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On Sufficient Conditions for Certain Subclass of Analytic Functions Defined by Differential Salagean Operator

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Abstract

By means of the Sălăgean differential operator we define a new class $\mathcal{BS}(m, \mu, \alpha)$ involving functions $f \in \mathcal{A}_n$. Parallel results, for some related classes including the class of starlike and convex functions respectively, are also obtained.

Keywords: *Analytic function, starlike function, convex function, Sălăgean differential operator.*
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1 Introduction and definitions

Let $\mathcal{A}_n$ denote the class of functions of the form

$$f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j \tag{1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U, $n \in \mathbb{N} = \{1, 2, ...\}$.

Let $\mathcal{S}_n$ denote the subclass of functions that are univalent in U.

By $\mathcal{S}_n^*(\alpha)$ we denote a subclass of $\mathcal{A}_n$ consisting of starlike univalent functions of order α, $0 \le \alpha < 1$ which satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U. \tag{2}$$

Further, a function f belonging to $\mathcal{S}_n$ is said to be convex of order α in U, if and only if

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha, \quad z \in U \tag{3}$$

for some α, $(0 \le \alpha < 1)$. We denote by $\mathcal{K}_n(\alpha)$ the class of functions in $\mathcal{S}_n$ which are convex of order α in U and denote by $\mathcal{R}_n(\alpha)$ the class of functions in $\mathcal{A}_n$ which satisfy

$$\operatorname{Re} f'(z) > \alpha, \quad z \in U. \tag{4}$$

It is well known that $\mathcal{K}_n(\alpha) \subset \mathcal{S}_n^*(\alpha) \subset \mathcal{S}_n$.

Let D^m be the Sălăgean differential operator [3], $D^m : \mathcal{A}_n \to \mathcal{A}_n$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, defined as

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= Df(z) = zf'(z) \\ D^m f(z) &= D(D^{m-1} f(z)), \quad z \in U. \end{aligned}$$

We note that if $f \in \mathcal{A}_n$, then

$$D^m f(z) = z + \sum_{j=n+1}^{\infty} j^m a_j z^j, \quad z \in U.$$

To prove our main theorem we shall need the following lemma.

Lemma 1. [*2*] *Let p be analytic in U with $p(0) = 1$ and suppose that*

$$\operatorname{Re}\left(1 + \frac{zp'(z)}{p(z)}\right) > \frac{3\alpha - 1}{2\alpha}, \quad z \in U. \tag{5}$$

Then $\operatorname{Re} p(z) > \alpha$ for $z \in U$ and $1/2 \le \alpha < 1$.

2 Main results

Definition 1. *We say that a function $f \in \mathcal{A}_n$ is in the class $\mathcal{BS}_n(m, \mu, \alpha)$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\mu \ge 0$, $\alpha \in [0, 1)$ if*

$$\left| \frac{D^{m+1} f(z)}{z} \left(\frac{z}{D^m f(z)} \right)^{\mu} - 1 \right| < 1 - \alpha, \quad z \in U. \tag{6}$$

Remark 1. *The family $\mathcal{BS}_n(m,\mu,\alpha)$ is a new comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, $\mathcal{BS}_n(0,1,\alpha)\equiv\mathcal{S}_n^*(\alpha)$, $\mathcal{BS}_n(1,1,\alpha)\equiv\mathcal{K}_n(\alpha)$ and $\mathcal{BS}_n(0,0,\alpha)\equiv\mathcal{R}_n(\alpha)$. Another interesting subclass is the special case $\mathcal{BS}_1(0,2,\alpha)\equiv\mathcal{B}(\alpha)$ which has been introduced by Frasin and Darus [1] and also the class $\mathcal{BS}_1(0,\mu,\alpha)\equiv\mathcal{B}(\mu,\alpha)$ which has been introduced by Frasin and Jahangiri [2].*

In this note we provide a sufficient condition for functions to be in the class $\mathcal{BS}_n(m,\mu,\alpha)$. Consequently, as a special case, we show that convex functions of order $1/2$ are also members of the above defined family.

Theorem 2. *For the function $f\in\mathcal{A}_n$, $n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, $\mu\geq 0$, $1/2\leq\alpha<1$ if*

$$\operatorname{Re}\left(\frac{D^{m+2}f(z)}{D^{m+1}f(z)}-\mu\frac{D^{m+1}f(z)}{D^{m}f(z)}+\mu\right)>\beta,\quad z\in U \tag{7}$$

where

$$\beta=\frac{3\alpha-1}{2\alpha}$$

then $f\in\mathcal{BS}_n(m,\mu,\alpha)$.

Proof. If we consider

$$p(z)=\frac{D^{m+1}f(z)}{z}\left(\frac{z}{D^{m}f(z)}\right)^{\mu} \tag{8}$$

then $p(z)$ is analytic in U with $p(0)=1$. A simple differentiation yields

$$\frac{zp'(z)}{p(z)}=\frac{D^{m+2}f(z)}{D^{m+1}f(z)}-\mu\frac{D^{m+1}f(z)}{D^{m}f(z)}+\mu-1. \tag{9}$$

Using (7) we get

$$\operatorname{Re}\left(1+\frac{zp'(z)}{p(z)}\right)>\frac{3\alpha-1}{2\alpha}.$$

Thus, from Lemma 1 we deduce that

$$\operatorname{Re}\left\{\frac{D^{m+1}f(z)}{z}\left(\frac{z}{D^{m}f(z)}\right)^{\mu}\right\}>\alpha.$$

Therefore, $f\in\mathcal{BS}_n(m,\mu,\alpha)$, by Definition 1. □

As a consequence of the above theorem we have the following interesting corollaries.

Corollary 3. *If $f \in \mathcal{A}_n$ and*

$$\operatorname{Re}\left\{\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right\} > -\frac{1}{2}, \quad z \in U \tag{10}$$

then

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2}, \quad z \in U. \tag{11}$$

That is, f is convex of order $\frac{1}{2}$.

Corollary 4. *If $f \in \mathcal{A}_n$ and*

$$\operatorname{Re}\left\{\frac{2z^2f''(z)+z^3f'''(z)}{zf'(z)+z^2f''(z)}\right\} > -\frac{1}{2}, \quad z \in U \tag{12}$$

then

$$\operatorname{Re}\left\{f'(z)+zf''(z)\right\} > \frac{1}{2}, \quad z \in U. \tag{13}$$

Corollary 5. *If $f \in \mathcal{A}_n$ and*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2}, \quad z \in U \tag{14}$$

then

$$\operatorname{Re} f'(z) > \frac{1}{2}, \quad z \in U. \tag{15}$$

In another words, if the function f is convex of order $\frac{1}{2}$ then $f \in \mathcal{BS}_n(0,0,\frac{1}{2}) \equiv \mathcal{R}_n\left(\frac{1}{2}\right)$.

Corollary 6. *If $f \in \mathcal{A}_n$ and*

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right\} > -\frac{3}{2}, \quad z \in U \tag{16}$$

then f is starlike of order $\frac{1}{2}$.

3 Open Problem

The open problem is to define a generic class of analytic functions such that the class $\mathcal{BS}_n(m,\mu,\alpha)$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\mu \geq 0$, $\alpha \in [0,1)$ is contained inside and is possible to obtain. Compare the new results with the results given by [2].

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