On an Univalent Integral Operator

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Abstract

The integral operator denoted by $I(f_1, f_2, \ldots, f_m)$ given in Definition 2 was introduced in [8]. Also, certain sufficient conditions of univalence of this operator were given there. In this paper, a different approach for proving the univalence of this operator is taken.

Keywords: analytic function, univalent function, differential operator, integral operator.

2000 Mathematical Subject Classification: 30C45, 30A20, 34A40.

1 Introduction and preliminaries

Let $U$ denote the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

and

$$\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}.$$

Let $\mathcal{H}(U)$ denote the space of holomorphic functions in $U$ and let

$$A_n = \{f \in \mathcal{H}(U), \ f(z) = z + a_{n+1}z^{n+1} + \ldots, \ z \in U\}$$

with $A_1 = A$.

Let

$$S = \{f \in A : f \text{ is univalent in } U\}.$$
**Definition 1.** (St. Ruscheweyh [16]). For \( f \in A, n \in \mathbb{N} \cup \{0\}, \) let \( R^n \) be the operator defined by
\[
R^0 f(z) = f(z) \\
(n + 1)R^{n+1} f(z) = z[R^n f(z)]' + nR^n f(z), \quad z \in U.
\]

**Remark 1.** If \( f \in A \)
\[
f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad z \in U
\]
then
\[
R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^n a_j z^j, \quad z \in U. \tag{1}
\]

In order to prove our main results, we shall use the following lemma:

**Lemma A.** [12] Let \( \alpha \) and \( c \) be complex numbers with \( \Re \alpha > 0, |c| \leq 1, c \neq -1 \) and \( f(z) = z + a_2 z^2 + \ldots \) be a regular function in \( U \). If
\[
\left| ce^{-2\alpha \alpha} + (1 - e^{-2\alpha \alpha}) \frac{e^{-t \alpha} f''(e^{-t \alpha} z)}{\alpha f'(e^{-t \alpha} z)} \right| \leq 1
\]
holds for every \( z \in U \) and \( t \geq 0 \), then the function
\[
F_\alpha(z) = \left[ \alpha \int_0^z u^{\alpha-1} f'(u) du \right]^{\frac{1}{\alpha}}
\]
is regular and univalent in \( U \).

In [8] we have defined the following integral operator:

**Definition 2.** [8] Let \( n, m \in \mathbb{N} \cup \{0\}, i \in \{1, 2, 3, \ldots, m\}, \alpha_i \in \mathbb{C}, A^m = A \times A \times \cdots \times A \). We let \( I : A^m \to A \) be the integral operator given by
\[
I(f_1, f_2, \ldots, f_m)(z) = F(z) \tag{2}
\]
\[
= \left[ \alpha \int_0^z t^{\alpha-1} \left( \frac{R^n f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{R^n f_m(t)}{t} \right)^{\alpha_m} dt \right]^{\frac{1}{\alpha}},
\]
where \( f_i \in A, i \in \{1, 2, 3, \ldots, n\} \) and \( R^n \) is the Ruscheweyh differential operator (Definition 1).

**Remark 2.** In [8], the author has shown that this operator is a generalization
of other operators studied in [1], [2], [3], [4], [5], [6], [7], [9], [10], [11], [12], [13], [14], [15].

**Remark 3.** In [8], the author has stated and proven the following theorems:

**Theorem 1.** Let \( n, m \in \mathbb{N} \cup \{0\}, \alpha \in \mathbb{C} \) with \( \text{Re} \alpha > 0 \), \( f_i \in A, \alpha_i \in \mathbb{C}, i \in \{1, 2, \ldots, m\} \) with \( |\alpha_1| + |\alpha_2| + \cdots + |\alpha_m| \leq 1 \).

If
\[
\left| \frac{z(R^n f_i(z))'}{R^n f_i(z)} - 1 \right| \leq 1, \quad z \in U, \ i \in \{1, 2, \ldots, m\}
\]
then \( F(z) \) given by (2) belongs to the class \( S \).

**Theorem 2.** Let \( n, m \in \mathbb{N} \cup \{0\}, \alpha, \beta \in \mathbb{C}, \) with \( \text{Re} \beta \geq \text{Re} \alpha > 0 \), let \( f_i \in A \) and let \( \alpha_i \in \mathbb{C}, i \in \{1, 2, \ldots, m\} \), with \( |\alpha_1| + |\alpha_2| + \cdots + |\alpha_m| \leq 1 \).

