Int. J. Open Problems Complex Analysis, Vol. 1, No. 1, June 2009
ISSN 2074-2827;
www.i-csrs.org

Bounded Turning for Generalized Integral Operator

Maslina Darus[1], Rabha W. Ibrahim[2] and Iqbal H. Jebril[3]

[1,2]School of Mathematical Sciences, Faculty of Science and Technology
Universiti Kebangsaan Malaysia, Bangi 43600 Selangor D. Ehsan, Malaysia
e-mail:[1]maslina@ukm.my(corresponding author) and [2]rabhaibrahim@yahoo.com
[3]Department of Mathematics, Faculty of Science
King Faisal University, Saudi Arabia
e-mail:[3]ijebril@kfu.edu.sa

Abstract

Modified integral operators on Fox-Wright functions are given. We determine conditions under which the partial sums of this integral operator of bounded turning are also of bounded turning. Further, an application of Cesáro means for this class is illustrated.

Keywords: *Integral operators; Fox-Wright functions; Hadamard products, Cesáro means; Partial Sums; Bounded Turning.*
AMS Mathematics Subject Classification (2000): 30C45.

1 Introduction and Definitions

Let $\mathcal{H}$ be the class of functions analytic in U and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U. \tag{1.1}$$

Consider two functions $f, g \in \mathcal{A}$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then their convolution or Hadamard product $f(z) * g(z)$ is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \ \ z \in U.$$

For several functions $f_1(z), ..., f_m(z) \in \mathcal{A}$, we can write

$$f_1(z) * ... * f_m(z) = z + \sum_{n=2}^{\infty} (a_{1n}...a_{mn}) z^n, \ \ , \ z \in U.$$

For complex parameters

$$\alpha_1, ..., \alpha_q \quad (\frac{\alpha_j}{A_j} \neq 0, -1, -2, ... \ ; j = 1, ..., q)$$

and

$$\beta_1, ..., \beta_p \quad (\frac{\beta_j}{B_j} \neq 0, -1, -2, ... \ ; j = 1, ..., p),$$

we state the Fox-Wright generalization ${}_q\Psi_p[z]$ of the hypergeometric ${}_qF_p$ function (see [1-3])as

$$\begin{aligned} {}_q\Psi_p \left[\begin{array}{ll} (\alpha_1, A_1), ..., (\alpha_q, A_q); & \\ & z \\ (\beta_1, B_1), ..., (\beta_p, B_p); & \end{array} \right] &= {}_q\Psi_p[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}; z] \\ &:= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + nA_1)...\Gamma(\alpha_q + nA_q)}{\Gamma(\beta_1 + nB_1)...\Gamma(\beta_p + nB_p)} \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{q} \Gamma(\alpha_j + nA_j)}{\prod_{j=1}^{p} \Gamma(\beta_j + nB_j)} \frac{z^n}{n!} \end{aligned}$$

where $A_j > 0$ for all $j = 1, ..., q$, $B_j > 0$ for all $j = 1, ..., p$ and $1 + \sum_{j=1}^{p} B_j - \sum_{j=1}^{q} A_j \geq 0$ for suitable values $|z|$. For the special case, where $A_j = 1$ for all $j = 1, ..., q$, and $B_j = 1$ for all $j = 1, ..., p$ the following relationship holds:

$${}_qF_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_p; z) = \Omega {}_q\Psi_p[(\alpha_j, 1)_{1,q}; (\beta_j, 1)_{1,p}; z],$$

$$q \leq p + 1; q, p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \ \ z \in U$$

where

$$\Omega := \frac{\Gamma(\beta_1)...\Gamma(\beta_p)}{\Gamma(\alpha_1)...\Gamma(\alpha_q)}.$$

Let

$$\begin{aligned} \Phi(z) :&= z\, {}_q\Psi_p[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}; z] \underbrace{*...*}_{k-times} z\, {}_q\Psi_p[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}; z] \\ &= z + \sum_{n=2}^{\infty} \Big[\frac{\prod_{j=1}^{q} \Gamma(\alpha_j + (n-1)A_j)}{\prod_{j=1}^{p} \Gamma(\beta_j + (n-1)B_j)} \frac{1}{(n-1)!} \Big]^k z^n, \ \ k \in \mathbb{N}_0. \end{aligned}$$

