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Radius Properties of Certain Analytic Functions

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Abstract

For analytic functions $f(z)$ ***normalized with*** $f(0) = 0$ ***and*** $f'(0) = 1$ ***in the open unit disk*** $\mathbb{U}$ ***, a class*** $\mathcal{U}_3(\lambda)$ ***of*** $f(z)$ ***satisfying some conditions is introduced. The object of the present paper is to discuss the problem such that*** $\frac{1}{\alpha}f(\alpha z) \in \mathcal{U}_3(\lambda)$ ***for*** $f(z) \in \mathcal{S}$. ***Also for our result, an open problem concern in Hölder inequality is given.***

Keywords: *Analytic function, univalent function, Caushy-Schwarz inequality, Hölder inequality.*
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1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of $f(z)$ which are univarent in $\mathbb{U}$. For $f(z) \in \mathcal{A}$, we say that $f(z) \in \mathcal{U}_3(\lambda)$ if it satisfies $\frac{f(z)}{z} \neq 0$ $(z \in \mathbb{U})$ and

$$\left| z^4 \left(\frac{1}{f(z)} - \frac{1}{z} \right)''' \right| \leqq \lambda \quad (z \in \mathbb{U}) \tag{1.2}$$

for some real $\lambda > 0$.

Let us consider a function $f(z)$ given by

$$f_\delta(z) = \frac{z}{(1-z)^\delta} \quad (\delta \in \mathbb{R}). \tag{1.3}$$

Then we have that

$$f_\delta(z) = \frac{z}{1 + \sum_{n=1}^{\infty} a_n z^n} \tag{1.4}$$

with

$$a_n = (-1)^n \binom{\delta}{n}. \tag{1.5}$$

It follows that

$$\left| z^4 \left(\frac{1}{f_\delta(z)} - \frac{1}{z} \right)''' \right| \leqq \sum_{n=1}^{\infty} (n-1)(n-2)(n-3)|a_n||z|^n \tag{1.6}$$

$$< \sum_{n=1}^{\infty} (n-1)(n-2)(n-3)|a_n|$$

for $z \in \mathbb{U}$.

Therefore, if $\delta = 3$, then

$$\left| z^4 \left(\frac{1}{f_3(z)} - \frac{1}{z} \right)''' \right| \leqq 0,$$

or $f_3(z) \in \mathcal{U}_3(\lambda)$ for $\lambda \geq 0$, if $\delta = 4$, then

$$\left| z^4 \left(\frac{1}{f_4(z)} - \frac{1}{z} \right)''' \right| < 6,$$

or $f_4(z) \in \mathcal{U}_3(6)$, if $\delta = 5$, then

$$\left| z^4 \left(\frac{1}{f_5(z)} - \frac{1}{z} \right)''' \right| < 54,$$

or $f_5(z) \in \mathcal{U}_3(54)$.

Obradović and Ponnusamy [2] have studied the subclass $\mathcal{U}_1(\lambda)$ of $\mathcal{A}$ consisting of $f(z)$ satisfying

$$\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U})$$

and

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| \leqq \lambda \quad (z \in \mathbb{U}) \tag{1.7}$$

which is equivalent to

$$\left| z^2 \left(\frac{1}{z} - \frac{1}{f(z)} \right)' \right| \leqq \lambda \quad (z \in \mathbb{U}). \tag{1.8}$$

2 Main result

To discuss our problem for the class $\mathcal{U}_3(\lambda)$,we have to recall here the following lemma by Goodman [1].

Lemma 1 *If $f(z) \in \mathcal{S}$ and*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \tag{2.1}$$

then, we have

$$\sum_{n=1}^{\infty} (n-1)|b_n|^2 \leqq 1 \tag{2.2}$$

Moreover,we need the following lemma.

Lemma 2 *Let $f(z) \in \mathcal{A}$ and*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \neq 0 \qquad (z \in \mathbb{U}).$$

If $f(z)$ satisfies

$$\sum_{n=3}^{\infty} (n-1)(n-2)(n-3)|b_n| \leqq \lambda, \tag{2.3}$$

then, $f(z) \in \mathcal{U}_3(\lambda)$.

Proof Since

$$\left|-z^4\left(\frac{1}{f(z)}-\frac{1}{z}\right)'''\right| = \left|z^4\left(\frac{1}{f(z)}-\frac{1}{z}\right)'''\right| < \sum_{n=3}^{\infty}(n-1)(n-2)(n-3)|b_n|, \quad (2.4)$$

if $f(z)$ satisfies the inequality (2.3),then $f(z) \in \mathcal{U}_3(\lambda)$.

