Certain Subclasses of Harmonic Multivalent Functions Involving an Integral Operator

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Abstract

For $p \geq 1$ ($p \in \mathbb{N} := \{1, 2, 3, \cdots \}$), let $\mathcal{H}_p$ denote the set of all normalized multivalent harmonic functions $f$, sense preserving in the open unit disk $\Delta$. Let $S_{p,l}^{n,\lambda}(\alpha, \beta), (0 \leq \alpha < p; l, \lambda, \beta \geq 0; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ denote the family of multivalent harmonic functions satisfying:

$$\Re \left\{ (1 - \beta) \frac{\Theta_{p,\lambda}^{n,l} f(z)}{z^p} + \beta \frac{(\Theta_{p,\lambda}^{n,l} f(z))'}{p z^{p-1}} \right\} > \frac{\alpha}{p}; \ (z \in \Delta),$$

where the operator $\tilde{\Theta}_{p,\lambda}^{n,l} f(z) : \mathcal{H}_p \to \mathcal{H}_p$ is introduced as a generalized Catas operator (cf.[12]). In the present paper various basic properties such as coefficient conditions, growth theorem, covering theorem, extreme points, inclusion relations, neighborhood property
and radius of starlikeness are discussed for the subclass $SH^{n,l}_{p,\lambda}(\alpha, \beta)$ of $SH^{n,l}_{p,\lambda}(\alpha, \beta)$. Apart from this, we also show that the newly defined subclasses are closed under convolution, convex combination and invariant under generalized Bernardi-Libera-Livingston operator.

Keywords: Harmonic multivalent functions, Coefficient estimates, Distortion inequality, convolution, convex combination, integral operator, neighborhood, radius of starlikeness.

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1 Introduction and Preliminaries

Let $S_H$ be the class of all complex valued, harmonic, orientation-preserving, univalent functions $f$ in the open unit disk $\Delta$ normalized by $f(0) = f_z(0) = 0$. Clunie and Sheil-Small [14] studied the family $S_H$ together with some geometric subclasses of $S_H$. They proved that although $S_H$ is not compact, it is normal with respect to the topology of uniform convergence on compact subsets of $\Delta$. Moreover, the subclass $S^0_H$ of $S_H$ consisting of the functions having the property $f_z(0) = 0$ is compact [19]. This pioneering work of Clunie and Sheil-Small on harmonic univalent mappings gave rise to the birth of theory of harmonic univalent mappings. This theory has recently attracted the function theorists to look at harmonic analogues of the theory of analytic univalent or multivalent functions in the open unit disk. For systematic study we refer [1], [20], [21], [24], [25], and [28] (also see recent books [10], [11] and [19]).

For $p \geq 1$ ($p \in \mathbb{N} := \{1, 2, 3, \cdots \}$), we denote by $A_p$ the class of all functions which are analytic multivalent in $\Delta$ and $H_p$, the set of all multivalent harmonic functions $f = h + \overline{g}$, where $h$ and $g$ are of the form:

$$h(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad g(z) = \sum_{k=p}^{\infty} b_k z^k,$$

and $|b_p| < 1$, \hspace{1cm} (1)

which are analytic multivalent harmonic functions and sense preserving in the open unit disk $\Delta$ normalized by the conditions $h(0) = h'(0) = \cdots = h^{(p-1)}(0) = h^{(p)}(0) - p! = 0$ and $g(0) = g'(0) = \cdots = g^{p-1}(0) = 0$. The function $h$ is called analytic part of $f$ and $g$ is called the co analytic part of $f$. The above defined class $H_p$ has been introduced and studied extensively by Jahangiri and Ahuja [2](also see [22]). It is well known that, every analytic function is complex valued harmonic. Thus $A_p$ is a subset of $H_p$.

Following the work of Silverman [29], several subclasses of analytic functions with negative coefficients have been introduced and studied (cf.[7], [15],
Certain subclasses of harmonic multivalent functions. In this work, we introduce following new subclasses of multivalent harmonic functions associated with generalized multiplier transformation.

**Definition 1.1** For \( f = h + \bar{g} \) given by (1), let \( SH_{p,\lambda}^{n,l}(\alpha, \beta) \) denote the family of multivalent harmonic functions satisfying the following:

\[
\Re \left\{ (1 - \beta) \frac{\Theta_{p,\lambda}^{n,l}f(z)}{zp} + \beta \frac{(\Theta_{p,\lambda}^{n,l}f(z))'}{pz^{p-1}} \right\} > \frac{\alpha}{p};
\]

\((z \in \Delta; 0 \leq \alpha < p; n \in \mathbb{N}_0; l, \lambda, \beta \geq 0)\),

where the operator \( \Theta_{p,\lambda}^{n,l}f(z) : \mathcal{H}_p \to \mathcal{H}_p \) is defined by

\[
\Theta_{p,\lambda}^{n,l}f(z) = \Theta_{p,\lambda}^{n,l}h(z) + (-1)^n \Theta_{p,\lambda}^{n,l}g(z),
\]

\[
\Theta_{p,\lambda}^{n,l}h(z) = z^p + \sum_{k=p+1}^{\infty} \left( \frac{p+l+\lambda(k-p)}{p+l} \right)^n a_k z^k
\]

and

\[
\Theta_{p,\lambda}^{n,l}g(z) = \sum_{k=p}^{\infty} \left( \frac{p+l+\lambda(k-p)}{p+l} \right)^n b_k z^k.
\]

Unless otherwise mentioned through out this paper we shall assume that \( p, n \in \mathbb{N}_0 \) are fixed, \( 0 \leq \alpha < p \) and \( l, \lambda, \beta \geq 0 \).