We introduce a function $[\Phi(z)]^{-1}$ given by

$$\begin{aligned}\Phi(z) * [\Phi(z)]^{-1} &= \frac{z}{(1-z)^{\lambda+1}}\\ &= z + \sum_{n=2}^{\infty} \frac{(\lambda+1)_{n-1}}{(n-1)!},\ (\lambda > -1)\end{aligned}$$

and obtain the following generalized operator:

$$I_\lambda^k[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}]f(z) = [\Phi(z)]^{-1} * f(z), \tag{1.2}$$

where $f \in \mathcal{A}$, $z \in U$ and

$$[\Phi(z)]^{-1} = z + \sum_{n=2}^{\infty} \Big[\frac{\prod_{j=1}^{p} \Gamma(\beta_j + (n-1)B_j)}{\prod_{j=1}^{q} \Gamma(\alpha_j + (n-1)A_j)}(n-1)!\Big]^k \frac{(\lambda+1)_{n-1}}{(n-1)!} z^n.$$

A computation gives us

$$\begin{aligned}I_\lambda^k[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}]f(z) &= z + \sum_{n=2}^{\infty} \Big[\frac{\prod_{j=1}^{p} \Gamma(\beta_j + (n-1)B_j)}{\prod_{j=1}^{q} \Gamma(\alpha_j + (n-1)A_j)}(n-1)!\Big]^k\\ &\times \frac{(\lambda+1)_{n-1}}{(n-1)!} a_n z^n\end{aligned} \tag{1.3}$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n = 0 \\ a(a+1)...(a+n-1), & n = \{1, 2, ...\}. \end{cases}$$

Remark 1.1. When $k = 1$, the operator (3) reduces to the integral operator defined by the authors [4], and in fact, is a generalization of the Noor integral operator defined by a hypergeometric functions [5]. Note also a special case of the operator (3) can be found in [6] by Carlson and Shaffer.
The following result follows from (3:

Lemma 1.1. Let $f \in \mathcal{A}$ for all $z \in U$ then

(1) $I_0^0[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}]f(z) = f(z)$.
(2) $I_1^0[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}]f(z) = zf'(z)$.

For $0 \le \mu < 1$, let $B(\mu)$ denote the class of functions f of the form (1) so that $\Re\{f'\} > \mu$ in U. The functions in $B(\mu)$ are called functions of bounded turning (c.f. [7, Vol. II]). Nashiro-Warschowski Theorem (see e.g. [7, Vol. I]) stated that the functions in $B(\mu)$ are univalent and also close-to-convex in U.

In the sequel we need to the following results.

Lemma 1.2. [8] For $z \in U$ we have

$$\Re\Big\{\sum_{n=1}^{j}\frac{z^n}{n+2}\Big\} > -\frac{1}{3}, \quad (z \in U).$$

Lemma 1.3. [7, Vol. I] Let $P(z)$ be analytic in U, such that $P(0) = 1$, and $\Re(P(z)) > \frac{1}{2}$ in U. For functions Q analytic in U the convolution function $P * Q$ takes values in the convex hull of the image on U under Q.

2 Main results

To make use of Lemma 1.2 and Lemma 1.3, we illustrate the conditions under which the $m-$th partial sums (4) of the integral operator (3) of bounded turning are also of bounded turning. The $m-$th partial sums of the operators (3) are given by

$$P_m(z) = z + \sum_{n=2}^{m} H_{n-1}^{k}\frac{(\lambda+1)_{n-1}}{(n-1)!}a_n z^n, \quad (z \in U), \tag{2.1}$$

where

$$H_{n-1}^{k} = \Big[\frac{\prod_{j=1}^{p}\Gamma(\beta_j+(n-1)B_j)}{\prod_{j=1}^{q}\Gamma(\alpha_j+(n-1)A_j)}(n-1)!\Big]^k$$

Theorem 2.1. Assume that $\lambda = 0$ and $H_{n-1}^{k} > 1$. Let $f \in \mathcal{A}$. If $\frac{1}{2} < \mu < 1$ and $f(z) \in B(\mu)$, then $P_m(z) \in B\big(\frac{2+\mu}{3}\big)$.

