\beta)-(\alpha+\beta)} \\
\sum_{k=2}^{\infty} [k]_q a_k &\leq \frac{[2]_q (1-\alpha)}{[2]_q^n \left[[2]_q (1+\beta)-(\alpha+\beta)\right]}, \qquad (65)
\end{aligned}$$

which, in view of the (60), proves Theorem 8.

Now we determine the neighborhood for the class $T_q(n, \alpha, \beta, \xi)$ which we define as follows.

A function $f \in T$ is said to be in the class $T_q(n, \alpha, \beta, \xi)$ if there exists a function $g \in T_q(n, \alpha, \beta)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \xi_q \quad (z \in \mathbb{U},\ 0 \leq \xi_q < 1). \qquad (66)$$

Theorem 9. If $g \in T_q(n, \alpha, \beta)$ and

$$\xi_q = 1 - \frac{\delta_q [2]_q^n \left[[2]_q (1+\beta)-(\alpha+\beta)\right]}{2\left\{[2]_q^n \left[[2]_q (1+\beta)-(\alpha+\beta)\right]-(1-\alpha)\right\}}. \qquad (67)$$

Then $N_{k,q,\delta}(g) \subset T_q(n, \alpha, \beta, \xi)$.

proof Suppose that $f \in N_{k,q,\delta}(g)$ then we find from (65) that

$$\sum_{k=2}^{\infty} [k]_q |a_k - b_k| \leq \delta_q,$$

which implies that the coefficient inequality

$$\sum_{k=2}^{\infty} |a_k - b_k| \leq \frac{\delta_q}{2}.$$

Next, since $g \in T_q(n, \alpha, \beta)$, we have

$$\sum_{k=2}^{\infty} b_k \leq \frac{1-\alpha}{[2]_q^n \left[[2]_q (1+\beta)-(\alpha+\beta)\right]},$$

so that

$$\begin{aligned}\left|\frac{f(z)}{g(z)}-1\right| &< \frac{\sum\limits_{k=2}^{\infty}|a_k-b_k|}{1-\sum\limits_{k=2}^{\infty}b_k} \\ &\leq \frac{\delta_q}{2}\times\frac{[2]_q^n\left[[2]_q(1+\beta)-(\alpha+\beta)\right]}{[2]_q^n\left[[2]_q(1+\beta)-(\alpha+\beta)\right]-(1-\alpha)} \\ &\leq 1-\xi_q.\end{aligned}$$

Provided that ξ_q is given precisely by (67). Thus, by definition, $f\in T_q(n,\alpha,\beta,\xi)$ for ξ_q given by (67), which completes the proof of Theorem 9.

Remark 1.

For different values of n, q, α and β in our results, we have results for the special classes defined in the introduction.

Remark 2.

Theorems 8 and 9 correct the results of [29, Theorems 3.1 and 3.2].

6 Open problem

The authors suggest to study the quasi Hadamard products of functions in this class.

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