It is observed that the operator \( \Theta_{p,\lambda}^{n,l} \) given in (3) unifies various classical and recently studied operators which have significant applications in Geometric function theory, for example:

For \( p = 1 \), \( \Theta_{p,\lambda}^{n,l} \) reduces to the modified multiplier transform introduced and studied by Catas [12]. For \( l = 0 \), \( \Theta_{p,\lambda}^{n,l} \) gives Modified El-Ashwah-Aouf operator [5]. The condition \( l = 0 \) and \( \lambda = 1 \), reduces \( \Theta_{p,\lambda}^{n,l} \) to the operator defined by Aouf and Mostafa[4]. For \( p = \lambda = 1 \) and \( l = 0 \), we obtain the modified Sălăgean operator [27]. Again, for \( p = \lambda = 1 \), we get the modified operator due to Cho and Srivastava [13], and for \( p = 1 \) and \( l = 0 \), the operator \( \Theta_{p,\lambda}^{n,l} \) generalizes to modified operator studied by Al-Oboudi [8]. It is worthy to mention here that, for \( n = 0 \), the newly defined class \( SH_{p,\lambda}^{n,l}(\alpha, \beta) \) generalizes the classical subclass which is introduced and studied by Ahuja and Jahangiri [3]. For \( \lambda = 1 \), the operator \( \Theta_{p,\lambda}^{n,l} \) reduces to the operator \( D_{p}^{n,l} \) which is introduced by EL-Ashwah and Aouf [6].

In the present investigation, we also define the following subclass of \( SH_{p,\lambda}^{n,l}(\alpha, \beta) \), with the additional features as follows:
Definition 1.2 The subclass $\overline{SH}^{n,l}_{p,\lambda}(\alpha, \beta)$ of $SH^{n,l}_{p,\lambda}(\alpha, \beta)$ contains the harmonic mappings $f_n(z) = h(z) + g_n(z)$, where

$$h(z) = z^p - \sum_{k=p+1}^{\infty} |a_k|z^k, \quad g_n(z) = (-1)^n \sum_{k=p}^{\infty} |b_k|z^k; \quad |b_p| < 1; \quad (z \in \Delta).$$

(4)

The class $\overline{SH}^{n,l}_{p,\lambda}(\alpha, \beta)$ reduces to the subfamily $\overline{H}^n(1, l, \beta, \alpha)$, studied by EL-Ashwah and Aouf [6], for $n = 0$ and $\lambda = 1$. When $n = 0$, $l = 0$ and $\lambda = 1$, the defined class $\overline{SH}^{n,l}_{p,\lambda}(\alpha, \beta)$, becomes the class $\mathcal{H}_p\mathcal{R}(1, \beta, \alpha)$ introduced and studied by Ahuja and Jahangiri [3] (see also [30]). When $\beta = 0$ and $n = 0$, the class $\overline{SH}^{n,l}_{p,\lambda}(\alpha, \beta)$, consists of multivalent functions $f$, where $\Re \left( \frac{pf(z)}{z^p} \right) > \alpha$. Therefore, in the present investigation we extend the results of [6], [3] and [30] to the class $\overline{SH}^{n,l}_{p,\lambda}(\alpha, \beta)$ and also obtain some additional results in different prospective.

This paper is organized as follows: The main results are given in section 2 which is further subdivided into subsections in accordance with the content of the results therein. In the subsection 2.1, the necessary and sufficient conditions for functions to be in the newly defined classes are discussed. We investigated the growth theorem in subsection 2.2 and determined the extreme points of the defined class in subsection 2.3. The inclusion properties in the context of convolution and convex combination are discussed in subsection 2.4. Furthermore, the inclusion property based on neighborhood properties of the class is given in subsection 2.5. In subsection 2.6, we studied the radius of starlikeness of the introduced subclass. Lastly, the problems for further investigation, which may be studied as a sequel to our investigations are given in the section 3.

2 Main Results

2.1 Coefficient Estimates

We begin with the following sufficient condition for a function $f \in \mathcal{H}_p$ to be in the class $SH^{n,l}_{p,\lambda}(\alpha, \beta)$.