If
\[
\left| \frac{z(R^n f_i(z))'}{R^n f_i(z)} - 1 \right| \leq 1, \quad z \in U, \ i \in \{1, 2, \ldots, m\}
\]
where \( R^n \) is the Ruschweyh differential operator, then the function given by
\[
(2') \quad F_\beta(z) = \left[ \beta \int_0^z t^{\beta-1} \left( \frac{R^n f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{R^n f_m(t)}{t} \right)^{\alpha_m} dt \right]^{\frac{1}{n}}
\]
belongs to the class \( S \).

**Theorem 3.** Let \( n, m \in \mathbb{N} \cup \{0\}, \mu > 0, \alpha \in \mathbb{C} \) with \( \text{Re} \alpha > 0 \), \( f_i \in A, \alpha_i \in \mathbb{C}, i \in \{1, 2, \ldots, m\} \) with \( |\alpha_1| + |\alpha_2| + \cdots + |\alpha_m| \leq \frac{1}{2\mu + 1} \).

If
\[
(i) \quad |R^n f_i(z)| \leq \mu,
(ii) \quad \left| \frac{z^2(R^n f_i(z))'}{R^n f_i(z)} - 1 \right| \leq 1, \quad z \in U, \ i \in \{1, 2, \ldots, m\}
\]
where \( R^n \) is the Ruschweyh differential operator, then the function \( F(z) \) given by (2) belongs to the class \( S \).

**Theorem 4.** Let \( n, m \in \mathbb{N} \cup \{0\}, \mu > 0, \alpha, \beta \in \mathbb{C}, \) with \( \text{Re} \beta \geq \text{Re} \alpha > 0 \), let \( f_i \in A \) and let \( \alpha_i \in \mathbb{C}, i \in \{1, 2, \ldots, m\} \), with \( |\alpha_1| + |\alpha_2| + \cdots + |\alpha_m| \leq \frac{1}{2\mu + 1} \).

If
\[
(i) \quad |R^n f_i(z)| \leq \mu,
(ii) \quad \left| \frac{z^2(R^n f_i(z))'}{R^n f_i(z)} - 1 \right| \leq 1, \quad z \in U, \ i \in \{1, 2, \ldots, m\}
\]
where \( R^n \) is the Ruschweyh differential operator, then the function \( F(z) \) given by (2’) belongs to the class \( S \).
2 Main results

Theorem 5. Let \(n, m \in \mathbb{N} \cup \{0\}\), \(\alpha\) be a complex number with \(\text{Re} \alpha > 0\) and \(c\) a complex number with \(|c| \leq 1\), \(c \neq -1\). If \(f_k \in A\), \(\alpha_k \in \mathbb{C}\), \(k \in \{1, 2, \ldots, m\}\) and if

\[
|c| |z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{z}{\alpha} \cdot \frac{f''(z)}{f'(z)} \leq 1, \tag{3}
\]

holds for all \(z \in U\), where \(f\) is given by

\[
f(z) = \int_0^z \left( \frac{R^n f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{R^n f_m(t)}{t} \right)^{\alpha_m} dt, \quad z \in U, \tag{4}
\]

with \(R^n\) given by Definition 1, then the function \(F\) given by (2), belongs to the class \(S\).

Proof. By differentiating (4), we obtain

\[
f'(z) = \left( \frac{R^n f_1(z)}{z} \right)^{\alpha_1} \cdots \left( \frac{R^n f_m(z)}{z} \right)^{\alpha_m}
\]

\[= 1 + p_1 z + p_2 z^2 + \ldots, \quad z \in U. \tag{5}\]

From (3) and (5) we have that \(f'(0) = 1\) and \(f'(z) \neq 0, z \in U\).

Since \(f'(z) \neq 0\), we can use (5) and obtain

\[
\log f'(z) = \alpha_1 [\log R^n f_1(z) - \log z] + \cdots + 
\]

\[+ \alpha_m [\log R^n f_m(z) - \log z], \quad z \in U. \tag{6}\]

By differentiating (6), after a short calculation, we have

\[
\frac{zf''(z)}{f'(z)} = \alpha_1 \left[ \frac{z(R^n f_1(z))'}{R^n f_1(z)} - 1 \right] + \cdots + \alpha_m \left[ \frac{z(R^n f_m(z))'}{R^n f_m(z)} - 1 \right]. \tag{7}\]

