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On an Univalent Integral Operator

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Abstract

The integral operator denoted by $I(f_1, f_2, \ldots, f_m)$ given in Definition 2 was introduced in [8]. Also, certain sufficient conditions of univalence of this operator were given there. In this paper, a different approach for proving the univalence of this operator is taken.

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1 Introduction and preliminaries

Let U denote the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : \ |z| < 1\}$$

and

$$\overline{U} = \{z \in \mathbb{C} : \ |z| \leq 1\}.$$

Let $\mathcal{H}(U)$ denote the space of holomorphic functions in U and let

$$A_n = \{f \in \mathcal{H}(U), \ f(z) = z + a_{n+1}z^{n+1} + \ldots, \ z \in U\}$$

with $A_1 = A$.

Let

$$S = \{f \in A : \ f \text{ is univalent in } U\}.$$

Definition 1. (St. Ruscheweyh [16]). For $f \in A$, $n \in \mathbb{N} \cup \{0\}$, let R^n be the operator defined by $R^n : A \to A$

$$R^0 f(z) = f(z)$$
$$(n+1)R^{n+1} f(z) = z[R^n f(z)]' + nR^n f(z), \ z \in U.$$

Remark 1. If $f \in A$

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad z \in U$$

then

$$R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} a_j z^j, \quad z \in U. \tag{1}$$

In order to prove our main results, we shall use the following lemma:

Lemma A. [12] *Let α and c be complex numbers with* $\operatorname{Re} \alpha > 0$, $|c| \le 1$, $c \ne -1$ *and* $f(z) = z + a_2 z^2 + \dots$ *be a regular function in* U. *If*

$$\left| ce^{-2t\alpha} + (1 - e^{-2t\alpha}) \frac{e^{-t} z f''(e^{-t} z)}{\alpha f'(e^{-t} z)} \right| \le 1$$

holds for every $z \in U$ *and* $t \ge 0$, *then the function*

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u) du \right]^{\frac{1}{\alpha}}$$

is regular and univalent in U.

In [8] we have defined the following integral operator:

Definition 2. [8] Let $n, m \in \mathbb{N} \cup \{0\}$, $i \in \{1, 2, 3, \dots, m\}$, $\alpha_i \in \mathbb{C}$, $A^m = \underbrace{A \times A \times \dots \times A}_{m \ times}$. We let $I : A^m \to A$ be the integral operator given by

$$I(f_1, f_2, \dots, f_m)(z) = F(z) \tag{2}$$

$$= \left[\alpha \int_0^z t^{\alpha-1} \left(\frac{R^n f_1(t)}{t} \right)^{\alpha_1} \dots \left(\frac{R^n f_m(t)}{t} \right)^{\alpha_m} dt \right]^{\frac{1}{\alpha}},$$

where $f_i \in A$, $i \in \{1, 2, 3, \dots, n\}$ and R^n is the Ruscheweyh differential operator (Definition 1).

Remark 2. In [8], the author has shown that this operator is a generalization of other operators studied in [1], [2], [3], [4], [5], [6], [7], [9], [10], [11], [12], [13], [14], [15].

Remark 3. In [8], the author has stated and proven the following theorems:

Theorem 1. *Let* $n, m \in \mathbb{N} \cup \{0\}, \alpha \in \mathbb{C}$ *with* $\operatorname{Re} \alpha > 0$, $f_i \in A$, $\alpha_i \in \mathbb{C}$, $i \in \{1, 2, \ldots, m\}$ *with* $|\alpha_1| + |\alpha_2| + \cdots + |\alpha_m| \leq 1$.

If

$$\left| \frac{z(R^n f_i(z))'}{R^n f_i(z)} - 1 \right| \leq 1, \quad z \in U, \ i \in \{1, 2, \ldots, m\}$$

then $F(z)$ *given by (2) belongs to the class* S.

Theorem 2. *Let* $n, m \in \mathbb{N} \cup \{0\}, \alpha, \beta \in \mathbb{C}$, *with* $\operatorname{Re} \beta \geq \operatorname{Re} \alpha > 0$, *let* $f_i \in A$ *and let* $\alpha_i \in \mathbb{C}$, $i \in \{1, 2, \ldots, m\}$, *with* $|\alpha_1| + |\alpha_2| + \cdots + |\alpha_m| \leq 1$.