Theorem 2.1 Let $f = h + \overline{g}$ in $\mathcal{H}_p$ be given by (1). Then $f \in SH^{n,l}_{p,\lambda}(\alpha, \beta)$.
Certain subclasses of harmonic multivalent functions.

if the following inequality holds true:

\[
\sum_{k=p}^{\infty} [(k - p)\beta + p]\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n |a_k| + \sum_{k=p}^{\infty} |(k + p)\beta - p|\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n |b_k| \leq (p - \alpha). \tag{5}
\]

Sharpness in (5) occurs for the function

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{x_k}{(k - p)\beta + p}\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n z^k + \sum_{k=p}^{\infty} \frac{y_k}{(k + p)\beta - p}\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n z^k,
\]

where

\[
\sum_{k=p+1}^{\infty} |x_k| + \sum_{k=p}^{\infty} |y_k| = p - \alpha.
\]

**Proof.** Let \(\Phi(z) = (1 - \beta) \tilde{\Theta}^{n,l}_{p,\lambda,f(z)} + \beta \frac{(\tilde{\Theta}^{n,l}_{p,\lambda,f(z)})'}{z^{p+1}}\). It is enough to show that for \(0 \leq \alpha < p\), we have

\[
|p - \alpha + p\Phi(z)| \geq |p + \alpha - p\Phi(z)|. \tag{6}
\]

Now upon using equations (2) to (6) and substituting in \(\Phi(z)\), we have

\[
|p - \alpha + p\Phi(z)| \geq 2p - \alpha - \sum_{k=p+1}^{\infty} [(k - p)\beta + p]\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n |a_k||z|^{k-p}
\]

\[
- \sum_{k=p}^{\infty} |(k + p)\beta - p|\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n |b_k||z|^{k-p}, \tag{7}
\]

and

\[
|p + \alpha - p\Phi(z)| \leq \alpha + \sum_{k=p+1}^{\infty} [(k - p)\beta + p]\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n |a_k||z|^{k-p}
\]

\[
+ \sum_{k=p}^{\infty} |(k + p)\beta - p|\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n |b_k||z|^{k-p}. \tag{8}
\]
Now (6), (7) and (8) yields
\[ |p - \alpha + p\Phi(z)| - |p + \alpha - p\Phi(z)| \geq 2 \left[ p - \alpha \right. \\
- \sum_{k=p+1}^{\infty} \left[ (k - p)\beta + p \right] \left( \frac{p + l + \lambda(k - p)}{p + l} \right)^{n} |a_{k}| |z|^{k-p} \\
- \sum_{k=p}^{\infty} \left[ (k + p)\beta - p \right] \left( \frac{p + l + \lambda(k - p)}{p + l} \right)^{n} |b_{k}| |z|^{k-p} \]
which is greater than equal to zero only if (5) holds true. This completes the proof of Theorem 2.1.

In the following theorem, we show that (5) is both necessary and sufficient for a function \( f \) to be in \( \hat{S}_{p,\lambda}^{n,l}(\alpha, \beta) \).

**Theorem 2.2** Let \( f_{n} = h + g_{n} \in \mathcal{H}_{p} \), where \( h \) and \( g_{n} \) be of the form (4).

Then \( f_{n} \in \hat{S}_{p,\lambda}^{n,l}(\alpha, \beta) \) if and only if
\[
\sum_{k=p+1}^{\infty} \left[ (k - p)\beta + p \right] \left( \frac{p + l + \lambda(k - p)}{p + l} \right)^{n} |a_{k}| \\
+ \sum_{k=p}^{\infty} \left[ (k + p)\beta - p \right] \left( \frac{p + l + \lambda(k - p)}{p + l} \right)^{n} |b_{k}| \leq p - \alpha. \quad (9)
\]

**Proof.** The sufficient condition is proved as in Theorem 2.1. Thus we prove the necessary part. We have
\[
\Re \left\{ (1 - \beta) \frac{\Theta_{p,\lambda}^{n,l}h(z)}{z^{p}} + \beta \left( \frac{\Theta_{p,\lambda}^{n,l}f(z)}{z^{p-1}} \right)' \right\} > \frac{\alpha}{p}.
\]

Therefore, upon suitable substitutions, we get
\[
\Re \left\{ (1 - \beta) \frac{\Theta_{p,\lambda}^{n,l}h(z)}{z^{p}} + (1)^{n} \Theta_{p,\lambda}^{n,l}g(z) \\
+ \beta \left( \frac{\Theta_{p,\lambda}^{n,l}h(z) + (1)^{n} \Theta_{p,\lambda}^{n,l}g(z)'}{z^{p-1}} \right) \right\} \\
\geq 1 - \frac{1}{p} \sum_{k=p+1}^{\infty} \left[ (k - p)\beta + p \right] \left( \frac{p + l + \lambda(k - p)}{p + l} \right)^{n} |a_{k}| |z|^{k-p} \\
- \frac{1}{p} \sum_{k=p}^{\infty} \left[ (k + p)\beta - p \right] \left( \frac{p + l + \lambda(k - p)}{p + l} \right)^{n} |b_{k}| |z|^{k-p} \geq \frac{\alpha}{p}. \quad (10)
\]
Taking $z$ over positive real axis, $(0 \leq z = r < 1)$, equation (10) gives

$$1 - \frac{1}{p} \sum_{k=p+1}^{\infty} ((k - p)\beta + p) \left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n |a_k||r|^{k-p}$$

$$- \frac{1}{p} \sum_{k=p}^{\infty} |(k + p)\beta - p| \left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n |b_k||r|^{k-p} \geq \frac{\alpha}{p}.$$ 

Therefore, for $r \to 1^-$, we have the desired inequality (9). This completes the proof of Theorem 2.2.