Our main result is contained in

Theorem 1 *Let* $f(z) \in \mathcal{S}$ *and* $\alpha \in \mathbb{U}$. *Then the function* $\frac{1}{\alpha}f(\alpha z)$ *belongs to the class* $\mathcal{U}_3(\lambda)$ *for* $0 \leqq |\alpha| \leqq |\alpha_0(\lambda)|$, *where* $|\alpha_0| = |\alpha_0(\lambda)|$ *is the smallest root of the equation*

$$(36-\lambda^2)|\alpha|^{12}+(72+6\lambda^2)|\alpha|^{10}+(12-15\lambda^2)|\alpha|^8+20\lambda^2|\alpha|^6-15\lambda^2|\alpha|^4+6\lambda^2|\alpha|^2-\lambda^2 = 0 \quad (2.5)$$

in $0 < |\alpha| < 1$.

Proof Since

$$\frac{z}{f(z)} \neq 0 \ (z \in \mathbb{U})$$

for $f(z) \in \mathcal{S}$, if we write

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n,$$

then we have

$$\frac{z}{\frac{1}{\alpha}f(\alpha z)} = 1 + \sum_{n=1}^{\infty} \alpha^n b_n z^n \quad (2.6)$$

for $0 < |\alpha| < 1$. Lemma 1 gives us that

$$\sum_{n=3}^{\infty}(n-1)|b_n|^2 \leqq \sum_{n=1}^{\infty}(n-1)|b_n|^2 \leqq 1. \quad (2.7)$$

Therefore, we have to show that

$$\sum_{n=1}^{\infty}(n-1)(n-2)(n-3)|\alpha^n b_n| \leqq \lambda \quad (2.8)$$

to prove that $\frac{1}{\alpha}f(\alpha z) \in \mathcal{U}_3(\lambda)$.

Applying the Cauchy-Schwarz inequality for the left hand of the inequality (2.8), we see that

$$\begin{aligned}
&\sum_{n=1}^{\infty}(n-1)(n-2)(n-3)|\alpha^n b_n| \\
= &\sum_{n=1}^{\infty}((n-1)(n-2)^2(n-3)^2|\alpha|^{2n})^{\frac{1}{2}}((n-1)|b_n|^2)^{\frac{1}{2}} \\
\leqq &\left(\sum_{n=1}^{\infty}(n-1)(n-2)^2(n-3)^2|\alpha|^{2n}\right)^{\frac{1}{2}} \\
= &\frac{2|\alpha|^4\sqrt{3(1+6|\alpha|^2+3|\alpha|^4)}}{(1-|\alpha|^2)^3}. \qquad (2.9)
\end{aligned}$$

Let us consider the complex number α $(0<|\alpha|<1)$ such that

$$\frac{2|\alpha|^4\sqrt{3(1+6|\alpha|^2+3|\alpha|^4)}}{(1-|\alpha|^2)^3} = \lambda \qquad (2.10)$$

It follows from (2.10) that

$$h(|\alpha|) = (36-\lambda^2)|\alpha|^{12} + (72+6\lambda^2)|\alpha|^{10} + (12-15\lambda^2)|\alpha|^8 + 20\lambda^2|\alpha|^6 -$$

$$h(15\lambda^2|\alpha|^4 + 6\lambda^2|\alpha|^2 - \lambda^2 = 0$$

Note that $h(0) = -\lambda^2 < 0$, $h(1) = 120 > 0$.

Thus, $h(|\alpha|) = 0$ has a root of $|\alpha_0| = |\alpha_0(\lambda)|$ in $0 < |\alpha| < 1$. This complete the proof of the theorem.

Remark1 If we put $\alpha = \frac{1}{2}e^{i\theta}$ in (2.5), then we have

$$\lambda = \frac{2\sqrt{129}}{27} = 0.84148\cdots .$$

If we make $\lambda = 1$ in (2.5), then the equation

$$35|\alpha|^{12} + 36|\alpha|^{10} - 3|\alpha|^8 + 20|\alpha|^6 - 15|\alpha|^4 + 36|\alpha|^2 - 1 = 0$$

has a root $|\alpha_0|$ such that $0.1676 < |\alpha_0| < 0.1678$.

3 Open problem

For the proof of Theorem 1, we apply Cauchy-Schwarz inequality given by

$$\sum |a_n||b_n| \leqq \left(\sum |a_n|^2\right)^{\frac{1}{2}} \left(\sum |b_n|^2\right)^{\frac{1}{2}}.$$

But we know that Hölder inequality given by

$$\sum |a_n||b_n| \leqq \left(\sum |a_n|^p\right)^{\frac{1}{p}} \left(\sum |b_n|^q\right)^{\frac{1}{q}} \qquad \left(p > 0, q > 0, \frac{1}{p} + \frac{1}{q} \geqq 1\right)$$

is the generalization inequality of Cauchy-Schwarz inequality. Therefore, if we find some application of Hölder inequality for the proof of Theorem 1 instead of Cauchy-Schwarz inequality, then we derive new result which is the generalization of Theorem 1.

References

[1] A. W. Goodman, Univalent Functions, Vol.I and Vol.II, Mariner, Tampa, Florida, 1983

[2] M. Obradocć and S. Ponnusamy, Radius properties for subclasses of univalent functions, Analysis **25**(2005), 183 - 188