If

$$\left| \frac{z(R^n f_i(z))'}{R^n f_i(z)} - 1 \right| \leq 1, \quad z \in U, \ i \in \{1, 2, \ldots, m\}$$

where R^n *is the Ruschweyh differential operator, then the function given by*

$$(2') \qquad F_\beta(z) = \left[\beta \int_0^z t^{\beta - 1} \left(\frac{R^n f_1(t)}{t} \right)^{\alpha_1} \cdots \left(\frac{R^n f_m(t)}{t} \right) dt \right]^{\frac{1}{\beta}}$$

belongs to the class S.

Theorem 3. *Let* $n, m \in \mathbb{N} \cup \{0\}, \mu > 0, \alpha \in \mathbb{C}$ *with* $\operatorname{Re} \alpha > 0$, $f_i \in A$, $\alpha_i \in \mathbb{C}$, $i \in \{1, 2, \ldots, m\}$ *with* $|\alpha_1| + |\alpha_2| + \cdots + \|\alpha_m| \leq \dfrac{1}{2\mu + 1}$.

If

(i) $|R^n f_i(z)| \leq \mu$,

(ii) $\left| \dfrac{z^2 (R^n f_i(z))'}{[R^n f_i(z)]^2} - 1 \right| \leq 1$, $z \in U$, $i \in \{1, 2, \ldots, m\}$

where R^n *is the Ruschweyh differential operator, then the function* $F(z)$ *given by (2) belongs to the class* S.

Theorem 4. *Let* $n, m \in \mathbb{N} \cup \{0\}, \mu > 0, \alpha, \beta \in \mathbb{C}$, *with* $\operatorname{Re} \beta \geq \operatorname{Re} \alpha > 0$, *let* $f_i \in A$ *and let* $\alpha_i \in \mathbb{C}$, $i \in \{1, 2, \ldots, m\}$, *with* $|\alpha_1| + |\alpha_2| + \cdots + |\alpha_m| \leq \dfrac{1}{2\mu + 1}$.

If

(i) $|R^n f_i(z)| \leq \mu$,

(ii) $\left| \dfrac{z^2 (R^n f_i(z))'}{[R^n f_i(z)]^2} - 1 \right| \leq 1$, $z \in U$, $i \in \{1, 2, \ldots, m\}$

where R^n *is the Ruschweyh differential operator, then the function* $F(z)$ *given by (2') belongs to the class* S.

2 Main results

Theorem 5. *Let* $n, m \in \mathbb{N} \cup \{0\}$, α *be a complex number with* $\operatorname{Re}\alpha > 0$ *and* c *a complex number with* $|c| \le 1$, $c \ne -1$. *If* $f_k \in A$, $\alpha_k \in \mathbb{C}$, $k \in \{1, 2, \dots, m\}$ *and if*

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{z}{\alpha} \cdot \frac{f''(z)}{f'(z)} \right| \le 1, \tag{3}$$

holds for all $z \in U$, *where* f *is given by*

$$f(z) = \int_0^z \left(\frac{R^n f_1(t)}{t} \right)^{\alpha_1} \dots \left(\frac{R^n f_m(t)}{t} \right)^{\alpha_m} dt, \quad z \in U, \tag{4}$$

with R^n *given by Definition 1, then the function* F *given by (2), belongs to the class* S.

Proof. By differentiating (4), we obtain

$$f'(z) = \left(\frac{R^n f_1(z)}{z} \right)^{\alpha_1} \dots \left(\frac{R^n f_m(z)}{z} \right)^{\alpha_m} \tag{5}$$

$$= 1 + p_1 z + p_2 z^2 + \dots, \quad z \in U.$$

From (3) and (5) we have that $f'(0) = 1$ and $f'(z) \ne 0$, $z \in U$.
Since $f'(z) \ne 0$, we can use (5) and obtain

$$\log f'(z) = \alpha_1[\log R^n f_1(z) - \log z] + \dots + \tag{6}$$

$$+\alpha_m[\log R^n f_m(z) - \log z], \quad z \in U.$$

By differentiating (6), after a short calculation, we have

$$\frac{zf''(z)}{f'(z)} = \alpha_1 \left[\frac{z(R^n f_1(z))'}{R^n f_1(z)} - 1 \right] + \dots + \alpha_m \left[\frac{z(R^n f_m(z))'}{R^n f_m(z)} - 1 \right]. \tag{7}$$