### 2.2 Growth and Covering Results for the Class $\overline{SH}_{p,\lambda}^{n,l}(\alpha, \beta)$

**Theorem 2.3** Let $f_n = h + g_n \in H_p$, where $h$ and $g_n$ be of the form (4), with $\frac{p(2\beta-1)}{p-\alpha}|b_p| < 1$ and $\beta \geq 1$. If $f \in \overline{SH}_{p,\lambda}^{n,l}(\alpha, \beta)$ then for $|z| = r < 1$, the following inequalities holds true:

$$(1 + |b_p|)r^p - \frac{p - \alpha}{(\beta + p)\left(\frac{p + l + \lambda}{p + l}\right)^n} \left(1 - \frac{p(2\beta - 1)}{p - \alpha}|b_p|\right) r^{p+1} \leq |f_n(z)|; \quad (11)$$

$$|f_n(z)| \leq (1 + |b_p|)r^p + \frac{p - \alpha}{(\beta + p)\left(\frac{p + l + \lambda}{p + l}\right)^n} \left(1 - \frac{p(2\beta - 1)}{p - \alpha}|b_p|\right) r^{p+1}. \quad (12)$$

**Proof.** Let $f_n \in \overline{SH}_{p,\lambda}^{n,l}(\alpha, \beta)$, where $f_n(z) = h(z) + g_n(z)$. In view of (4), we get

$$f_n(z) = z^p + (-1)^n|b_p|z^p - \sum_{k=p+1}^{\infty} |a_k|z^k + (-1)^n \sum_{k=p+1}^{\infty} |b_k|z^k.$$
\[ |f_n(z)| = z^p + (-1)^n|b_p|z^p - \sum_{k=p+1}^{\infty} |a_k|z^k + (-1)^n \sum_{k=p+1}^{\infty} |b_k|z^k \]

\[ \leq (1 + |b_p|)r^p + r^{p+1} \sum_{k=p+1}^{\infty} (|a_k| + |b_k|) \]

\[ = (1 + |b_p|)r^p + \frac{(p - \alpha)r^{p+1}}{(\beta + p)\left(\frac{p + l + \lambda}{p + l}\right)^\alpha} \sum_{k=p+1}^{\infty} \left(\frac{\beta + p}{p - \alpha}\right) \left(\frac{p + l + \lambda}{p + l}\right)^n (|a_k| + |b_k|) \]

\[ \leq (1 + |b_p|)r^p + \frac{(p - \alpha)r^{p+1}}{(\beta + p)\left(\frac{p + l + \lambda}{p + l}\right)^\alpha} \left\{ \sum_{k=p+1}^{\infty} \left(\frac{(k + p)\beta - p}{p - \alpha}\right) \left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n |a_k| \right\} \]

\[ + \sum_{k=p+1}^{\infty} \left|\frac{(k + p)\beta - p}{p - \alpha}\right| \left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n |b_k| \]

\[ \leq (1 + |b_p|)r^p + \frac{p - \alpha}{(\beta + p)\left(\frac{p + l + \lambda}{p + l}\right)^\alpha} \left[ 1 - \frac{p(2\beta - 1)}{p - \alpha} |b_p| \right] r^{p+1}, \]

which is the desired inequality (12). The left hand inequality (11) can be proved on similar lines and hence omitted. This completes the proof of Theorem 2.3.

The inequality (11) of Theorem 2.3, upon simplification yields:

\[ \frac{A - B}{C} < |f_n(z)|, \]

where \( A := (\beta + p)(p + l + \lambda)^n - (p + l)^n(p - \alpha), \quad B := ((\beta + p)(p + l + \lambda)^n - p(p + l)^n(2\beta - 1))|b_p| \) and \( C := (\beta + p)(p + l + \lambda)^n. \) Thus, we have the following covering result:

**Theorem 2.4** Let \( f_n(z) = h(z) + g_n(z) \) be in \( \overline{SH}^{n,l}_{p,\lambda}(\alpha, \beta) \) where \( h \) and \( g_n \) are given by (4). Then

\[ \{ w : |w| < \left( \frac{A - B}{C} \right) \} \subset f_n(\Delta). \]

### 2.3 Extreme Points for the Class \( \overline{SH}^{n,l}_{p,\lambda}(\alpha, \beta) \)

We recall that, a function \( f \in \mathcal{F} \subset S_H^0 \) is called an *extreme point* of \( \mathcal{F} \) if, for \( 0 < \mu < 1, \) \( f = \mu f_1 + (1 - \mu)f_2 \) implies \( f_1 = f_2 = f \) for all \( f_1 \) and \( f_2 \) in \( \mathcal{F}. \) The intersection of closed convex subsets of \( S_H^0 \) which contain \( \mathcal{F} \) is called the *closed convex hull* of \( \mathcal{F}. \) We denote by \( \text{cl}\mathcal{F}, \) the set of all closed convex hull of \( \mathcal{F}. \)
Suppose that $f_n = h + \overline{f}_n \in \mathcal{H}_p$ be defined by (4). Then, for $\beta > 1/2$, $f_n \in \text{clSH}_{p,\lambda}(\alpha, \beta)$ if and only if $f_n = \sum_{k=0}^{\infty} (x_k h_k(z) + y_k g_k(z))$, where $h_p(z) = z^p$, $x_k, y_k \geq 0$, and $x_p = 1 - \sum_{k=p+1}^{\infty} x_k - \sum_{k=p}^{\infty} y_k$.

