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Radius Problems of Certain Analytic Functions

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Abstract

For analytic functions $f(z)$ ***normalized by*** $f(0) = 0$ ***and*** $f'(0) = 1$ ***in the open unit disk*** $\mathbb{U}$***, a class*** $P_4(\lambda)$ ***of*** $f(z)$ ***defined by some inequality for*** $f(z)$ ***is introduced. In the present paper, we discuss the problem such that*** $\frac{1}{\alpha}f(\alpha z)\in P_4(\lambda)$ ***for*** $f(z)\in\mathcal{S}$***. Also for our result, an open problem concern in Hölder inequality is given.***

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1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z\in\mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$. For $f(z)\in\mathcal{A}$, we say that $f(z)\in P_4(\lambda)$ if $f(z)$ satisfies $\dfrac{f(z)}{z} \neq 0 \quad (z\in\mathbb{U})$ and

$$\left| \left(\frac{z}{f(z)} \right)'''' \right| \leqq \lambda \qquad (z\in\mathbb{U}) \tag{1.2}$$

for some real $\lambda > 0$. Obradović and Ponnusamy [2] have studied the subclass $P_2(\lambda)$ of $\mathcal{A}$ consisting of $f(z)$ satisfying $\dfrac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U})$ and

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leqq \lambda \qquad (z \in \mathbb{U}) \tag{1.3}$$

for some real $\lambda > 0$.
Let us consider a function $f(z)$ given by

$$f(z) = \frac{z}{(1-z)^{\delta}} \qquad (\delta \geqq 0). \tag{1.4}$$

Then, we see that $\dfrac{f(z)}{z} = \dfrac{1}{(1-z)^{\delta}} \neq 0 \quad (z \in \mathbb{U})$,

$$\left| \left(\frac{z}{f(z)} \right)'' \right| = \left| \delta(\delta-1)(1-z)^{\delta-2} \right| < \delta(\delta-1)2^{\delta-2} \tag{1.5}$$

for $\delta \geqq 2$, and

$$\left| \left(\frac{z}{f(z)} \right)'''' \right| = \left| \delta(\delta-1)(\delta-2)(\delta-3)(1-z)^{\delta-4} \right| < \delta(\delta-1)(\delta-2)(\delta-3)2^{\delta-4} \tag{1.6}$$

for $\delta \geqq 4$. Therefore, Koebe function $f(z) = \dfrac{z}{(1-z)^2}$ belongs to $P_2(2)$ and $P_4(\lambda)$ for any $\lambda > 0$.
If we consider

$$f(z) = \frac{z}{\sum\limits_{k=0}^{n} z^k},$$

then

$$\left| \left(\frac{z}{f(z)} \right)'''' \right| = \left| \sum_{k=4}^{n} \frac{k!}{(k-4)!} z^{k-4} \right| < \sum_{k=4}^{n} \frac{k!}{(k-4)!} = \frac{(n-3)(n-2)(n-1)n(n+1)}{5}.$$

Therefor $f(z) \in P_4\left(\dfrac{(n-3)(n-2)(n-1)n(n+1)}{5} \right)$.

2 Main result

To consider our problem for the class $P_4(\lambda)$, we need the following lemma due to Goodman [1].

Lemma 1 *If $f(z) \in \mathcal{S}$ and*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \tag{2.1}$$

then we have

$$\sum_{n=1}^{\infty} (n-1)\left|b_n\right|^2 \leqq 1. \tag{2.2}$$

Further, we need the following lemma.

Lemma 2 *Let $f(z) \in \mathcal{A}$ and $\dfrac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \neq 0 \quad (z \in \mathbb{U})$. If $f(z)$ satisfies*

$$\sum_{n=4}^{\infty} \frac{n!}{(n-4)!}\left|b_n\right| \leqq \lambda, \tag{2.3}$$

then $f(z) \in P_4(\lambda)$.

Proof. We note that

$$\left| \left(\frac{z}{f(z)} \right)'''' \right| < \sum_{n=4}^{\infty} \frac{n!}{(n-4)!}\left|b_n\right|. \tag{2.4}$$

Thus, if $f(z)$ satisfies the inequality (2.3), then $f(z) \in P_4(\lambda)$. $\square$

Now, we derive

Theorem 1 *Let $f(z) \in \mathcal{S}$ and $\alpha \in \mathbb{C}$ $(|\alpha| < 1)$. Then the function $\dfrac{1}{\alpha} f(\alpha z)$ belongs to the class $P_4(\lambda)$ for $0 < |\alpha| \leqq |\alpha_0(\lambda)|$, where $|\alpha_0| = |\alpha_0(\lambda)|$ is the smallest root of the equation*

$$\lambda^2|\alpha|^{16} - (8\lambda^2 + 288)|\alpha|^{14} + (28\lambda^2 - 2496)|\alpha|^{12} - (56\lambda^2 + 2064)|\alpha|^{10}$$
$$+ (70\lambda^2 - 192)|\alpha|^8 - 56\lambda^2|\alpha|^6 + 28\lambda^2|\alpha|^4 - 8\lambda^2|\alpha|^2 + \lambda^2 = 0 \tag{2.5}$$

in $0 < |\alpha| < 1$.