Let

$$w(z,t) = ce^{-2t\alpha} + (1 - e^{-2t\alpha}) \frac{1}{\alpha} \cdot \frac{e^{-t} z f''(e^{-t}z)}{f'(e^{-t}z)}, \quad z \in \overline{U},\ t \ge 0, \tag{8}$$

where

$$\frac{e^{-t} z f''(e^{-t}z)}{f'(e^{-t}z)} = \sum_{k=1}^{m} \alpha_k \left[\frac{e^{-t} z [R^n f_k(e^{-t}z)]'}{R^n f_k(e^{-t}z)} - 1 \right], \quad z \in \overline{U},\ t \ge 0. \tag{9}$$

We evaluate

$$|w(z,t)| = \left| ce^{-2t\alpha} + (1 - e^{-2t\alpha}) \frac{e^{-t}}{\alpha} \cdot \frac{z f''(e^{-t}z)}{f'(e^{-t}z)} \right| \tag{10}$$

$$< \max_{|z|=1} |w(z,t)| = |w(e^{i\theta},t)| = \left| ce^{-2t\alpha} + (1-e^{-2t\alpha}) \frac{e^{-t}e^{i\theta} f''(e^{-t}e^{i\theta})}{\alpha f'(e^{-t}e^{i\theta})} \right|$$

Let

$$\zeta = e^{-t}e^{i\theta}, \tag{11}$$

from which we have that

$$|\zeta| = |e^{-t}e^{i\theta}| = |e^{-t}| = e^{-t}, \quad t > 0.$$

Using this in (10), we obtain

$$|w(z,t)| < \left| ce^{-t\cdot 2\alpha} + (1-e^{-t\cdot 2\alpha}) \frac{e^{-t}e^{i\theta} f''(e^{-t}e^{i\theta})}{\alpha f'(e^{-t}e^{i\theta})} \right| \tag{12}$$

$$= \left| c|\zeta|^{2\alpha} + (1-|\zeta|^{2\alpha}) \frac{\zeta f''(\zeta)}{\alpha f'(\zeta)} \right|$$

Since $|\zeta| = e^{-t} < 1$, we have that $\zeta \in U$. Then ζ can be replaced by $z \in U$ and (12) becomes:

$$|w(z,t)| < \left| c|z|^{2\alpha} + (1-|z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right|. \tag{13}$$

Using (3) we obtain

$$|w(z,t)| < \left| c|z|^{2\alpha} + (1-|z|^{2\alpha}) \frac{zf''(z)}{f'(z)} \right| \le 1, \tag{14}$$

for all $z \in U$ and $t > 0$.

If $t = 0$, then from (13) we have

$$|w(z,0)| = \left| c + (1-1) \frac{e^{i\theta} f''(e^{i\theta})}{\alpha f'(e^{i\theta})} \right| = |c|.$$

Using the conditions from Theorem 1, we have $|c| \le 1$, and hence

$$|w(z,0)| = |c| \le 1, \quad z \in \overline{U}. \tag{15}$$

From (14) and (15) we obtain

$$|w(z,t)| \le 1, \quad z \in \overline{U},\ t \ge 0.$$

Applying Lemma A, we obtain that the function F given by (2) belongs to the class S.

Example 1. Let $n \in \mathbb{N} \cup \{0\}$, $m = 2$, $\alpha = 5 + 4i$, $c = \dfrac{1}{3} + \dfrac{i\sqrt{3}}{3}$, $|c| = \dfrac{2}{3} < 1$, $f_1(z) = z + a_2 z$, $R^n f_1(z) = z + (n+1)a_2 z^2$, $f_2(z) = z + b_2 z^2$, $R^n f_2(z) = z + (n+1)b_2 z^2$, $a_2 \in \mathbb{C}$, $b_2 \in \mathbb{C}$, $z \in U$, $\alpha_1 = 2 + i$, $\alpha_2 = 3 - 4i$.

If

$$\left| \left(\frac{1}{3} + i\frac{\sqrt{3}}{3} \right) |z|^{10+8i} + (1 - |z|^{10+8i}) \frac{zf''(z)}{(5+4i)f'(z)} \right| \leq 1, \quad z \in U,$$

where $\dfrac{zf''(z)}{f'(z)}$ is given by (7), then from Theorem 5, we have

$$F(z) = \left[(5+4i) \int_0^z t^{4+4i}(1 + (n+1)a_2 t)(1 + (n+1)b_2 t)^{3-4i} dt \right]^{\frac{1}{5+4i}} \in S,$$

for all $z \in U$.