\[ h_k(z) = z^p - \frac{p - \alpha}{((k - p)\beta + p)\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n} z^k, \quad (k = p + 1, p + 2, \cdots) \quad \text{and} \]

\[ g_k(z) = z^p - (-1)^n \frac{p - \alpha}{|k + p|\beta - p\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n} z^k, \quad (k = p, p + 1, p + 2, \cdots). \]

In particular, \{h_k\}, \{g_k\} are the extreme points of the class $\text{SH}_{p,\lambda}^{n, l}(\alpha, \beta)$.

**Proof.** Suppose that

\[ f_n(z) = \sum_{k=p}^{\infty} (x_k h_k(z) + y_k g_k(z)). \]

By proper substitutions and application of Theorem 2.2, we get

\[
\sum_{k=p+1}^{\infty} ((k - p)\beta + p)\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n \left(\frac{p - \alpha}{((k - p)\beta + p)\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n} x_k \right)
\]

\[ + \sum_{k=p}^{\infty} |(k + p)\beta - p|\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n \left(\frac{p - \alpha}{|(k + p)\beta - p|\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n} y_k \right) \]

\[ = (p - \alpha) \left( \sum_{k=p+1}^{\infty} x_k + \sum_{k=p}^{\infty} y_k \right) = (p - \alpha)(1 - x_p) \leq p - \alpha. \]

This implies that $f_n \in \text{clSH}_{p,\lambda}^{n, l}(\alpha, \beta)$. Conversely, let $f_n \in \text{clSH}_{p,\lambda}^{n, l}(\alpha, \beta)$,

\[ x_k = \frac{((k - p)\beta + p)\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n}{p - \alpha} |a_k|; \quad (k = p + 1, p + 2, \cdots), \]

and

\[ y_k = \frac{|(k + p)\beta - p|\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^n}{p - \alpha} |b_k|; \quad (k = p, p + 1, \cdots). \]
From Theorem 2.2, we have $0 \leq x_k \leq 1, \ (k = p + 1, p + 2, \cdots)$ and $0 \leq y_k \leq 1, \ (k = p, p + 1, \cdots)$. Thus, we have

$$f_n(z) = z^p - \sum_{k=p+1}^{\infty} |a_k|z^k - (-1)^n \sum_{k=p}^{\infty} |b_k|z^k$$

$$= z^p - \sum_{k=p+1}^{\infty} \frac{p - \alpha}{|((k - p)\beta + p)\left(\frac{p+l+\lambda(k-p)}{p+l}\right)|} x_k z^k$$

$$- (-1)^n \sum_{k=p}^{\infty} \frac{p - \alpha}{|(k+p)\beta - p)\left(\frac{p+l+\lambda(k-p)}{p+l}\right)|} y_k \bar{z}^k,$$

which upon simplification becomes

$$f_n(z) = z^p - \sum_{k=p+1}^{\infty} \frac{(z^p - h_k(z))x_k z^k - \sum_{k=p}^{\infty}(z^p - g_k(z))y_k \bar{z}^k}{1 - \sum_{k=p+1}^{\infty} x_k - \sum_{k=p}^{\infty} y_k} z^p + \sum_{k=p+1}^{\infty} h_k(z)x_k z^k + \sum_{k=p}^{\infty} g_k(z)y_k \bar{z}^k$$

$$= \sum_{k=p}^{\infty} [x_k h_k(z) + y_k g_k(z)].$$

This completes the proof of Theorem 2.5.

### 2.4 Inclusion Properties for the Class $\overline{SH}_{n,l}^{p,\lambda}(\alpha, \beta)$

For the functions $f_m, F_m \in \mathcal{H}_p$ which are of the forms:

$$f_m(z) = z^p - \sum_{k=p+1}^{\infty} |a_k|z^k + (-1)^n \sum_{k=p}^{\infty} |b_k|\bar{z}^k$$

(13)

and

$$F_m(z) = z^p - \sum_{k=p+1}^{\infty} |A_k|z^k + (-1)^n \sum_{k=p}^{\infty} |B_k|\bar{z}^k,$$

(14)

we define the function $(f_m * F_m)(z)$ by

$$(f_m * F_m)(z) = z^p - \sum_{k=p+1}^{\infty} |a_k A_k|z^k + (-1)^n \sum_{k=p}^{\infty} |b_k B_k|\bar{z}^k.$$
Therefore, for the convolution \( f_m * F_m \), since \( 0 \leq \alpha < \alpha_1 < p \) and \( f_m \in \overline{SH}^{n,l}_{p,\alpha}(\alpha, \beta) \) we get
\[
\sum_{k=p+1}^{\infty} |(k+\beta - p)| \left( \frac{p+\lambda(k-p)}{p} \right)^n |a_k| + \sum_{k=p}^{\infty} |(k+\beta - p)| \left( \frac{p+\lambda(k-p)}{p+1} \right)^n |b_k| 
\leq \frac{1}{1 - \alpha_1}
\]

Therefore \( f_m * F_m \in \overline{SH}^{n,l}_{p,\alpha}(\alpha, \beta) \subset \overline{SH}^{n,l}_{p,\alpha}(\alpha, \beta) \). This completes the proof of Theorem 2.6.