Proof. Let f be of the form (1) and $f(z) \in B(\mu)$ that is

$$\Re\{f'(z)\} > \mu, \ (\frac{1}{2} < \mu < 1, \ z \in U).$$

Implies

$$\Re\{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\} > \mu > \frac{1}{2}.$$

Now for $\frac{1}{2} < \mu < 1$ we have

$$\Re\Big\{1 + \sum_{n=2}^{\infty} a_n \frac{n}{1-\mu} z^{n-1}\Big\} > \Re\Big\{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\Big\}.$$

It is clear that

$$\Re\Big\{1+\sum_{n=2}^{\infty}\frac{nH^{k}_{n-1}}{1-\mu}a_n z^{n-1}\Big\}>\frac{1}{2}. \tag{2.2}$$

Applying the convolution properties of power series to $P'_m(z)$, we may write

$$\begin{aligned} P'_m(z) &= 1+\sum_{n=2}^{m} H^{k}_{n-1}\frac{(\lambda+1)_{n-1}}{(n-1)!}na_n z^{n-1}\\ &= \Big[1+\sum_{n=2}^{m}\frac{nH^{k}_{n-1}}{(1-\mu)}a_n z^{n-1}\Big]*\Big[1+\sum_{n=2}^{m}(1-\mu)z^{n-1}\Big] \\ &:= P(z)*Q(z). \end{aligned} \tag{2.3}$$

In virtue of Lemma 1.2 and for $j=m-1$, we obtain

$$\Re\Big\{\sum_{n=2}^{k}\frac{z^{n-1}}{n+1}\Big\}\geq -\frac{1}{3}. \tag{2.4}$$

Since

$$\Re\Big\{\sum_{n=2}^{k}z^{n-1}\Big\}\geq \Re\Big\{\sum_{n=2}^{k}\frac{z^{n-1}}{n+1}\Big\}. \tag{2.5}$$

Then we have

$$\Re\Big\{\sum_{n=2}^{k}z^{n-1}\Big\}\geq -\frac{1}{3}. \tag{2.6}$$

A computation gives

$$\Re\Big\{Q(z)\Big\}=\Re\Big\{1+\sum_{n=2}^{k}(1-\mu)z^{n-1}\Big\}>\frac{2+\mu}{3}.$$

On the other hand, the power series

$$P(z)=\Big[1+\sum_{n=2}^{m}\frac{nH^{k}_{n-1}}{(1-\mu)}a_n z^{n-1}\Big],\ \ (z\in U)$$

satisfies: $P(0)=1$ and

$$\Re\Big\{P(z)\Big\}=\Re\Big\{1+\sum_{n=2}^{\infty}\frac{nH^{k}_{n-1}}{1-\mu}a_n z^{n-1}\Big\}>\frac{1}{2},\ \ (z\in U).$$

Therefore, by Lemma 1.3, we have

$$\Re\Big\{P'_m(z)\Big\}>\frac{2+\mu}{3},\ \ (z\ \in U).$$

This completes the proof of Theorem 2.1.

Next we determine the bounded turning for the Cesáro sums of order ν where $\nu \in \mathbb{N} \cup \{0\}$ of the operator (3).