Let

\[
w(z, t) = c e^{-2\alpha t} + (1 - e^{-2\alpha t}) \frac{1}{\alpha} \cdot \frac{e^{-t} z f''(e^{-t} z)}{f'(e^{-t} z)}, \quad z \in \overline{U}, \ t \geq 0, \tag{8}\]

where

\[
\frac{e^{-t} z f''(e^{-t} z)}{f'(e^{-t} z)} = \sum_{k=1}^m \alpha_k \left[ \frac{e^{-t} z [R^n f_k(e^{-t} z)]'}{R^n f_k(e^{-t} z)} - 1 \right], \quad z \in \overline{U}, \ t \geq 0. \tag{9}\]

We evaluate

\[
|w(z, t)| = \left| c e^{-2\alpha t} + (1 - e^{-2\alpha t}) \frac{e^{-t} z f''(e^{-t} z)}{f'(e^{-t} z)} \right| \tag{10}\]
\[
< \max_{|z|=1} |w(z, t)| = |w(e^{i\theta}, t)| = \left| ce^{-2t\alpha} + (1 - e^{-2t\alpha}) \frac{e^{-t} f''(e^{-t} e^{i\theta})}{\alpha f'(e^{-t} e^{i\theta})} \right|
\]

Let
\[
\zeta = e^{-t} e^{i\theta},
\]
from which we have that
\[
|\zeta| = |e^{-t} e^{i\theta}| = |e^{-t}| = e^{-t}, \quad t > 0.
\]

Using this in (10), we obtain
\[
|w(z, t)| < \left| ce^{-t} 2\alpha + (1 - e^{-t} 2\alpha) \frac{\zeta f''(\zeta)}{\alpha f'(\zeta)} \right| \tag{12}
\]

Since \(|\zeta| = e^{-t} < 1\), we have that \(\zeta \in U\). Then \(\zeta\) can be replaced by \(z \in U\) and (12) becomes:
\[
|w(z, t)| < \left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{f'(z)} \right|. \tag{13}
\]

Using (3) we obtain
\[
|w(z, t)| < \left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{f'(z)} \right| \leq 1, \tag{14}
\]
for all \(z \in U\) and \(t > 0\).

If \(t = 0\), then from (13) we have
\[
|w(z, 0)| = \left| c + (1 - 1) \frac{e^{i\theta} f''(e^{i\theta})}{\alpha f'(e^{i\theta})} \right| = |c|.
\]

Using the conditions from Theorem 1, we have \(|c| \leq 1\), and hence
\[
|w(z, 0)| = |c| \leq 1, \quad z \in \overline{U}. \tag{15}
\]

From (14) and (15) we obtain
\[
|w(z, t)| \leq 1, \quad z \in \overline{U}, \quad t \geq 0.
\]

Applying Lemma A, we obtain that the function \(F\) given by (2) belongs to the class \(S\).
Example 1. Let \( n \in \mathbb{N} \cup \{0\} \), \( m = 2 \), \( \alpha = 5 + 4i \), \( c = \frac{1}{3} + \frac{i\sqrt{3}}{3} \), \( |c| = \frac{2}{3} < 1 \), \( f_1(z) = z + a_2z^2 \), \( R^n f_1(z) = z + (n + 1)a_2z^2 \), \( f_2(z) = z + b_2z^2 \), \( R^n f_2(z) = z + (n + 1)b_2z^2 \), \( a_2 \in \mathbb{C} \), \( b_2 \in \mathbb{C} \), \( z \in U \), \( \alpha_1 = 2 + i \), \( \alpha_2 = 3 - 4i \).

If
\[
\left| \left( \frac{1}{3} + i\frac{\sqrt{3}}{3} \right) |z|^{10 + 8i} + (1 - |z|^{10 + 8i}) \frac{zf''(z)}{(5 + 4i)f'(z)} \right| \leq 1, \quad z \in U,
\]

where \( \frac{zf''(z)}{f'(z)} \) is given by (7), then from Theorem 5, we have

\[
F(z) = \left( 5 + 4i \right) \int_0^z t^{4 + 4i} (1 + (n + 1)a_2t)(1 + (n + 1)b_2t)^{3 - 4i} dt \right]^{\frac{1}{5 + 4i}} \in S,
\]

for all \( z \in U \).

Theorem 6. Let \( n, m \in \mathbb{N} \cup \{0\} \), \( \alpha \) and \( c \) be complex numbers with \( \text{Re} \alpha > 0 \), \( |c| \leq 1 \), \( c \neq -1 \) and let \( f_k \in A \), \( \alpha_k \in \mathbb{C} \), \( k \in \{1, 2, \ldots, m\} \).