Next, to study the properties of convex combination, we recall the following:

Let \( f_{n_i}(z) \in \overline{SH}^{n,l}_{p,\alpha}(\alpha, \beta), \quad i = 1, 2, 3, \ldots \). Suppose \( f_{n_i}(z) \) is written as,
\[
f_{n_i}(z) = z^p - \sum_{k=p+1}^{\infty} |a_{k_i}| z^k + (-1)^n \sum_{k=p}^{\infty} |b_{k_i}| \bar{z}^k, \quad (z \in \Delta).
\]

The convex combination of \( f_{n_i}, \quad i = 1, 2, 3, \ldots \) is given by the expression, for \( \sum_{i=1}^{\infty} t_i = 1, \quad 0 \leq t_i \leq 1 \), as:
\[
\sum_{i=1}^{\infty} t_i f_{n_i}(z) = \sum_{i=1}^{\infty} t_i \left( z^p - \sum_{k=p+1}^{\infty} |a_{k_i}| z^k + (-1)^n \sum_{k=p}^{\infty} |b_{k_i}| \bar{z}^k \right) 
= \sum_{i=1}^{\infty} t_i z^p - \sum_{k=p+1}^{\infty} \sum_{i=1}^{\infty} t_i |a_{k_i}| z^k + (-1)^n \sum_{i=1}^{\infty} \sum_{k=p}^{\infty} t_i |b_{k_i}| \bar{z}^k 
= z^p - \sum_{k=p+1}^{\infty} \left( \sum_{i=1}^{\infty} t_i |a_{k_i}| \right) z^k + (-1)^n \sum_{k=p}^{\infty} \left( \sum_{i=1}^{\infty} t_i |b_{k_i}| \right) \bar{z}^k.
\]

We have the following inclusion Theorem.
Theorem 2.7 The class $\widehat{SH}_{p,\lambda}(\alpha, \beta)$ is closed under convex combination.

Proof. Let $f_n(z) \in \widehat{SH}_{p,\lambda}(\alpha, \beta), \ i = 1, 2, 3, \ldots$ To prove that $\sum_{i=1}^{\infty} t_i f_n(z)$ also belongs to the class $\widehat{SH}_{p,\lambda}(\alpha, \beta)$ we show that necessary and sufficient conditions of Theorem 2.2 is true for (15).

Indeed, Theorem 2.2 yields:

$$\begin{align*}
&\sum_{k=p+1}^{\infty} ((k-p)\beta + p) \left( \frac{p+l+\lambda (k-p)}{p+l} \right)^n \left( \sum_{i=1}^{\infty} t_i |a_k| \right) \\
&+ \sum_{k=p}^{\infty} |(k+p)\beta - p| \left( \frac{p+l+\lambda (k-p)}{p+l} \right)^n \left( \sum_{i=1}^{\infty} t_i |b_k| \right) \\
&= \sum_{i=1}^{\infty} t_i \left\{ \sum_{k=p+1}^{\infty} ((k-p)\beta + p) \left( \frac{p+l+\lambda (k-p)}{p+l} \right)^n |a_k| \\
&+ \sum_{k=p}^{\infty} |(k+p)\beta - p| \left( \frac{p+l+\lambda (k-p)}{p+l} \right)^n |b_k| \right\} \\
&\leq \sum_{i=1}^{\infty} t_i (p - \alpha) = (p - \alpha).
\end{align*}$$

This completes the proof of Theorem 2.7.

The following class preserving property immediately follows from the Theorem 2.2 and hence we state the result without proof.

Theorem 2.8 If $f_n(z) = h(z) + g_n(z)$ defined by (4) is in $\widehat{SH}_{p,\lambda}(\alpha, \beta)$, then $I_c(f_n), \ c > -1$ also belongs to the class $\widehat{SH}_{p,\lambda}(\alpha, \beta)$, where $I_c(f_n)$ is the well known generalized Bernardi-Libera-Livingston integral operator (cf.[9] and [23]) is defined as:

$$I_c(f_n) = \frac{c+p}{z^c} \int_0^z t^{c-1} f_n(t) dt \quad c > -1.$$
For \( f_n(z) = h(z) + g_n(z) \) defined by (4) belonging to \( \tilde{SH}_{p,\lambda}^{n,l}(\alpha, \beta) \), the neighborhood of \( f \) in \( \tilde{SH}_{p,\lambda}^{n,l}(\alpha, \beta) \) is defined by

\[
N_\delta(f, s) := \left\{ s \in \tilde{SH}_{p,\lambda}^{n,l}(\alpha, \beta) : \sum_{k=p+1}^{\infty} k (|a_k| - |A_k|) + |(b_k - |B_k|)) + p(|b_p - |B_p|)| \leq \delta \right\}.
\]