Theorem 6. *Let $n, m \in \mathbb{N} \cup \{0\}$, α and c be complex numbers with* $\operatorname{Re} \alpha > 0$, $|c| \leq 1$, $c \neq -1$ *and let $f_k \in A$, $\alpha_k \in \mathbb{C}$, $k \in \{1, 2, \ldots, m\}$.*

If

$$\left| (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1 - |c|, \tag{16}$$

where $\dfrac{zf''(z)}{f'(z)}$ is given by (7), holds for all $z \in U$, then the function F given by (2) is univalent in U.

Proof. If $z \in \mathbb{C}$, $|z| \leq 1$, then $|z|^{2\alpha} \leq 1$. Using $|z|^{2\alpha} \leq 1$ and the conditions from Theorem 2, we have

$$\begin{aligned} \left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right| &\leq |c||z|^{2\alpha} + \left| (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right| \\ &\leq |c| + 1 - |c| = 1. \end{aligned} \tag{17}$$

Using Theorem 5, we obtain that F given by (2) belongs to the class S. □

Example 2. Let $n \in \mathbb{N} \cup \{0\}$, $m = 2$, $\alpha = 3 - 2i$, $c = \dfrac{1}{5} + i\dfrac{\sqrt{3}}{5}$, $|c| = \dfrac{2}{5}$,

$$f(z) = z + a_2 z^2 + a_3 z^3, \quad f_2(z) = z + b_2 z^2 + b_3 z^3,$$

$$R^n f_1(z) = z + (n+1)a_2 z^2 + \frac{(n+1)(n+2)}{2} a_3 z^3,$$

$$R^n f_2(z) = z + (n+1)b_2 z^2 + \frac{(n+1)(n+2)}{2} b_3 z^3,$$

$a_2, a_3, b_2, b_3 \in \mathbb{C}$, $z \in U$, $\alpha_1 = 1 - i$, $\alpha_2 = 2 + i$. If

$$\left|(1-|z|^{(6-4i)})\frac{zf''(z)}{(3-4i)f'(z)}\right| \le \frac{3}{5},$$

for all $z \in U$, where $\dfrac{zf''(z)}{f'(z)}$ is given by (7), from Theorem 6 we have

$$F(z) = \left[(3-2i)\int_0^z t^{(2-2i)}\left(1+(n+1)a_2t+\frac{(n+1)(n+2)}{2}a_3t^2\right)^{(1-i)} \cdot \right.$$

$$\left. \cdot \left(1+(n+1)b_2t+\frac{(n+1)(n+2)}{2}b_3t^2\right)^{2+i} dt\right]^{\frac{1}{3-2i}} \in S,$$

for all $z \in U$.

Theorem 7. *Let $n, m \in \mathbb{N} \cup \{0\}$, α and c be complex numbers with* $\mathrm{Re}\,\alpha > 0$, $|c| \le 1$, $c \ne -1$ *and let $f_k \in A$, $\alpha_k \in \mathbb{C}$, $k \in \{1, 2, \ldots, m\}$.*
If

$$i)\quad |\alpha_1| + |\alpha_2| + \cdots + |\alpha_m| \le 1 - |c|$$

$$ii)\quad \left|\frac{1-|z|^{2\alpha}}{\alpha}\left[\frac{z(R^nf_k(z))'}{R^nf_k(z)} - 1\right]\right| \le 1, \quad z \in U,\ k \in \{1, 2, \ldots, m\}, \tag{18}$$

where R^n is the Ruscheweyh differential operator, then the function F given by (2) belongs to the class S.

Proof. In order to prove the theorem, we evaluate:

$$\left|c|z|^{2\alpha} + \frac{1-|z|^{2\alpha}}{\alpha}\cdot\frac{zf''(z)}{f'(z)}\right| \le |c||z|^{2\alpha}| + \left|\frac{1-|z|^{2\alpha}}{\alpha}\cdot\frac{zf''(z)}{f'(z)}\right| \tag{19}$$

$$\le |c| + \left|\frac{1-|z|^{2\alpha}}{\alpha}\cdot\frac{zf''(z)}{f'(z)}\right|, \quad z \in U$$