If
\[
\left| (1 - |z|^{2\alpha}) \frac{zf''(z)}{af'(z)} \right| \leq 1 - |c|, \tag{16}
\]

where \( \frac{zf''(z)}{f'(z)} \) is given by (7), holds for all \( z \in U \), then the function \( F \) given by (2) is univalent in \( U \).

Proof. If \( z \in \mathbb{C} \), \( |z| \leq 1 \), then \( |z|^{2\alpha} \leq 1 \). Using \( |z|^{2\alpha} \leq 1 \) and the conditions from Theorem 2, we have

\[
\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{af'(z)} \right| \leq |c||z|^{2\alpha} + \left| (1 - |z|^{2\alpha}) \frac{zf''(z)}{af'(z)} \right| \tag{17}
\]

\[
\leq |c| + 1 - |c| = 1.
\]

Using Theorem 5, we obtain that \( F \) given by (2) belongs to the class \( S \). \( \square \)

Example 2. Let \( n \in \mathbb{N} \cup \{0\} \), \( m = 2 \), \( \alpha = 3 - 2i \), \( c = \frac{1}{5} + \frac{i\sqrt{3}}{5} \), \( |c| = \frac{2}{5} \),

\[
f(z) = z + a_3z^2 + a_3z^3, \quad f_2(z) = z + b_3z^2 + b_3z^3,
\]

\[
R^n f_1(z) = z + (n + 1)a_2z^2 + \frac{(n + 1)(n + 2)}{2}a_3z^3,
\]

\[
R^n f_2(z) = z + (n + 1)b_2z^2 + \frac{(n + 1)(n + 2)}{2}b_3z^3,
\]
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\[ a_2, a_3, b_2, b_3 \in \mathbb{C}, \ z \in U, \ \alpha_1 = 1 - i, \ \alpha_2 = 2 + i. \text{ If} \]

\[ \left| (1 - |z|^{6-4i}) \frac{zf''(z)}{(3-4i)f'(z)} \right| \leq \frac{3}{5}, \]

for all \( z \in U \), where \( \frac{zf''(z)}{f'(z)} \) is given by (7), from Theorem 6 we have

\[ F(z) = \left[ (3-2i) \int_0^z t^{(2-2i)} \left( 1 + (n+1)a_2t + \frac{(n+1)(n+2)}{2}a_3t^2 \right)^{1-i} \right. \]

\[ \cdot \left. \left( 1 + (n+1)b_2t + \frac{(n+1)(n+2)}{2}b_3t^2 \right)^{2+i} dt \right] \frac{1}{i - 2} \in S, \]

for all \( z \in U \).

**Theorem 7.** Let \( n, m \in \mathbb{N} \cup \{0\} \), \( \alpha \) and \( c \) be complex numbers with \( \text{Re} \ \alpha > 0 \), \( |c| \leq 1 \), \( c \neq -1 \) and let \( f_k \in A \), \( \alpha_k \in \mathbb{C} \), \( k \in \{1, 2, \ldots, m\} \). If

\[ i) \ \ |\alpha_1| + |\alpha_2| + \ldots + |\alpha_m| \leq 1 - |c| \]

\[ ii) \ \ \left| \frac{1 - |z|^{2\alpha}}{\alpha} \cdot \left( z(R^n f_k(z))' - 1 \right) \right| \leq 1, \]

(18)

where \( R^n \) is the Ruscheweyh differential operator, then the function \( F \) given by (2) belongs to the class \( S \).

**Proof.** In order to prove the theorem, we evaluate:

\[ \left| cz^{2\alpha} + \frac{1 - |z|^{2\alpha}}{\alpha} \cdot \frac{zf''(z)}{f'(z)} \right| \leq |c||z|^{2\alpha} + \left| \frac{1 - |z|^{2\alpha}}{\alpha} \cdot \frac{zf''(z)}{f'(z)} \right| \]