In particular for \( e(z) = z^p \), we have

\[
N_\delta(e, s) := \left\{ s \in \tilde{SH}_{p,\lambda}^{n,l}(\alpha, \beta) : \sum_{k=p+1}^{\infty} k (|A_k| + |B_k|) + p|B_p| \leq \delta \right\}.
\]

We have the following inclusion result:

**Theorem 2.9** Let \( s(z) \) given by (16) be in the class \( \tilde{SH}_{p,\lambda}^{n,l}(\alpha, \beta) \). Then for \( 1/2 \leq \beta \leq 2 \),

\[
\tilde{SH}_{p,\lambda}^{n,l}(\alpha, \beta) \subset N_\delta(e, s),
\]

where \( \delta = 2(p - \alpha) + (1 - 2\beta)p|B_p| \).

**Proof.** Let \( s(z) \in \tilde{SH}_{p,\lambda}^{n,l}(\alpha, \beta) \). From Theorem 2.2, we get

\[
\sum_{k=p+1}^{\infty} ((k - p)\beta + p) \left( \frac{p + l + \lambda(k - p)}{p + l} \right)^n |A_k| + \sum_{k=p}^{\infty} ((k + p)\beta - p) \left( \frac{p + l + \lambda(k - p)}{p + l} \right)^n |B_k| \leq p - \alpha.
\]

Upon simplification, the above inequality reduced to

\[
\beta \left( \frac{p + l + \lambda}{p + l} \right)^n \sum_{k=p+1}^{\infty} k(|A_k| + |B_k|) + (2\beta - 1)p|B_p| - (\frac{p + l + \lambda}{p + l})^n \sum_{k=p+1}^{\infty} p(\beta - 1)|A_k| + p(1 - \beta)|B_k| \leq p - \alpha.
\]

Note that \( \beta \left( \frac{p + l + \lambda}{p + l} \right)^n \geq 1 \) for all \( \beta \geq 1 \). Therefore, the above inequality becomes

\[
\sum_{k=p+1}^{\infty} k(|A_k| + |B_k|) + p|B_p| \leq (p - \alpha) + \left( \frac{p + l + \lambda}{p + l} \right)^n \sum_{k=p+1}^{\infty} p(\beta - 1)|A_k| + p(1 - \beta)|B_k|.
\]  (17)
Further, applications of (9), we also have
\[
\left( \frac{p + l + \lambda}{p + l} \right)^n \left\{ \sum_{k=p+1}^{\infty} p|A_k| + 2p|B_k| \right\} \leq (p - \alpha) + (1 - 2\beta)p|B_p|.
\]  
(18)

It is observed that for \( \beta \in [1/2, 2] \), the inequalities (17) and (18) yield
\[
\sum_{k=p+1}^{\infty} k(|A_k| + |B_k|) + p|B_p| \leq 2(p - \alpha) + (1 - 2\beta)p|B_p|.
\]

Hence, \( s(z) \in N_\delta(e, s) \). This completes the proof of Theorem 2.9.

### 2.6 Radius of Starlikeness for the Class \( \overline{SH}_{p,\lambda}^{n,l}(\alpha, \beta) \)

Finally, to study the radius of starlikeness, recalling, a function \( f \in \mathcal{H}_p \) is said to be starlike of order \( \mu \), \( (0 \leq \mu < 1) \) if and only if \( \Re\left( \frac{J_z f(z)}{f(z)} \right) > \mu, \ (z \in \Delta) \), where \( J_z f(z) := zh'(z) - zg'(z) \), or equivalently following inequality holds true:
\[
\left| \frac{J_z f(z) - (1 + \mu)f(z)}{J_z f(z) + (1 - \mu)f(z)} \right| < 1, \quad (z \in \Delta).
\]  
(19)

Let \( \Delta_r := \{ z \in \mathbb{C} : |z| = r < 1 \} \), the radius of starlikeness of the class \( \mathcal{F} \) is given by
\[
r^*_\mu(\mathcal{F}) := \inf_{f \in \mathcal{F}} \left\{ \sup_{r \in [0, 1]} (r \in [0, 1] : f \text{ is starlike of order } \mu \text{ in } \Delta_r) \right\}.
\]

The following result determine the radius of starlikeness of order \( \mu \) \( (0 \leq \mu < 1) \), for the class \( \overline{SH}_{p,\lambda}^{n,l}(\alpha, \beta) \).

**Theorem 2.10** Let \( 0 \leq \mu < 1 \) and \( n \) be an even number. Then the radius of starlikeness of order \( \mu \) for the class \( \overline{SH}_{p,\lambda}^{n,l}(\alpha, \beta) \) is given by
\[
r^*_{\mu} = \inf_{k > p} \left\{ \min \left\{ \frac{\gamma_1 \eta^n}{p - \mu}, \frac{\gamma_2 \eta^n}{p - \mu} \right\} \right\} \frac{1}{\eta^p},
\]  
(20)

where \( \gamma_1 := (k - p)\beta + p \), \( \gamma_2 := |(k + p)\beta - p| \) and \( \eta := \frac{p + l + \lambda(k - p)}{p + l} \).