Proof. Since $\dfrac{z}{f(z)} \neq 0 \quad (z \in \mathbb{U})$ for $f(z) \in \mathcal{S}$, if we write

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n,$$

then

$$\frac{z}{\frac{1}{\alpha} f(\alpha z)} = 1 + \sum_{n=1}^{\infty} \alpha^n b_n z^n \tag{2.6}$$

for $0 < |\alpha| < 1$. It follows that

$$\sum_{n=4}^{\infty} (n-1)\left|b_n\right|^2 \leqq \sum_{n=1}^{\infty} (n-1)\left|b_n\right|^2 \leqq 1 \tag{2,7}$$

from Lemma 1. To show that $\frac{1}{\alpha} f(\alpha z) \in P_4(\lambda)$, we have to prove that

$$\sum_{n=4}^{\infty} \frac{n!}{(n-4)!} \left|\alpha^n b_n\right| \leqq \lambda \tag{2.8}$$

by means of Lemma 2. Indeed, applying the Cauchy-Schwarz inequality for the left hand of (2.8), we obtain that

$$\begin{aligned}
\sum_{n=4}^{\infty} \frac{n!}{(n-4)!} \left|\alpha^n b_n\right| &= \sum_{n=4}^{\infty} \left(n^2 (n-1)(n-2)^2 (n-3)^2 |\alpha|^{2n}\right)^{\frac{1}{2}} \left((n-1)|b_n|^2\right)^{\frac{1}{2}} \\
&\leqq \left(\sum_{n=4}^{\infty} n^2 (n-1)(n-2)^2 (n-3)^2 |\alpha|^{2n}\right)^{\frac{1}{2}} \left(\sum_{n=4}^{\infty} (n-1)|b_n|^2\right)^{\frac{1}{2}} \\
&\leqq \left(\sum_{n=4}^{\infty} n^2 (n-1)(n-2)^2 (n-3)^2 |\alpha|^{2n}\right)^{\frac{1}{2}} \\
&= \frac{4|\alpha|^4 \sqrt{3\left(6|\alpha|^6 + 52|\alpha|^4 + 43|\alpha|^2 + 4\right)}}{\left(1-|\alpha|^2\right)^4} . \qquad (2.9)
\end{aligned}$$

Now, we consider the complex number α $\big(0 < |\alpha| < 1\big)$ such that

$$\frac{4|\alpha|^4 \sqrt{3\left(6|\alpha|^6 + 52|\alpha|^4 + 43|\alpha|^2 + 4\right)}}{\left(1-|\alpha|^2\right)^4} = \lambda. \tag{2.10}$$

This give that

$$h(|\alpha|) = \lambda^2|\alpha|^{16} - (8\lambda^2 + 288)|\alpha|^{14} + (28\lambda^2 - 2496)|\alpha|^{12} - (56\lambda^2 + 2064)|\alpha|^{10}$$
$$+(70\lambda^2 - 192)|\alpha|^8 - 56\lambda^2|\alpha|^6 + 28\lambda^2|\alpha|^4 - 8\lambda^2|\alpha|^2 + \lambda^2 = 0.$$

Noting that $h(0) = \lambda^2 > 0$ and $h(1) = -5040 < 0$, $h(|\alpha|) = 0$ has a root $|\alpha_0| = |\alpha_0(\lambda)|$ in $0 < |\alpha| < 1$. This completes the proof of the theorem. □

Remark 1 If we take $\alpha = \frac{1}{2}e^{i\theta}$ in (2.5), then we have

$$\lambda = \frac{8\sqrt{386}}{27} = 5.821 \cdots .$$

If we put $\lambda = 1$ in (2.5), then we see that

$$|\alpha|^{16}+280|\alpha|^{14}-2468|\alpha|^{12}+2008|\alpha|^{10}-122|\alpha|^8-56|\alpha|^6+28|\alpha|^4-8|\alpha|^2+1 = 0$$

has a root $|\alpha_0|$ such that $0.414 < |\alpha_0| < 0.415$.

3 Open problem

For the proof of Theorem 1, we apply Cauchy-Schwarz inequality given by

$$\sum |a_n||b_n| \leqq \left(\sum |a_n|^2\right)^{\frac{1}{2}} \left(\sum |b_n|^2\right)^{\frac{1}{2}}.$$

But we know that Hölder inequality given by

$$\sum |a_n||b_n| \leqq \left(\sum |a_n|^p\right)^{\frac{1}{p}} \left(\sum |b_n|^q\right)^{\frac{1}{q}} \qquad \left(p > 0, q > 0, \frac{1}{p} + \frac{1}{q} \geqq 1\right)$$

is the generalization inequality of Cauchy-Schwarz inequality. Therefore, if we find some application of Hölder inequality for the proof of Theorem 1 instead of Cauchy-Schwarz inequality, then we derive new result which is the generalization of Theorem 1.

References

[1] A.W.Goodman, Univalent Functions, Vol.I and II, Mariner, Tampa, Florida, 1983.

[2] M. Obradovć and S. Ponnusamy, Radius properties for subclasses of univalent functions,Analysis **25**(2005), 183-188.