(19)

\[ \leq |c| + \left| \frac{1 - |z|^{2\alpha}}{\alpha} \cdot \frac{zf''(z)}{f'(z)} \right|, \quad z \in U \]

\[ \left| \frac{1 - |z|^{2\alpha}}{\alpha} \cdot \frac{zf''(z)}{f'(z)} \right| = \left| 1 - \frac{|z|^{2\alpha}}{\alpha} \cdot \alpha_1 \left[ \frac{(R^n f_1(z))'}{R^n f_1(z)} - 1 \right] \right| + \frac{1 - |z|^{2\alpha}}{\alpha} \cdot \alpha_2 \left[ \frac{(R^n f_2(z))'}{R^n f_2(z)} - 1 \right] + \ldots + \frac{1 - |z|^{2\alpha}}{\alpha} \cdot \alpha_m \left[ \frac{(R^n f_m(z))'}{R^n f_m(z)} - 1 \right] \]

(20)

\[ \leq |\alpha_1| \left| 1 - \frac{|z|^{2\alpha}}{\alpha} \cdot \frac{(R^n f_1(z))'}{R^n f_1(z)} - 1 \right| + |\alpha_2| \left| 1 - \frac{|z|^{2\alpha}}{\alpha} \cdot \frac{(R^n f_2(z))'}{R^n f_2(z)} - 1 \right| + \ldots + |\alpha_m| \left| 1 - \frac{|z|^{2\alpha}}{\alpha} \cdot \frac{(R^n f_m(z))'}{R^n f_m(z)} - 1 \right| \leq |\alpha_1| + \cdots + |\alpha_m| \leq 1 - |c|. \]
Using (20) in (19), we obtain
\[ \left| c \cdot |z|^{2\alpha} + \frac{1 - |z|^{2\alpha}}{\alpha} \cdot \frac{zf''(z)}{f'(z)} \right| \leq c + 1 - |c| = 1. \]

By applying Theorem 5, we obtain that function \( F \) given by (2) belongs to the class \( S \).

**Example 3.** Let \( n \in \mathbb{N} \cup \{0\} \), \( m = 2 \), \( \alpha = 4 + 7i \), \( c = \frac{1}{4} - \frac{i\sqrt{3}}{4} \), \( |c| = \frac{1}{2} \), \( \alpha_1 = \frac{1}{8} + \frac{i\sqrt{3}}{8} \), \( \alpha_2 = \frac{1}{10} + \frac{i\sqrt{3}}{10} \), \( |\alpha_1| = \frac{1}{5} \), \( |\alpha_2| = \frac{9}{20} < \frac{1}{2} \), \( f_1(z) = z + a_2z^2 \), \( R^nf_1(z) = z + (n + 1)a_2z^2 \), \( f_2(z) = z + b_2z^2 \), \( R^nf_2(z) = z + (n + 1)b_2z^2 \), \( a, b \in \mathbb{C} \), \( |a| \leq \frac{1}{2(n+1)} \), \( |b| \leq \frac{1}{2(n+1)} \), \( z \in U \).

We evaluate (18i) and (18ii)

\[(18i) \quad |\alpha_1| + |\alpha_2| = \frac{9}{20} < \frac{1}{2} \]

\[(18ii) \quad \left| 1 - \frac{|z|^{2(4+7i)}}{4 + 7i} \left[ \frac{z(1 + 2(n+1)a_2z)}{z + (n+1)a_2z^2} - 1 \right] \right| \]

\[= \frac{1 - |z|^{2(4+7i)}}{\sqrt{65}} \left| \frac{(n+1)a_2z}{1 + (n+1)a_2z} \right| \leq \frac{1 - |z|}{\sqrt{65}} \leq \frac{1}{\sqrt{65}} < 1. \]

\[\left| 1 - \frac{|z|^{2(4+7i)}}{4 + 7i} \left[ \frac{z(1 + 2(n+1)b_2z)}{z + (n+1)b_2z^2} - 1 \right] \right| = \frac{1 - |z|^{2(4+7i)}}{\sqrt{65}} \left| \frac{(n+1)b_2z}{1 + (n+1)b_2z} \right| \]

\[\leq \frac{1 - |z|}{\sqrt{65}} \leq \frac{1}{\sqrt{65}} < 1. \]

Using Theorem 7, we have

\[ F(z) = \left[ (4 + 7i) \int_{0}^{z} t^{3+7i}(1 + (n+1)a_2t) \left( \frac{t^{\frac{1}{2} + \frac{i\sqrt{3}}{4}}} {\left( \frac{1}{2} + \frac{i\sqrt{3}}{4} \right)} \right) \cdot (1 + (n+1)b_2t) \left( \frac{1}{\sqrt{65}} \right) dt \right]^{\frac{1}{1+7i}} \in S, \]

for all \( z \in U \).

**An open problem.** Similar results can be obtained by using other differential operators.
References