**Proof.** Let \( f_n(z) = h(z) + g_n(z) \in \overline{SH}_{p,\lambda}^{n,l}(\alpha, \beta) \), where \( h(z) \) and \( g_n(z) \) be of the form (4). Then, for \( |z| = r < 1 \), we have
\[
\left| \frac{J_z f(z) - (1 - \mu)f(z)}{J_z f(z) + (1 - \mu)f(z)} \right| = \frac{(p - 1 - \mu) - \sum_{k=p+1}^{\infty} (k - 1 - \mu)|a_k|z^{k-p} - (-1)^{n-1} \sum_{k=p+1}^{\infty} (k + 1 + \mu)|b_k|z^{k-p}}{(p + 1 - \mu) - \sum_{k=p+1}^{\infty} (k + 1 - \mu)|a_k|z^{k-p} - (-1)^{n-1} \sum_{k=p+1}^{\infty} (k - 1 + \mu)|b_k|z^{k-p}} \leq \frac{\Gamma_1 - \Gamma_2}{\Gamma_3 - \Gamma_4},
\]  
(21)
Certain subclasses of harmonic multivalent functions.

where

\[
\begin{align*}
\Gamma_1 &= (p - 1 - \mu) - (-1)^{n-1}(p + 1 + \mu)|b_p| \\
\Gamma_2 &= \sum_{k=p+1}^{\infty} ((k - 1 - \mu)|a_k| + (-1)^{n-1}(k + 1 + \mu)|b_k|)r^{k-p} \\
\Gamma_3 &= (p + 1 - \mu) - (-1)^{n-1}(p - 1 + \mu)|b_p| \\
\Gamma_4 &= \sum_{k=p+1}^{\infty} ((k + 1 - \mu)|a_k| + (-1)^{n-1}(k - 1 + \mu)|b_k|)r^{k-p}.
\end{align*}
\]

From the (19) and (21), it is cleared that \( f \) is starlike of order \( \mu \) in \( \Delta_r \) if and only if

\[
(p - 1 - \mu) - \sum_{k=p+1}^{\infty} (k - 1 - \mu)|a_k|r^{k-p} - (-1)^n \sum_{k=p}^{\infty} (k + 1 + \mu)|b_k|r^{k-p} \leq (p + 1 - \mu) - \sum_{k=p+1}^{\infty} (k + 1 - \mu)|a_k|r^{k-p} - (-1)^n \sum_{k=p}^{\infty} (k - 1 + \mu)|b_k|r^{k-p}.
\]

Which on simplification reduces to

\[
\sum_{k=p+1}^{\infty} |a_k|r^{k-p} + \sum_{k=p}^{\infty} (-1)^n|b_k|r^{k-p} \leq 1. \tag{22}
\]

But, from Theorem 2.2, we have

\[
\sum_{k=p+1}^{\infty} \frac{\gamma_1\eta^n}{p - \alpha}|a_k| + \sum_{k=p}^{\infty} \frac{\gamma_2\eta^n}{p - \alpha}|b_k| \leq 1. \tag{23}
\]

Therefore, comparing (22) and (23), we must have

\[
r^{k-p} \leq \frac{\gamma_1\eta^n}{p - \alpha} \quad \text{and} \quad r^{k-p} \leq \frac{\gamma_2\eta^n}{p - \alpha}, \quad \text{for } k = p + 1, p + 2, \ldots.
\]

This implies that

\[
r \leq \left( \min \left\{ \frac{\gamma_1\eta^n}{p - \alpha}, \frac{\gamma_2\eta^n}{p - \alpha} \right\} \right)^{\frac{1}{k-p}}, \quad \text{for } k = p + 1, p + 2, \ldots.
\]

Thus, the radius of starlikeness of order \( \mu \) for the class \( \overline{SH}_{\mu}|_{p,\lambda}(\alpha, \beta) \) is given by (20). This completes the proof of Theorem 2.10.
3 Open Problems

The author(s) suggest to study the following aspects:

- The determination of the disc, in the domain set, where the functions are fully convex of order \( \alpha \) (0 \( \leq \alpha < 1 \)), fully starlike of order \( \alpha \) are interesting future aspects of investigation. In this context, the author(s) have given a theoretical viewpoint of the radius of starlikeness of order \( \mu \) (0 \( \leq \mu < 1 \)) for the family \( \mathcal{SH}_{p,\lambda}^{n,\ell}(\alpha, \beta) \). The exact, numerical evaluation of such radius is challenging.

- The geometric nature of the image (whether it is fully starlike, fully convex or uniformly convex) of the functions belonging to the defined classes need to settle down.

- Following the Bieberbach Conjecture for the family \( \mathcal{S}_H \) (or \( \mathcal{S}^n_H \)), it also interesting to investigate the sharp bounds of \( n^{th} \) coefficients of both analytic and co analytic part of function belonging to the newly defined classes.

References


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