$$\sigma_m^\nu(z, I_\lambda^k[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}]f(z)) = \sigma_m^\nu * I_\lambda^k[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}]f(z)$$

$$= \sum_{n=0}^{k} \frac{\begin{pmatrix} m-n+\nu \\ m-n \end{pmatrix}}{\begin{pmatrix} m+\nu \\ m \end{pmatrix}} \Big[\frac{\prod_{j=1}^{p}\Gamma(\beta_j + (n-1)B_j)}{\prod_{j=1}^{q}\Gamma(\alpha_j + (n-1)A_j)}(n-1)!\Big]^k \frac{(\lambda+1)_{n-1}}{(n-1)!} a_n z^n \tag{2.7}$$

where $\begin{pmatrix} a \\ b \end{pmatrix} = \frac{a!}{b!(a-b)!}$. We observe that

$$\frac{\begin{pmatrix} m-n+\nu \\ m-n \end{pmatrix}}{\begin{pmatrix} m+\nu \\ m \end{pmatrix}} = \frac{m!(m-n+\nu)!}{(m-n)!(m+\nu)!} \leq 1 \tag{2.8}$$

for $\nu \geq 0$ *and* $n = 0, 1, ..., m$. In the same manner of Theorem 2.1, we pose the following result.

Theorem 2.2. Let λ and H_{n-1}^k as in Theorem 2.1. If $\frac{1}{2} < \mu < 1$ and $f(z) \in B(\mu)$, then $\sigma_m^\nu(z, I_\lambda^k[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}]f(z)) \in B\big(\frac{2+\mu}{3}\big)$.

In [9] the authors determined the Cesáro means for operators containing Fox-Wright functions.

Theorem 2.3. Let $f_j(z) \in B(\mu),\ \ j = 1, ..., m,\ \ 0 \leq \mu < 1$. Then the arithmetic mean of $f_j(z)$ defined by

$$F(z) = \frac{1}{m}\sum_{j=1}^{m} f_j(z),\ \ (z \in U)$$

is also in $B(\mu)$.

Proof. Since for all $j = 1, ..., m$,

$$\Re\{f_j'(z)\} > \mu,\ \ (0 \leq \mu < 1,\ \ z \in U)$$

then

$$\Re\{F'(z)\} = \frac{1}{m}\Re\{\sum_{j=1}^{m} f_j'(z)\} = \frac{1}{m}\sum_{j=1}^{m}\Re\{f_j'(z)\} > \mu.$$

Hence $F(z) \in B(\mu)$.
In the same way of Theorem 2.3, we introduce the following result.

Theorem 2.4. Let $f_j(z) \in B(\mu)$, $j = 1, 2$, $0 \le \mu < 1$. Then the weighted mean of f_1 and f_2 defined by

$$W_m(z) = \frac{1}{2}[(1-m)f_1(z) + (1+m)f_2(z)], \ \ (z \in U)$$

is also in $B(\mu)$.

3 Open Problem

The definitions and theorems we establish can be extended into N-symmetric functions, N-conjugate symmetric functions and many others.

Acknowledgement: The authors were supported in part by ScienceFund: 04-01-02-SF0425, MOSTI, Malaysia.

References

[1] E. M. Wright, The asymptotic expansion of the generalized hypergeometric function, J. London Math. Soc., 10(1935), 286-293.

[2] E. M. Wright, The asymptotic expansion of the generalized hypergeometric function, J. London Math. Soc.,(Ser.2)46(1940), 389-408.

[3] C. Fox, The asymptotic expansion of the generalized hypergeometric function, J. London Math. Soc., (Ser.2), 27(1928), 389-400.

[4] R. W. Ibrahim, M. Darus, New classes of analytic functions involving generalized Noor Integral operator, J.Ineq. and Appl.10.1155 (2008), 1-14.

[5] K. Noor, Integral operators defined by convolusion with hypergeometric functions, App. Math. Comp.,182(2006), 1872-1881.

[6] C. Carlson, D. Shaffer, Starlike and prestarlike hypergeometric functions, SIAM J Math.Anal. 15(1984), 737-745.

[7] A. W. Goodman, Univalent Functions, Vols. I and II, Polygonal Publishing House, Washington, New Jersey, 1983.

[8] J. M. Jahangiri, K. Farahmad, Partial sums of functions of bounded turning, J. Inequal. Pure and Appl. Math., 4(4) Art. 79, (2003), 1-3.

[9] M. Darus, R. W. Ibrahim, On Cesáro means for Fox-Wright functions, J. Math. and Stat.: 4(3): (2008), 156-160.