$$\left|\frac{1-|z|^{2\alpha}}{\alpha}\cdot\frac{zf''(z)}{f'(z)}\right| = \left|\frac{1-|z|^{2\alpha}}{\alpha}\cdot\alpha_1\left[\frac{(R^nf_1(z))'}{R^nf_1(z)} - 1\right]\right. \tag{20}$$

$$\left. + \frac{1-|z|^{2\alpha}}{\alpha}\cdot\alpha_2\left[\frac{(R^nf_2(z))'}{R^nf_2(z)} - 1\right] + \cdots + \frac{1-|z|^{2\alpha}}{\alpha}\cdot\alpha_m\left[\frac{(R^nf_m(z))'}{R^nf_m(z)} - 1\right]\right|$$

$$\le |\alpha_1|\left|\frac{1-|z|^{2\alpha}}{\alpha}\left[\frac{(R^nf_1(z))'}{R^nf_1(z)} - 1\right]\right| + |\alpha_2|\left|\frac{1-|z|^{2\alpha}}{\alpha}\left[\frac{(R^nf_2(z))'}{R^nf_2(z)} - 1\right]\right| +$$

$$+ \cdots + |\alpha_m|\left|\frac{1-|z|^{2\alpha}}{\alpha}\left[\frac{(R^nf_m(z))'}{R^nf_m(z)} - 1\right]\right| \le |\alpha_1| + \cdots + |\alpha_m| \le 1 - |c|.$$

Using (20) in (19), we obtain

$$\left|c|z|^{2\alpha} + \frac{1-|z|^{2\alpha}}{\alpha} \cdot \frac{zf''(z)}{f'(z)}\right| \leq c + 1 - |c| = 1.$$

By applying Theorem 5, we obtain that function F given by (2) belongs to the class S.

Example 3. Let $n \in \mathbb{N} \cup \{0\}$, $m = 2$, $\alpha = 4+7i$, $c = \frac{1}{4} - \frac{i\sqrt{3}}{4}$, $|c| = \frac{1}{2}$, $\alpha_1 = \frac{1}{8} + \frac{i\sqrt{3}}{8}$, $|\alpha_1| = \frac{1}{4}$, $\alpha_2 = \frac{1}{10} + \frac{i\sqrt{3}}{10}$, $|\alpha_2| = \frac{1}{5}$, $|\alpha_1| + |\alpha_2| = \frac{9}{20} < \frac{1}{2}$, $f_1(z) = z + a_2z^2$, $R^n f_1(z) = z + (n+1)a_2z^2$, $f_2(z) = z + b_2z^2$, $R^n f_2(z) = z + (n+1)b_2z^2$, $a, b \in \mathbb{C}$, $|a_2| \leq \frac{1}{2(n+1)}$, $|b_2| \leq \frac{1}{2(n+1)}$, $z \in U$.

We evaluate (18i) and (18ii)

$$(18i) \qquad |\alpha_1| + |\alpha_2| = \frac{9}{20} < \frac{1}{2}$$

$$(18ii) \qquad \left|\frac{1-|z|^{2(4+7i)}}{4+7i}\left[\frac{z(1+2(n+1)a_2z)}{z+(n+1)a_2z^2} - 1\right]\right|$$

$$= \frac{1-|z|^{2(4+7i)}}{\sqrt{65}}\left|\frac{(n+1)a_2z}{1+(n+1)a_2z}\right| \leq \frac{1-|z|}{\sqrt{65}} \leq \frac{1}{\sqrt{65}} < 1.$$

$$\left|\frac{1-|z|^{2(4+7i)}}{4+7i}\left[\frac{z(1+2(n+1)b_2z)}{z+(n+1)b_2z^2} - 1\right]\right| = \frac{1-|z|^{2(4+7i)}}{\sqrt{65}}\left|\frac{(n+1)b_2z}{1+(n+1)b_2z}\right|$$

$$\leq \frac{1-|z|}{\sqrt{65}} \leq \frac{1}{\sqrt{65}} < 1.$$

Using Theorem 7, we have

$$F(z) = \left[(4+7i)\int_0^z t^{3+7i}(1+(n+1)a_2t)^{\left(\frac{1}{8}+\frac{i\sqrt{3}}{8}\right)}\cdot\right.$$

$$\left.\cdot(1+(n+1)b_2t)^{\left(\frac{1}{10}+\frac{i\sqrt{3}}{10}\right)}dt\right]^{\frac{1}{4+7i}} \in S,$$

for all $z \in U$.

An open problem. Similar results can be obtained by using other differential operators.

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